Bol. Soc. Paran. Mat. (3s.) **v. 2025 (43)** : 1–16.
©SPM – E-ISSN-2175-1188 ISSN-0037-8712
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.65722

A Study on the Sensitivity Analysis of the Parameter of the Generalized Mean-Variance-Skewness Model for Portfolio Selection Problem

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ABSTRACT: In this paper, the concept of h-zigzag uncertainty distribution as a generalization of the zigzag uncertainty distribution is introduced. The same has been applied to the construction of a generalized mean-variance-skewness model of portfolio optimization. The introduced distribution has the unique feature of accommodating different types of distributions comprising of two line segments with a mere change in the value of the parameter h. The method of solution to the proposed model has been illustrated by constructing a portfolio selection problem based on an expert's opinion and available information collected from the National Stock Exchange (NSE), India. A sensitivity analysis of the optimal solution with respect to the parameter h has been performed, and observations are analyzed. The optimal solutions for different values of h are compared. The results developed in this article generalize and unify several existing results established by researchers.

Key Words: Portfolio optimization problem, uncertainty distribution, chance distribution, uncertain random variable, skewness.

Contents

1. Introduction

In portfolio selection problem one wishes to get a list of stocks for which the total return earned is maximized and at the same time the concomitant risk in investment becomes minimum. In 1952, Markowitz [20] did fundamental research work for modern finance theory and then the analysis of modern portfolio selection problem started. In his model, he maximized the expected return of the investment and minimized the risk, measured by the total variance of the returns earned from all the stocks.

In portfolio selection problems, the expected value and variance of the returns earned from the stocks are generally calculated by considering the returns as random variables. Such estimation of expected value and variance requires enough historical data to evaluate the parameters of the probability distributions of the random returns. But in reality, there are some stocks in the share market which have recently been entered, and they naturally lack sufficient past records. Also, there are some new share markets that have come into existence in the recent past. Some of these new stocks may also be promising in respect of giving higher returns. In such cases, an alternative option is to estimate returns by taking experts' considerate opinions.

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Submitted November 04, 2022. Published May 23, 2025
2010 *Mathematics Subject Classification:* 74Pxx, 90-xx, 90C30, 91G10.

Considering the returns of newly listed stocks as fuzzy variables, some researchers have developed a fuzzy portfolio optimization theory. They constructed the membership function of the fuzzy variables by taking experts' opinions. In their works, we can find the applications of possibility theory [2,33], fuzzy set theory [6] and credibility theory [7,26]. These are the three main methods on which the study of fuzzy optimization theory is based. If one can estimate the parameters needed to construct the membership functions accurately, then the fuzzy methods of portfolio optimization give an alternative way of dealing with the stocks, lack of data. However, the fuzzy methods have some drawbacks too. A paradox will appear when fuzzy variables are used to describe the return of the stock. One may refer to [9].

To deal with the inherent uncertainty existing in the human thinking process in an alternative way besides probability theory and fuzzy set theory, Liu [15] introduced the uncertainty theory in 2007. Lots of research work has been done to develop the uncertainty theory [3,5,30,31] and its application in real-life situations [13,15,17,24]. Uncertainty theory can also be used to model and solve portfolio selection problems. To this end, pioneering work has been done by Qin et al. [27]. They applied the uncertainty theory to develop a mean-variance model of portfolio selection. Further, Huang [8] proposed a risk-index model for portfolio selection where returns from the securities have lack of historical data and are estimated on the basis of experts' opinions. He also developed a new measure for adjudicating the portfolio investment in a safer way. Huang and Ying [9] established risk index based models for portfolio adjusting problem with uncertain returns subject to experts' evaluation. They have proposed two portfolio optimization models for an existing portfolio in two cases. Their study reveals that the expected security return is smaller in the case when the risk-less assets can only be lent than in the case when they can be both borrowed and lent. To measure risk, Liu and Qin [19] introduced semi-absolute deviation of uncertain variable and constructed a model, namely return-risk model and then Qin et al. [25] put forward an uncertain portfolio adjusting model using semi-absolute deviation. Along with these optimization models, Huang and Qiao [10] proposed a risk index model for multi-period uncertain portfolio selection, and Zhu [34] introduced the uncertain optimal control with an application to a portfolio selection. The list of such works using uncertainty theory is long. Thus, it is evident that portfolio selection problems considering the security return as an uncertain variable is a vibrant area of research.

It is well known that, skewness is one of the measures of the asymmetry factor in a distribution. Skewness is widely used in various real-life problems, including portfolio selection problems. A decision maker always prefers a positive skewness in return distribution, as a positive skewness ensures a greater return than the expected return. To a decision maker who gives minimization of risk as a priority prefers positive skewness in return distribution. Research carried out in this direction is as follows:

Kunno and Suzuki [11] introduced a mean-variance-skewness model for portfolio selection, where they maximized the third order moment of rate of return subject to some constraints on the mean and variance of the return. Chen et al. [3] applied intuitionistic fuzzy sets to reformulate the optimization problem, considering degree of membership as well as non-membership, and applied this concept to the portfolio selection problem. Bhattacharyya et al. [1] introduced fuzzy cross-entropy, mean, variance, skewness models for portfolio selection, where they maximized expected return as well as skewness of the return and minimized portfolio variance and cross entropy. They also observed that maximizing skewness actually generates a larger payoff. Ray and Majumder [28] proposed a new non-Shannon fuzzy Mean–Variance–Skewness-entropy model considering stock returns as triangular fuzzy numbers. In their proposed model, they maximized mean and skewness and minimized portfolio variance and cross-entropy in terms of Burg. Considering returns as fuzzy variables Li et al. [14] also worked on a mean-variance-skewness model for portfolio selection. By using an improved genetic algorithm, in 2021, Mittal and Srivastava [22] constructed mean-variance-skewness portfolio optimization under an uncertain environment. They have also developed a hybrid intelligent algorithm to solve the portfolio problem. Other important works in this field that emphasized skewness maximization were those by Li and Shu [12]; Metaxiotis [21].

A further study of the portfolio selection problem has been undertaken in a generalized way in this paper. We have considered two stocks, one of which has enough past records and the other one is newly introduced to the market. For the stock having enough historical data, the returns are random in nature and the return distribution has been taken as standard normal. For the newly listed stock, the returns are assumed to be uncertain following a new uncertainty distribution, viz., h-zigzag, introduced in this paper. The total return, which is a combination of these two types of returns, follows an uncertain random distribution or chance distribution [17]. The Expected value, variance, and skewness of the uncertain random return have been obtained and are put into the form of theorems. The corresponding expressions for zigzag uncertainty distribution [12] then follow as a particular case of our results. Moreover, the new generalized uncertainty distribution is capable of reflecting the real situation more closely than other previously introduced methods due to the presence of the parameter h which can be chosen by the decision maker. The new distribution can represent symmetric as well as asymmetric (positively / negatively skewed) uncertain return distribution, depending on the selection of the parameter h. It should be noted that all previous works involving a zigzag uncertainty return distribution function are symmetric in terms of the second parameter. Thus, it can be concluded that h-zigzag uncertainty distribution introduced in this paper is capable of reflecting uncertainty more aptly than zigzag uncertain distribution used by Li and Shu [12].

To achieve our aim, the paper is organized as follows: In Section 2, we put forward some definitions, preliminaries, and results related to uncertainty theory, which are relevant and necessary for our paper. Next, in Section 3, we introduced a new uncertainty distribution, viz., h-zigzag uncertainty distribution. Then the expected value, variance, and skewness of uncertain random return are obtained. In Section 4, we stated the problem of optimal portfolio selection where the returns are random and uncertain. In the proposed model, the total skewness of the uncertain random return is maximized keeping its expected value and variance at some pre-assigned levels. In Section 5, we have performed the sensitivity analysis with respect to the parameter h and thus obtained the optimal solution to the problem stated in Section 4 and also illustrated the procedure for solving the models. Solutions of the constructed problems have been obtained for different values of the parameter h and different assigned levels of the variance (measure of risk) and expected return. Section 6 contains the conclusions arrived at, and the references relevant to our study have been listed.

2. Definition and Preliminaries

Here we put forward some of the basic definitions and results regarding uncertainty theory which are relevant and will be used in our study ahead without any specific references.

For the sake of ease of the readers the definitions of uncertain measure, probability measure, uncertain variable, uncertain random variable and its distribution have been recollected first.

Definition 2.1 *[15] Let Γ be a non-empty set and $\mathcal{L}$ be a σ algebra over Γ. The elements of $\mathcal{L}$ is termed as events. A set function $\mathcal{M}\colon \mathcal{L} \to [0,1]$ is called an uncertain measure if it satisfies the following three axioms:*

(1) $\mathcal{M}\{\Gamma\} = 1$, *for the universal set* Γ.

(2) $\mathcal{M}\{A\} + \mathcal{M}\{A^c\} = 1$, *for any* $A \in \mathcal{L}$, A^c *denotes as usual the compliment of* A *in* Γ.

(3) *For every countable sequence of events* $A_1, A_2, ... \in \mathcal{L}$, *we have*

$$\mathcal{M}\left\{ \bigcup_{i=1}^{\infty} A_i \right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{A_i\}.$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ *is called an uncertainty space.*

Next, we place some important definitions in the realm of uncertainty theory.

Definition 2.2 *[15] An uncertain variable ξ is a measurable function from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \ \in \ B\} = \{\gamma \ \in \ \Gamma \mid \xi(\gamma) \in B\} \in \mathcal{L}.$$

Definition 2.3 *[15,16] For any real number x, the uncertainty distribution function Φ of an uncertain variable ξ defined on an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ is given by*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\},$$

where $\{\xi \leq x\} = \{\gamma \in \Gamma : \xi(\gamma) \leq x\}$.

It is said to be regular if it is continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1.$$

Next, we recall an important uncertain distribution due to Liu [15,16] called zigzag uncertainty distribution.
For the following uncertainty distribution we consider Γ to be the close interval $[a, c]$ of real numbers.

Definition 2.4 *[16] An uncertain variable ξ is called zigzag if it has a zigzag uncertainty distribution function given by*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a \ ; \\ \frac{x-a}{2(b-a)}, & \text{if } a \leq x \leq b \ ; \\ \frac{x+c-2b}{2(c-b)}, & \text{if } b \leq x \leq c \ ; \\ 1, & \text{if } x \geq c, \end{cases}$$

and we denote an uncertain zigzag variable by $\boldsymbol{Z}(a, b, c)$ and we write $\xi \sim \boldsymbol{Z}(a, b, c)$, where a, b, c are real numbers with $a < b < c$.

Definition 2.5 *[15] The expected value of an uncertain variable ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\, dx \ - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}\, dx,$$

provided that at least one of the two integrals exits finitely.

Definition 2.6 *[15] Let ξ be an uncertain variable with finite expected value e. Then the variance of ξ is given by*

$$V[\xi] = E[(\xi - e)^2]$$

Lemma 2.1 *[16] If ξ is an uncertain variable with finite expected value e, then*

$$V[a\xi + b] = a^2 V[\xi]$$

Definition 2.7 *[32] Let ξ be an uncertain variable with finite expected value e. Then the skewness of ξ is*

$$Sk[\xi] = E[(\xi - e)^3].$$

If we are to examine the simultaneous effect of randomness and uncertainty in some situation, then we have to mingle the probability space and uncertainty space. This is what is done in chance space.

Definition 2.8 *[17] Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be an uncertainty space and $(\Omega, \mathcal{A}, \mathcal{P},)$ be a probability space. The product $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P},)$ is called a chance space.*
It is known that in a probability space $(\Omega, \mathcal{A}, \mathcal{P})$, Ω is the sample space, $\mathcal{A}$ is the σ algebra of the subsets of Ω and $\mathcal{P}$ is the probability measure $\mathcal{P} \colon \mathcal{A} \to [0, 1]$.

Definition 2.9 *[17] Let* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P},)$ *is a chance space and let* $\Theta \in \mathcal{L} \times \mathcal{A}$, *i.e.,* $\Theta = G \times W; G \in \mathcal{L}, W \in \mathcal{A}$ *Then the chance measure of uncertain random event* Θ *is defined as*

$$ch\{\Theta\} = \int_0^1 \mathcal{P}\{\omega \in \Omega \mid \mathcal{M}\{\gamma \in \Gamma \mid (\gamma, \omega) \in \Theta\} \geq r\}\ dr.$$

Definition 2.10 *[17] An uncertain random variable is a measurable function* $\tilde{\xi}$ *from the chance space* $(\Gamma, \mathcal{L}, \mathcal{M}) \times (\Omega, \mathcal{A}, \mathcal{P},)$ *to the set of real numbers, i.e., for any Borel set* B *of set of real numbers* R, $\{\tilde{\xi} \in B\} = \{(\gamma, \omega) : \tilde{\xi}(\gamma, \omega) \in B\} \in \mathcal{L} \times \mathcal{A}$ *i.e., an event in the chance space, where* $\gamma \in \Gamma, \omega \in \Omega$. *Its chance distribution is defined by* $\Psi(x) = ch\{\tilde{\xi} \leq x\}$, *for any* $x \in R$,
where $\{\tilde{\xi} \leq x\} = \{(\gamma, \omega)\ \in \Theta \mid \gamma \in \Gamma, \omega \in \Omega$ *and* $\tilde{\xi}(\gamma, \omega) \leq x\} \in \mathcal{L} \times \mathcal{A}$.

Remark 2.1 *[18] Let* η *be a random variable and* τ *be an uncertain variable then* $\tilde{\xi} = \eta + \tau$ *is an uncertain random variable.*

Theorem 2.1 *[18] Let* $\alpha_1, \alpha_2, ..., \alpha_m$ *be independent random variables with probability distribution functions* $\phi_1, \phi_2, ..., \phi_m$ *respectively and let* $\beta_1, \beta_2, ..., \beta_n$ *be uncertain variables. Then the uncertain random variable* $\tilde{\xi} = f(\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n)$ *has a chance distribution*

$$\Psi(x) = \textstyle\int_{R^m} F(x; y_1, y_2, ..., y_m)\ d\phi_1(y_1)...d\phi_m(y_m)$$

where $F(x; y_1, y_2, ..., y_m)$ *is the uncertainty distribution of uncertain variable* $f(y_1, y_2, ..., y_m, \beta_1, \beta_2, ..., \beta_n)$ *and* $y_1, y_2, ..., y_m$ *are respectively the particular values of the random variables* $\alpha_1, \alpha_2, ..., \alpha_m$ *and* x *is a real number.*

It was also mentioned in [18] that if
$\tilde{\xi} = f(\alpha_1, \alpha_2, ..., \alpha_m, \beta_1, \beta_2, ..., \beta_n) = \alpha_1 + \alpha_2 + ... + \alpha_m + \beta_1 + \beta_2 + ... + \beta_n$
then $\tilde{\xi}$ has chance distribution

$$\Psi(x) = \int_{-\infty}^{+\infty} \Upsilon(x - y) d\phi(y) \tag{2.1}$$

where

$$\phi(y) = \int_{y_1 + y_2 + ... y_m \leq y} d\phi_1(y_1) d\phi_2(y_2)...d\phi_m(y_m)$$

is the probability distribution of $\alpha_1 + \alpha_2 + ... + \alpha_m$, and

$$\Upsilon(z) = \sup_{z_1 + z_2 + ... + z_n = z} \Upsilon_1(z_1) \wedge \Upsilon_2(z_2) \wedge ... \wedge \Upsilon_n(z_n)$$

is the uncertainty distribution $\beta_1 + \beta_2 + ... + \beta_n$.

Definition 2.11 *[17] Let* $\tilde{\xi}$ *be an uncertain random variable. Then its expected value is defined by*

$$E[\tilde{\xi}] = \int_0^{+\infty} ch\{\tilde{\xi} \geq x\}\, dx\ - \int_{-\infty}^0 ch\{\tilde{\xi} \leq x\}\, dx,$$

provided at least one of the two integrals exits finitely.

Theorem 2.2 *[18] If* α *is a random variable and* β *is an uncertain variable then the expected value of the uncertain random variable* $\alpha + \beta$ *is given by* $E[\alpha + \beta] = E[\alpha] + E[\beta]$.

Definition 2.12 *[17] Let* $\tilde{\xi}$ *be an uncertain random variable with finite expected value* $E[\tilde{\xi}] = e$. *Then its variance is defined by*

$$V[\tilde{\xi}] = E[(\tilde{\xi} - e)^2].$$

Theorem 2.3 *[29] Let* $\tilde{\xi}$ *be an uncertain random variable with finite expected value* $E[\tilde{\xi}] = e$ *and with chance distribution* Ψ *then*

$$V[\tilde{\xi}] = \int_{-\infty}^{\infty} (x - e)^2 d\Psi(x) \tag{2.2}$$

Definition 2.13 *[12] Let $\tilde{\xi}$ be an uncertain random variable with finite expected value $E[\tilde{\xi}] = e$. Then its skewness is defined by*

$$Sk[\tilde{\xi}] = E[(\tilde{\xi} - e)^3].$$

Theorem 2.4 *[12] Let $\tilde{\xi}$ be an uncertain random variable with a chance distribution Ψ and a finite expected value $E[\tilde{\xi}] = e$. Then the skewness $\tilde{\xi}$ is*

$$Sk[\tilde{\xi}] = \int_{-\infty}^{+\infty} (x - e)^3 d\Psi(x). \tag{2.3}$$

Theorem 2.5 *[12] Let $\tilde{\xi}$ be an uncertain random variable with a finite expected value $E[\tilde{\xi}]$. For any constants a and b, we have*

$$Sk[a\tilde{\xi} + b] = a^3 Sk[\tilde{\xi}].$$

With this mathematical background we now state the problem of portfolio selection under the simultaneous presence of random and uncertain returns.Also provide a method of solution of the same.

3. h-zigzag uncertainty distribution and its properties

In this section we first define h-zigzag uncertainty distribution and assume that uncertain return follows this distribution.

Definition 3.1 *An uncertain variable ξ is called h-zigzag if its uncertainty distribution function is given by*

$$\Phi(x) = \begin{cases} 0, & \text{if } x \leq a; \\ \frac{h(x-a)}{(b-a)}, & \text{if } a \leq x \leq b; \\ h + \frac{(1-h)(x-b)}{(c-b)}, & \text{if } b \leq x \leq c; \\ 1, & \text{if } x \geq c, \end{cases}$$

and we denote it by $\boldsymbol{Z_h}(a, b, c; h)$ and write $\xi \sim \boldsymbol{Z_h}(a, b, c; h)$, where a, b, c, h are real numbers with $a < b < c$ and $0 < h < 1$.

Clearly, a zigzag uncertain variable is a particular case of an h-zigzag uncertain variable for $h = 1/2$. h-zigzag uncertainty distribution is flexible, by varying h we can find different uncertainty distributions comprising two line segments. It is obvious that the decision-maker does not know in advance, which value of h is suitable for him or her in respect of giving objective values based on the experimental data set provided by an expert. Thus, it is better to choose different values of h and compare the results obtained. The most suitable one, considering expected value, variance, and skewness, may then be taken. For this, we need general expressions for expected value, variance, and skewness in terms of h. The theory developed on the portfolio selection problem in this paper actually generalizes the work of Li and Shu [12]. The results obtained by Li and Shu [12] follow directly from our result as a particular case, by putting $h = 1/2$.

Our aim is to find an optimal portfolio consisting of two stocks, one with a random return and the other with an uncertain return. In our proposed model, we achieve this by maximizing the skewness of the total return earned from the investment while keeping the expected return and variance of the uncertain random returns at some given levels u and v respectively.

3.1. Calculation of Expected Value, Variance and Skewness of Uncertain Random Return

Here we find out the expressions for the expected value, variance, and skewness of the uncertain random return using chance distribution. We assume that the random return follows standard normal distribution having density function f and distribution function ϕ, where $d\phi(z) = f(z)dz$ and $f(z) = \frac{1}{\sqrt{2\pi}} exp(-\frac{z^2}{2}), -\infty < z < +\infty$.

With the stated target, we first find the expected value and variance of an h-zigzag uncertain variable in the form of theorems.

Theorem 3.1 *The h-zigzag uncertain variable* $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ *has expected value is given by*

$$E[\beta] = \frac{ah + b + (1-h)c}{2}. \tag{3.1}$$

Proof: By the definition of expected value of an uncertain variable η we have for any real number x

$$\begin{aligned} E[\beta] &= \int_0^{+\infty} \mathcal{M}\{\eta \geq x\}dx - \int_{-\infty}^{0} \mathcal{M}\{\eta \leq x\}dx \\ &= \int_0^{+\infty} (1-\Phi(x))dx - \int_{-\infty}^{0} \Phi(x)dx. \end{aligned} \tag{3.2}$$

Now we consider the following exhaustive cases (recall $a < b < c$):
(i) $a \geq 0$,
(ii) $c \leq 0$,
(iii) $a \leq 0,\ b > 0$,
(iv) $b \leq 0,\ c > 0$.
Let us consider the case (i) and performing the indicated integration in equation (3.2), we get

$$\begin{aligned} E[\beta] &= \int_0^a dx + \int_a^b \left\{1 - h\left(\frac{x-a}{b-a}\right)\right\}dx + \int_b^c \left\{1 - \left(h + \frac{(1-h)(x-b)}{c-b}\right)\right\}dx \\ &= a + \frac{(2-h)(b-a)}{2} + \frac{(1-h)(c-b)}{2} \\ &= \frac{ah + b + c(1-h)}{2}. \end{aligned}$$

Similarly, in the other three cases also the same result follow.
Hence, $E[\beta] = \frac{ah+b+c(1-h)}{2}$. □

Theorem 3.2 *The h-zigzag uncertain variable* $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ *has the variance given by*

$$V[\beta] = \frac{h}{3}\Big[(b-a)^2 + 3(b-A)(a-A)\Big] + \frac{(1-h)}{3}\Big[(c-b)^2 + 3(c-A)(b-A)\Big],$$

where $A = \frac{ah+b+c(1-h)}{2}$.

Proof: Using Definition 2.6 and Theorem 3.1, the theorem can easily be proved. □

From the expression of the expected value of the uncertain variable β, it is clear that $a < E[\beta] < c$, since $a < b < c$ and $0 < h < 1$. But $E[\beta]$ may be greater than or equal to b or less than or equal to b depending on the magnitudes of $b-a$ and $c-b$.

Next, we state a result that follows as a corollary of Theorem 2.2. The result will be used in Theorems 3.2 and 3.3.

Corollary 3.1 *If* $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ *is h-zigzag uncertain variable, and* $\alpha \sim N(0,1)$*, the standard normal distribution, then the expected value of the uncertain random variable* $\tilde{\xi} = \alpha + \beta$ *is given by* $E[\tilde{\xi}] = E[\alpha+\beta] = E[\alpha] + E[\beta] = E[\beta]$.

Theorem 3.3 *If $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ is h-zigzag uncertain variable, and $\alpha \sim N(0,1)$ is random variable, then the variance of the uncertain random variable $\tilde{\xi} = \alpha + \beta$ is given by*

$$V[\tilde{\xi}] = 1 + \frac{h}{3}\Big[(b-a)^2 + 3(b-A)(a-A)\Big] + \frac{(1-h)}{3}\Big[(c-b)^2 + 3(c-A)(b-A)\Big],$$

where $A = \frac{ah+b+c(1-h)}{2}$.

Proof: From [18] it follows that, $\tilde{\xi} = \alpha + \beta$ has the chance distribution

$$\Psi(x) = \int_{-\infty}^{+\infty} \Upsilon(x-y)d\phi(y),$$

where ϕ is probability distribution function of the random variable α and Υ is the h-zigzag uncertainty distribution of the uncertain variable β.

$$= \int_{-\infty}^{+\infty} \Upsilon(x-y)f(y)dy, \quad d\phi(y) = f(y)dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x-c} \exp\left(-\frac{y^2}{2}\right) dy + \frac{1}{\sqrt{2\pi}} \int_{x-c}^{x-b} \left[h + \frac{(1-h)(x-y-b)}{c-b}\right] \exp\left(-\frac{y^2}{2}\right) dy$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{x-b}^{x-a} \left[\frac{h(x-y-a)}{b-a}\right] \exp\left(-\frac{y^2}{2}\right) dy$$

Differentiating both side of the above equation with respect to x, we have

$$\frac{d\Psi(x)}{dx} = \frac{h}{\sqrt{2\pi}(b-a)} \int_{x-b}^{x-a} \exp\left(-\frac{y^2}{2}\right) dy + \frac{1-h}{\sqrt{2\pi}(c-b)} \int_{x-c}^{x-b} \exp\left(-\frac{y^2}{2}\right) dy$$

Now using equation (2.2), we get

$$V[\tilde{\xi}] = \int_{-\infty}^{+\infty} \left(x - \frac{ah+b+c(1-h)}{2}\right)^2 d\Psi(x)$$

$$= \frac{h}{\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \int_{x-b}^{x-a} \left(x - \frac{ah+b+c(1-h)}{2}\right)^2 \exp\left(-\frac{y^2}{2}\right) dydx$$
$$+ \frac{1-h}{\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \int_{x-c}^{x-b} \left(x - \frac{ah+b+c(1-h)}{2}\right)^2 \exp\left(-\frac{y^2}{2}\right) dydx$$

By Changing the order of integration we get

$$V[\tilde{\xi}] = \frac{h}{\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left\{ \int_{y+a}^{y+b} \left(x - \frac{ah+b+c(1-h)}{2}\right)^2 dx \right\} dy$$
$$+ \frac{1-h}{\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left\{ \int_{y+b}^{y+c} \left(x - \frac{ah+b+c(1-h)}{2}\right)^2 dx \right\} dy$$

$$= \frac{h}{3\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \Big\{ 3y^2(b-a) + 3y[(b-A)^2 - (a-A)^2]$$
$$+ [(b-A)^3 - (a-A)^3] \Big\} dy$$
$$+ \frac{1-h}{3\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \Big\{ 3y^2(c-b) + 3y[(c-A)^2 - (b-A)^2]$$
$$+ 4y[(c-A)^3 - (b-A)^3] \Big\} dy$$

$$= 1 + \frac{h}{3}[(b-a)^2 + 3(b-A)(c-A)] + \frac{1-h}{3}[(c-b)^2 + 3(c-A)(b-A)],$$
$$\text{since, } \int_{-\infty}^{+\infty} y^2 \exp\left(-\frac{y^2}{2}\right) = \sqrt{2\pi} \text{ and } \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) = \sqrt{2\pi}.$$

□

Hence the theorem.

Now, variance of the random variable $\alpha \sim N(0,1)$ is 1 and from above, the variance of the uncertain random variable $\alpha+\beta$ using chance distribution is $1+\frac{h}{3}[(b-a)^2+3(b-A)(c-A)]+\frac{1-h}{3}[(c-b)^2+3(c-A)(b-A)]$ (Theorem 3.3). Further, from Theorem 3.2 it is seen that the variance of the uncertain variable $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ is $\frac{h}{3}[(b-a)^2 + 3(b-A)(c-A)] + \frac{1-h}{3}[(c-b)^2 + 3(c-A)(b-A)]$.
Therefore, combining the result we have, $V[\tilde{\xi}] = V[\alpha+\beta] = V[\alpha] + V[\beta]$, where $V[\alpha] = 1$ and $V[\beta] = \frac{h}{3}[(b-a)^2 + 3(b-A)(c-A)] + \frac{1-h}{3}[(c-b)^2 + 3(c-A)(b-A)]$, $A = \frac{ah+b+c(1-h)}{2}$. The above expression of $V[\tilde{\xi}]$ takes a simple form for specific values of h.

For example, if $h = 1/2$ then it becomes

$$V[\tilde{\xi}] = 1 + \frac{5(b-a)^2 + 5(c-b)^2 + 6(b-a)(c-b)}{48},$$

the same was obtained by Qin [23].

Corollary 3.2 *If $k_1,\ k_2$ be two real numbers and $\alpha \sim N(0,1)$, $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ then* $V[k_1\alpha + k_2\beta] = V[k_1\alpha] + V[k_2\beta] = k_1^2 V[\alpha] + k_2^2 V[\beta] = k_1^2 + k_2^2 V[\beta]$.

Proof: The proof follows directly from $V[\tilde{\xi}] = V[\alpha+\beta] = V[\alpha] + V[\beta]$ and $V[p\gamma + q] = p^2 V[\gamma]$, where $p,\ q$ are any real numbers and γ is a random (or uncertain) variable. □

Theorem 3.4 *If $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ is h-zigzag uncertain variable, and $\alpha \sim N(0,1)$, the standard normal distribution, then the skewness of the uncertain random variable $\tilde{\xi} = \alpha + \beta$ is given by*

$$Sk[\tilde{\xi}] = \frac{h(b+a-2A)}{4}\left[6 + (b-A)^2 + (a-A)^2\right]$$
$$+ \frac{(1-h)(c+b-2A)}{4}\left[6 + (c-A)^2 + (b-A)^2\right]$$

where $A = \frac{ah+b+c(1-h)}{2}$.

Proof: From [18] it follows that, $\tilde{\xi} = \alpha + \beta$ has the chance distribution

$$\Psi(x) = \int_{-\infty}^{+\infty} \Upsilon(x-y)d\phi(y),$$

where ϕ is probability distribution function of the random variable α and Υ is the h-zigzag uncertainty distribution of the uncertain variable β.

$$= \int_{-\infty}^{+\infty} \Upsilon(x-y)f(y)dy, \quad d\phi(y) = f(y)dy$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x-c} \exp\left(-\frac{y^2}{2}\right) dy + \frac{1}{\sqrt{2\pi}} \int_{x-c}^{x-b} \left[h + \frac{(1-h)(x-y-b)}{c-b}\right] \exp\left(-\frac{y^2}{2}\right) dy$$
$$+ \frac{1}{\sqrt{2\pi}} \int_{x-b}^{x-a} \left[\frac{h(x-y-a)}{b-a}\right] \exp\left(-\frac{y^2}{2}\right) dy$$

Differentiating both side of the above equation with respect to x, we have

$$\frac{d\Psi(x)}{dx} = \frac{h}{\sqrt{2\pi}(b-a)} \int_{x-b}^{x-a} \exp\left(-\frac{y^2}{2}\right) dy + \frac{1-h}{\sqrt{2\pi}(c-b)} \int_{x-c}^{x-b} \exp\left(-\frac{y^2}{2}\right) dy$$

Now using equation (2.3), we get

$$Sk[\tilde{\xi}] = \int_{-\infty}^{+\infty} \left(x - \frac{ah+b+c(1-h)}{2}\right)^3 d\Psi(x)$$

$$= \frac{h}{\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \int_{x-b}^{x-a} \left(x - \frac{ah+b+c(1-h)}{2}\right)^3 \exp\left(-\frac{y^2}{2}\right) dydx$$
$$+ \frac{1-h}{\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \int_{x-c}^{x-b} \left(x - \frac{ah+b+c(1-h)}{2}\right)^3 \exp\left(-\frac{y^2}{2}\right) dydx$$

By Changing the order of integration we get

$$Sk[\tilde{\xi}] = \frac{h}{\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left\{ \int_{y+a}^{y+b} \left(x - \frac{ah+b+c(1-h)}{2}\right)^3 dx \right\} dy$$
$$+ \frac{1-h}{\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \left\{ \int_{y+b}^{y+c} \left(x - \frac{ah+b+c(1-h)}{2}\right)^3 dx \right\} dy$$

$$= \frac{h}{4\sqrt{2\pi}(b-a)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \Big\{ 4y^3(b-a) + 6y^2[(b-A)^2 - (a-A)^2] + 4y[(b-A)^3 - (a-A)^3] + [(b-A)^4] - (a-A)^4 \Big\} dy$$
$$+ \frac{1-h}{4\sqrt{2\pi}(c-b)} \int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) \Big\{ 4y^3(c-b) + 6y^2[(c-A)^2 - (b-A)^2] + 4y[(c-A)^3 - (b-A)^3] + [(c-A)^4] - (b-A)^4 \Big\} dy$$

$$= \frac{h}{4(b-a)} [(b-A)^2 - (a-A)^2][6 + (b-A)^4 + (a-A)^4] + \frac{h}{4(c-b)} [(c-A)^2 - (b-A)^2][6 + (c-A)^4 + (b-A)^4]$$

$$= \frac{h(b+a-2A)}{4} \left[6 + (b-A)^2 + (a-A)^2\right] + \frac{(1-h)(c+b-2A)}{4} \left[6 + (c-A)^2 + (b-A)^2\right],$$

since, $\int_{-\infty}^{+\infty} y^2 \exp\left(-\frac{y^2}{2}\right) = \sqrt{2\pi}$ and $\int_{-\infty}^{+\infty} \exp\left(-\frac{y^2}{2}\right) = \sqrt{2\pi}$.

□

Hence the theorem.

Now, skewness of the random variable $\alpha \sim N(0,1)$ is 0 and from above, the skewness of the uncertain random variable $\alpha + \beta$ using chance distribution is $\frac{h(b+a-2A)}{4}\left[6 + (b-A)^2 + (a-A)^2\right] + \frac{(1-h)(c+b-2A)}{4}\left[6 + (c-A)^2 + (b-A)^2\right]$. Also, it can be checked that the skewness of the uncertain variable $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$ is $\frac{h(b+a-2A)}{4}\left[6 + (b-A)^2 + (a-A)^2\right] + \frac{(1-h)(c+b-2A)}{4}\left[6 + (c-A)^2 + (b-A)^2\right]$.

Therefore, $Sk[\tilde{\xi}] = Sk[\alpha + \beta] = Sk[\alpha] + Sk[\beta]$
where $Sk[\alpha] = 0$ and $Sk[\beta] = \frac{h(b+a-2A)}{4}\left[6 + (b-A)^2 + (a-A)^2\right]$
$+ \frac{(1-h)(c+b-2A)}{4}\left[6 + (c-A)^2 + (b-A)^2\right]$,
$A = \frac{ah+b+c(1-h)}{2}$.

For $h = 1/2$ the skewness of uncertain random variable $\tilde{\xi}$ is

$$Sk[\tilde{\xi}] = \frac{1}{32}(c-a)^2(a+c-2b)$$

the same was obtained by Li and Shu [12].

Corollary 3.3 *If $k_1,\ k_2$ be two real numbers and $\alpha \sim N(0,1),\ \beta \sim \boldsymbol{Z_h}(a,b,c;h)$ then $Sk[k_1\alpha + k_2\beta] = Sk[k_1\alpha] + Sk[k_2\beta] = k_2^3 Sk[\beta]$.*

Proof: It proof follows directly from $Sk[\tilde{\xi}] = Sk[\alpha+\beta] = Sk[\alpha]+Sk[\beta]$ and $Sk[p\gamma+q] = p^3 Sk[\gamma]$, where $p,\ q$ are any real numbers and γ is a random (or uncertain) variable. □

4. Portfolio Selection Problem

In this section, we consider a portfolio selection problem and solve it using our proposed model (4.1). For this, we consider a portfolio consisting of two stocks from NSE, India. The first stock gives a random return, and for the second, it is uncertain. Let y_1 and y_2 be the holding proportions of these stocks, respectively. Further, suppose that α and β respectively denote the rate of return from these stocks. Let the returns earned from them, respectively, follow standard normal and h-zigzag uncertainty distributions. Our ultimate goal is to find an optimal portfolio in this scenario. Now, in Section 1 it is already mentioned that positive skewness is always preferred by investors in the stock market. This is because positive skewness indicates greater overall returns than the average. Thus, an investor would naturally try to maximize the skewness of the total return of his or her portfolio. But, at the same time, it should ensure a minimum return and a prescribed tolerance limit for the risk (measured by variance) not to be exceeded. In our proposed model, this is achieved by maximizing the skewness of the total return and keeping its expected value and variance at a fixed level, subject to some constraints.

Now we present a generalized mean-variance-skewness model under uncertain random environment as follows:

$$\begin{cases} \max \ \ Sk[y_1\alpha + y_2\beta], \\ \text{s. t.} \ \ \ E[y_1\alpha + y_2\beta] \geq u, \\ \qquad\ \ V[y_1\alpha + y_2\beta] \leq v, \\ \qquad\ \ y_1 + y_2 = 1, \\ \qquad\ \ y_1, y_2 \geq 0. \end{cases} \tag{4.1}$$

where u and v represent the lower bound of expected return and the upper bound of risk, respectively.

Using corollaries 3.2A, 3.3A and 3.4A the formulated model (4.1) can be rewritten as

$$\begin{cases} \max \ \ y_2^3 Sk[\beta], \\ \text{s. t.} \ \ \ y_1E[\alpha] + y_2E[\beta] \geq u, \\ \qquad\ \ y_1^2V[\alpha] + y_2^2V[\beta] \leq v, \\ \qquad\ \ y_1 + y_2 = 1, \\ \qquad\ \ y_1, y_2 \geq 0. \end{cases} \tag{4.2}$$

Since $\alpha \sim N(0,1)$ and $\beta \sim \boldsymbol{Z_h}(a,b,c;h)$, the expression (4.2), further reduces to,

$$\begin{cases} \max \ \ y_2^3 Sk[\beta], \\ \text{s. t.} \ \ \ y_2E[\beta] \geq u, \\ \qquad\ \ y_1^2 + y_2^2V[\beta] \leq v, \\ \qquad\ \ y_1 + y_2 = 1, \\ \qquad\ \ y_1, y_2 \geq 0. \end{cases} \tag{4.3}$$

where $E[\beta] = \frac{ah+b+c(1-h)}{2}$, $V[\beta] = \frac{h}{3}[(b-a)^2 + 3(b-A)(c-A)] + \frac{1-h}{3}[(c-b)^2 + 3(c-A)(b-A)]$, $Sk[\beta] = \frac{h(b+a-2A)}{4}\Bigg[6 + (b-A)^2 + (a-A)^2\Bigg] + \frac{(1-h)(c+b-2A)}{4}\Bigg[6 + (c-A)^2 + (b-A)^2\Bigg]$ and $A = E[\beta]$.

Now, to evaluate $E[\beta],\ V[\beta]$ and $Sk[\beta]$ using the above expression we need to find the parameters

a, b and c of $\beta \sim \boldsymbol{Z_h}(a, b, c; h)$. These parameters of the newly emerged stock having uncertain return distribution are obtained by taking an expert's opinion.

According to an expert's opinion, the estimated values of the parameters using the moment method [16] for the selected stock are $a = -0.08$, $b = 0.05$ and $c = 0.32$. Hence, the parametric representation of the uncertain return of the stock follows the h-zigzag uncertainty distribution $\boldsymbol{Z_h}(-0.08, 0.05, 0.32; h)$.

5. Sensitivity analysis and Numerical solution of the portfolio selection problem

Here we performed the sensitivity analysis of the portfolio selection problem (4.3) by varying the parameter h, $0 < h < 1$. This exercise has been done for five different values of h: $h = 1/5,\ 1/3,\ 1/2,\ 2/3,\ 4/5$. For each value of h, the values of the other parameters a, b, and c are obtained using an expert's opinion. Then we construct five different numerical models of the portfolio selection problem, and these are solved using the "fmincon" function in Matlab. First we find the optimal solution of (4.3) for $h = 1/5$.

For $h = 1/5$ the model (4.3) becomes

$$\begin{cases} \max\ \ 0.00032\ y_2^3, \\ \text{s. t.}\ \ \ 0.145\ y_2 \geq u, \\ \qquad\quad y_1^2 + 0.0115\ y_2^2 \leq v, \\ \qquad\quad y_1 + y_2 = 1, \\ \qquad\quad y_1, y_2 \geq 0. \end{cases} \tag{5.1}$$

Considering different values of u and v, the solution of the above model (5.1) is shown in table 1.

Table 1: Solution of model (5.1) for different values of u and v

u	v	y_1	y_2
0.14	0.1	0	1
0.14	0.2	0	1
0.14	0.3	0	1
0.12	0.1	0	1
0.12	0.2	0	1
0.12	0.3	0	1
0.10	0.1	0	1
0.10	0.2	0	1
0.10	0.3	0	1

From table 1 it is seen that for all the cases the optimal solution is same, which is $y_1 = 0$ and $y_2 = 1$ with an expected return 0.145, variance 0.0115, skewness 0.00032. The result is not surprising. Since, the random part of the investment has the expected value 0, as $\alpha \sim N(0, 1)$, it is advisable that the investor should invest all his capital to the newly introduced stock only.

Naturally, we would like to see the nature of solutions for other four considered values of parameter

h viz., 1/3, 1/2, 2/3, and 4/5. For these values of h the proposed models are given below.

$$\begin{cases} \max \; 0.00002\, y_2^3, \\ \text{s. t.} \;\; 0.119\, y_2 \geq u, \\ \qquad y_1^2 + 0.0134\, y_2^2 \leq v, \\ \qquad y_1 + y_2 = 1, \\ \qquad y_1, y_2 \geq 0. \end{cases}$$

Model For $h = 1/3$

$$\begin{cases} \max \; 0.0007\, y_2^3, \\ \text{s. t.} \;\; 0.084\, y_2 \geq u, \\ \qquad y_1^2 + 0.0137\, y_2^2 \leq v, \\ \qquad y_1 + y_2 = 1, \\ \qquad y_1, y_2 \geq 0. \end{cases}$$

Model For $h = 1/2$

$$\begin{cases} \max \; 0.0012\, y_2^3, \\ \text{s. t.} \;\; 0.051\, y_2 \geq u, \\ \qquad y_1^2 + 0.0118\, y_2^2 \leq v, \\ \qquad y_1 + y_2 = 1, \\ \qquad y_1, y_2 \geq 0. \end{cases}$$

Model For $h = 2/3$

$$\begin{cases} \max \; 0.0012\, y_2^3, \\ \text{s. t.} \;\; 0.025\, y_2 \geq u, \\ \qquad y_1^2 + 0.0087\, y_2^2 \leq v, \\ \qquad y_1 + y_2 = 1, \\ \qquad y_1, y_2 \geq 0. \end{cases}$$

Model For $h = 4/5$

Solving the models we get the same optimal solution $y_1 = 0,\ y_2 = 1$ for different preassigned values of u and v. The optimal values of the objectives for the models are given below in tabulated form.

Table 2: The optimal objective values for different values of h

h	Expected return	Variance	Skewness
1/3	0.119	0.0134	0.00002
1/2	0.084	0.0137	0.0007
2/3	0.051	0.0118	0.0012
4/5	0.025	0.0087	0.0012

5.1. Observations from the sensitivity analysis

The sensitivity analysis of the parameter h of the h-zigzag uncertainty distribution reveals the following facts:
(i) With the decrease of the value of h, the expected return increases.
(ii) For $h > 1/2$, variance decreases with the increase in value of h. And for $h < 1/2$, variance decreases with the decrease in value of h. Thus variance attains its maximum at $h = 1/2$.
(iii) Further it is observed that skewness behaves randomly in respect of h, but it is always positive.

6. Conclusion

In this paper, we have examined the portfolio selection problem in a situation where randomness and uncertainty coexist. Here we have studied the expected value, variance, and skewness of an uncertain random variable in a generalized way. For this purpose, we have introduced the h-zigzag uncertainty distribution and calculated the statistics of the uncertain random variable using the chance distribution. For optimization of the portfolio, a generalized mean-variance-skewness model is proposed. In this model, the total skewness is maximized, ensuring that the total expected return and risk (measured by variance) do not, respectively, fall short and exceed certain pre-assigned values. Finally, we considered a portfolio selection problem consisting of two stocks: for one, the return is random and follows $N(0,1)$, and the other is uncertain. We randomly selected one newly listed stock. For it, taking an expert's opinion, the parameters a, b, c of the h-zigzag uncertainty distribution were estimated. Then, varying the parameter h, a sensitivity analysis is performed (Subsection 5.1). From the analysis, it was manifested that at $h = 1/2$,

the variance of the total return is the maximum, and hence predictions regarding the expected return are highly susceptible to variation. On the other hand, for the values of $h < 1/2$, the expected value increases and the variance also decreases. Thus, a decision-maker would make a choice about the newly entered stock in such a way that it satisfies the h-zigzag uncertainty distribution with some $h < 1/2$. To apply our proposed portfolio selection technique The decision-maker may proceed as follows regarding the selection of newly listed stock:
First, a finite number of newly emerged stocks are chosen at random. For each of them, using an expert's opinion, the parameters a_i, b_i, c_i are estimated. Then, for a fixed $h < 1/2$ and given limits of expected return and risk in the model, we calculate the maximum skewness of each of the selected stocks. The stock with the maximum skewness will be selected.

As a further scope of research, the case consisting of $m + n$ numbers of stocks where there are m stocks having uncertain returns and n stocks having random returns may be considered following respectively h-zigzag and standard normal distributions.

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