Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–15.
ISSN-0037-8712
doi:10.5269/bspm.65738

Limit study of an input stability problem on nanolayer

Tarik BOULAHROUZ *, Mohammed FILALI, Jamal MESSAHO and Najib TSOULI

ABSTRACT: This paper aims to discuss a stabilization problem for semilinear systems and to study the asymptotic behavior of a distributed system on an evolution domain, in a containing structure, of a nanolayer. The epi-convergence method is considered to find the limit problem with interface conditions; this approach studies the stability of the approximate problem associated with our initial problem. After that, we study the limit behavior. The received results are tested numerically.

Key Words: Semilinear systems, limit behavior, limit problems, feedback control, epi-convergence method.

Contents

1. Introduction

The concept of stability is one of the essential notions in system analysis. A state is said to be stable if the system evolves close to this state for small perturbations; for an unstable system, it is often asked if it is possible to stabilize it by feedback control on a geometric field Ω.

However, there are systems which are unstable in their Ω geometrical domains, but do not behave in the same way on the whole Ω; indeed, they can be stable in some regions of Ω (see example 4.2 [1]); we can then focus on the behavior of the state system only in an interior subregion of the nanolayer evolution domain; a nanolayer means a layer or a film on the surface of a solid or liquid of different physical and chemical nature that has nanoscale thickness. It deals with many real applications, for example, microelectronics, nanomedicine, nanotechnology, etc.

The regional control theory of distributed systems has yielded several impressive achievements. Amourous and al. [2]. El Jai and al. [3] provide a survey of these developments and a discussion of related issues, like observability, controllability, and optimal control.

Exponential stabilizability is treated by Triggiani [4] through a proper decomposition of the state space. The strong stabilization is studied by Balakrishnan [5] using the Riccati equation for steady state. Our goal is to prove the stability of a semilinear system via state feedback in the case of operators that generate a compact semigroup of contractions, with an explicit state feedback control given. Previously, this problem has been studied in a weak sense in [6,7,8]; in our case, we work with a B_ε region of the nanostructure, which will cause a problem during numerical resolution with the finite element method and specifically during the creation of the domain mesh, which will be very thin, so if we increase the nodes to have precision with a three-dimensional problem of nano order, we may have a numerical instability that can cause numerical explosions, thus the method's inefficiency.

The idea would be to look for another equivalent approximation model to work with the finite element method accurately to obtain the limit problem.

* Corresponding author

Submitted November 06, 2022. Published March 18, 2025
2010 *Mathematics Subject Classification:* 34G25, 35B40, 76M50, 93B52, 82B24.

This paper is organized as follows: in section 2, we show the strong stability by feedback control using a result of [9], which makes the approximating semi-linear system associated to the initial problem strongly stabilizable, then we prove a priori estimates. In the third section, we pass to the limit in the minimization problem associated with the approximate problem using the epi-convergence method and preliminary results. Finally, in section 4, one will give a numerical test illustrating the obtained theoretical results.

2. Study of the problem

Position of the problem

One considers a problem of semilinear evolution in a body that occupies a bonded domain, $\Omega \subset \mathbb{R}^3$, with a Lipschitz border $\partial\Omega$, composed of a nanolayer B_ε, with oscillating border $\Sigma_\varepsilon^\pm$, and a remaining region of Ω_ε (see Figure 1). The body occupying the domain Ω, with the control $u \in U_{ad}$, and the operator $L : L^p(B_\varepsilon) \to L^{p'}(B_\varepsilon)$ is nonlinear, where $Lz = |z|^{p-2}z$, and $\frac{5}{2} < p < 3$.
With the set of admissible controls

$$U_{ad} = \left\{ u \in L^\infty(]0,\infty[) : \|u(t)\|_{L^\infty(]0,\infty[)} \leq c \right\}.$$

Consider the following problem:

$$\begin{cases} \dot{z} = -\Delta z & \text{in } \Omega_\varepsilon^\infty \\ \dot{z} = -\frac{1}{\varepsilon^\alpha}\Delta z + u|z|^{p-2}z & \text{in } B_\varepsilon^\infty \\ z(t,x) = 0 & \text{on } \Gamma^\infty =]0,\infty[\times\partial\Omega \\ z(0,x) = z_0 & \text{in } \Omega \\ [z(t,x)] = 0 & \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm \\ \frac{\partial z}{\partial n}\Big|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}\frac{\partial z}{\partial n}\Big|_{B_\varepsilon} & \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases} \quad (\mathscr{P})$$

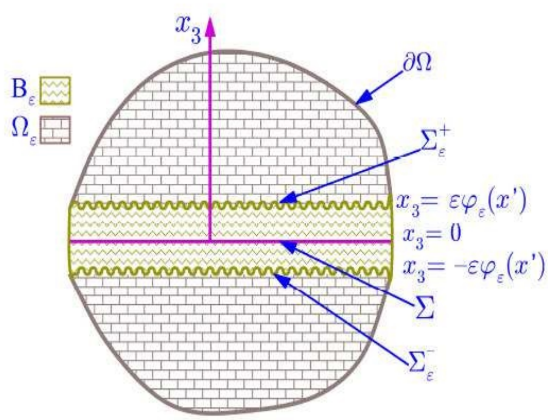


Figure 1: Domain Ω.

Notation and functional setting

- $\mathcal{Q} =]0,\infty[\times\Omega$, $\Omega_\varepsilon^\infty =]0,\infty[\times\Omega_\varepsilon$, $B_\varepsilon^\infty =]0,\infty[\times B_\varepsilon$.

- $L^p(0,\infty,X)$ has the norm,

$$\|z\|_{L^p(0,\infty,X)} = \left(\int_0^\infty \|z(t)\|_X^p dt\right)^{\frac{1}{p}}.$$

With X, a banach space.

- $[z]_{\Sigma_\varepsilon^\pm} = z|_{\overline{\Omega_\varepsilon}|_{\Sigma_\varepsilon^\pm}} - z|_{\overline{B_\varepsilon}|_{\Sigma_\varepsilon^\pm}}$.
-
$$\mathbb{G} = \begin{cases} z \in L^2(0,\infty,H_0^1(\Omega)) : \eta(\alpha)z(t)|_\Sigma \in H_0^1(\Sigma) & \text{if } \alpha \le 1, \\ z \in L^2(0,\infty,H_0^1(\Omega)) : z(t)|_\Sigma = C & \text{if } \alpha > 1. \end{cases}$$

$$\mathbb{D} = \begin{cases} \mathcal{D}(]0,\infty[\times\Omega) & \text{if } \alpha \le 1, \\ \{z \in \mathcal{D}(]0,\infty[\times\Omega) : z(t)|_\Sigma = C\} & \text{if } \alpha > 1. \end{cases}$$

We know that $\overline{\mathbb{D}} = \mathbb{G}$.

- Let us define the operator m^ε which transforms functions defined z on B_ε into functions defined on Σ by,
$$m^\varepsilon z(t,x_1,x_2) = \frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} z(t,x_1,x_2,x_3)\,dx_3.$$
- The inner product defines the real-valued function and will denote by,
$$\langle f,g\rangle = \int_\Omega f(x)g(x)dx.$$
- $(t,x) = (t,x',x_3)$, where $x' = (x_1,x_2), \lambda = 1,2,\ \nabla' = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right),\ Y =]0,1[\times]0,1[, \varphi : \mathbb{R}^2 \to [a_1,a_2]$ where φ is Y-periodic and $a_2 \ge a_1 > 0, \varphi_\varepsilon(x') = \varphi\left(\frac{x'}{\varepsilon}\right), \frac{\partial\varphi}{\partial x_\lambda} \in \mathcal{C}(\Sigma)\cap L^\infty(\Sigma), m(\varphi) = \int_Y \varphi(x')\,dx'$, $\eta(\alpha) = \lim_{\varepsilon\to 0}\varepsilon^{1-\alpha}$, with $\alpha \ge 0$.
In the following, C will denote any constant with respect to ε.

Study of the problem

Stability study Consider the following approximate problem;

$$\begin{cases} \dot{z_\varepsilon}(t,x) = -\Delta z_\varepsilon(t,x) \quad \text{in } \Omega_\varepsilon^\infty \\ \dot{z_\varepsilon}(t,x) = -\frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon(t,x) + u_\varepsilon|z_\varepsilon|^{p-2}z_\varepsilon \quad \text{in } B_\varepsilon^\infty \\ z_\varepsilon(t,x) = 0 \quad \text{on } \Gamma^\infty =]0,\infty[\times\partial\Omega \\ z_\varepsilon(0,x) = z_{0,\varepsilon} \quad \text{in } \Omega \\ [z_\varepsilon(t,x)] = 0 \quad \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm \\ \left.\frac{\partial z_\varepsilon(t,x)}{\partial n}\right|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}\left.\frac{\partial z_\varepsilon(t,x)}{\partial n}\right|_{B_\varepsilon} \quad \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases} \qquad (\mathscr{P}_\varepsilon)$$

We are interested in stabilizing;

$$\dot{z_\varepsilon}(t,x) = -\frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon(t,x) + u_\varepsilon|z_\varepsilon|^{p-2}z_\varepsilon \quad \text{in } B_\varepsilon^\infty. \tag{2.1}$$

We note $z_\varepsilon(t) = z_\varepsilon(.,t), u_\varepsilon(t) = u_\varepsilon(.,t)$ and $A = -\frac{1}{\varepsilon^\alpha}\Delta$. We pose $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$, where $\Omega \subset \mathbb{R}^3$.

A is the infinitesimal generator of the compact semigroup (see [10]), given by

$$S(t)z_\varepsilon = \sum_{n,m,k=0}^{\infty} e^{((\frac{n}{b_1}+\frac{m}{b_2}+\frac{k}{b_3})\pi)^2\frac{t}{\varepsilon^\alpha}}\langle\phi_{n,m,k}, z_\varepsilon\rangle\,\phi_{n,m,k},$$

where $\phi_{n,m,k}(x) = e_n(x_1)\,e_m(x_2)\,e_k(x_3)$, we have $e_0 = 1, e_n(x_i) = \sqrt{2}\cos\left(\frac{n\pi x_i}{b_i}\right), \forall n \in \mathbb{N}^*$, with $b_i = \beta_i - \alpha_i$, $i = 1,2,3$, such that $\Omega =]\alpha_1,\beta_1[\times]\alpha_2,\beta_2[\times]\alpha_3,\beta_3[$,

$L : H_0^1(B_\varepsilon) \cap L^p(B_\varepsilon) \to L^{p'}(B_\varepsilon)$, defined by

$$Lz_\varepsilon = |z_\varepsilon|^{p-2} z_\varepsilon, \quad \forall z_\varepsilon \in L^2(0,\infty; H_0^1(B_\varepsilon) \cap L^p(B_\varepsilon)),$$

is a derivable operator with a bounded derivative, indeed $L' : H_0^1(B_\varepsilon) \cap L^p(B_\varepsilon) \to L^2(B_\varepsilon)$;

$$L'z_\varepsilon = (p-1)|z_\varepsilon|^{p-2}, \quad \forall z_\varepsilon \in L^2(0,\infty; H_0^1(B_\varepsilon) \cap L^p(B_\varepsilon)).$$

Furthermore, for $p \in]\frac{5}{2}, 3[$ we have $L^2 \hookrightarrow L^{2p-4}$, and $\forall z_\varepsilon \in L^2(0,\infty; H_0^1(B_\varepsilon) \cap L^p(B_\varepsilon))$,

$$\int_{]0,\infty[} \|L'z_\varepsilon\|^2_{L^2(B_\varepsilon)} \leq C \int_{]0,\infty[} \|z_\varepsilon\|^{2p-4}_{L^{2p-4}(B_\varepsilon)} \leq C \int_{]0,\infty[} \|z_\varepsilon\|^{2p-4}_{L^2(B_\varepsilon)} \leq C \int_{]0,\infty[} \|z_\varepsilon\|^2_{H_0^1(B_\varepsilon)} \leq \infty.$$

Hence the boundedness of L' implies that L is Lipschitzian.

Furthermore, $\langle L(S(t)z_\varepsilon), S(t)z_\varepsilon\rangle = 0$, $\forall t \geq 0$, implies that

$$|S(t)z_\varepsilon|^p = |\sum_{n,m,k=0}^{\infty} e^{((\frac{n}{b_1}+\frac{m}{b_2}+\frac{k}{b_3})\pi)^2 \frac{t}{\varepsilon^\alpha}} \langle \phi_{n,m,k}, z_\varepsilon\rangle \, \phi_{n,m,k}|^p = 0, \qquad \forall n,m,k \geq 0, \forall t \geq 0.$$

Since all series converge, their general term tends to 0, which gives $z_\varepsilon = 0$.
According to theorem 5.4, the system (2.1) controlled by the feedback : $u_\varepsilon(t) = -r(z_\varepsilon)\langle L(z_\varepsilon), z_\varepsilon\rangle$ is strongly stabilizable, with $r(z_\varepsilon) = \frac{c}{1+\langle L(z_\varepsilon), z_\varepsilon\rangle^2}$, $\forall c > 0$.

A priori estimate

Approximate problem
The set $V = H_0^1(\Omega) \cap L^p(\Omega)$ is separable, with $H_0^1(\Omega) \cap L^p(\Omega)$ has the norm
$\|\cdot\|_{H_0^1(\Omega)\cap L^p(\Omega)} = \|\cdot\|_{H_0^1(\Omega)} + \|\cdot\|_{L^p(\Omega)}$, therefore, it admits a countable basis $\{w_1, w_2, w_3, \dots, w_n, \dots\}$, with $w_i \in V$, $\forall\, m$ $\{w_1, w_2, w_3, \dots, w_n\}$ is a free family, $H = Vect\{w_1, w_2, w_3, \dots, w_n, \dots\}$ is dense in V.

Let us consider in the spaces $V_m = Vect\{w_1, w_2, w_3, ..., w_m\}$ the following approximate problem;

$$\text{We put } z_\varepsilon(t) = \sum_{i=1}^{m} h_{i\varepsilon}(t) w_i \in V_m.$$

$$\begin{cases} < \dot{z_\varepsilon}, w_i >_{\Omega_\varepsilon} + < \Delta z_\varepsilon, w_i >_{\Omega_\varepsilon} = 0 \quad \text{in }]0,\infty[\\ < \dot{z_\varepsilon}, w_i >_{B_\varepsilon} = - < \frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon, w_i >_{B_\varepsilon} + < u_\varepsilon |z_\varepsilon|^{p-2} z_\varepsilon, w_i >_{B_\varepsilon} \quad \text{in }]0,\infty[\\ z_\varepsilon(t,x) = 0 \quad \text{on} \quad \Gamma^\infty =]0,\infty[\times\partial\Omega \\ z_\varepsilon(0,x) = \quad z_{0,\varepsilon} \quad \text{in } \Omega \\ [z_\varepsilon(t,x)] = 0 \quad \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm \\ \frac{\partial z_\varepsilon}{\partial n}\Big|_{\Omega_\varepsilon} = \frac{1}{\varepsilon^\alpha}\frac{\partial z_\varepsilon}{\partial n}\Big|_{B_\varepsilon} \quad \text{on }]0,\infty[\times\Sigma_\varepsilon^\pm. \end{cases} \qquad (\mathscr{P}_{m,\varepsilon})$$

With $< \cdot, \cdot >$ is a duality bracket.

From the results on systems of differential equations, we are sure that the problem $(\mathscr{P}_{m,\varepsilon})$ has a solution $z_\varepsilon(t)$ in an interval $[0, t_\varepsilon[$, with $t_\varepsilon \leq T$ such that $T \to \infty$ for $\varepsilon \to 0$.

Remark 2.1 *The following a priori estimates show that $t_\varepsilon = T$.*

A priori estimate

Lemma 2.1 *The family $(z_\varepsilon)_{\varepsilon>0}$ satisfies:*

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(B_\varepsilon)} \leq C\varepsilon^\alpha. \tag{2.2}$$

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C. \tag{2.3}$$

Moreover z_ε is bounded in $L^2(0,\infty, H_0^1(\Omega) \cap L^p(\Omega))$.

Proof:

Let the problem $(\mathscr{P}_{m,\varepsilon})$; multiply the equations defined on B_ε^∞ and $\Omega_\varepsilon^\infty$ by $h_{i\varepsilon}(t)$ and sum from $i=1$ to m, we get;

On the one hand in B_ε,

$$< \dot{z_\varepsilon}, z_\varepsilon >_{B_\varepsilon} + < \frac{1}{\varepsilon^\alpha}\Delta z_\varepsilon, z_\varepsilon >_{B_\varepsilon} = < u_\varepsilon |z_\varepsilon|^{p-2} z_\varepsilon, z_\varepsilon >_{B_\varepsilon} .$$

$$< \dot{z_\varepsilon}(t,x), z_\varepsilon >_{B_\varepsilon} - < \frac{1}{\varepsilon^\alpha}\nabla z_\varepsilon, \nabla z_\varepsilon >_{B_\varepsilon} - < u_\varepsilon |z_\varepsilon|^{p-2} z_\varepsilon, z_\varepsilon >_{B_\varepsilon} = 0 \quad \text{in} \quad]0,t_\varepsilon[\times B_\varepsilon$$

by a change of variable from t to $t_\varepsilon - t$ with $t_1 = t_\varepsilon - t \in]0,t_\varepsilon[$, we obtain in $]0,t_\varepsilon[\times B_\varepsilon$,

$$- < \dot{z_\varepsilon}(t_1,x), z_\varepsilon >_{B_\varepsilon} - < \frac{1}{\varepsilon^\alpha}\nabla z_\varepsilon, \nabla z_\varepsilon >_{B_\varepsilon} - < u_\varepsilon |z_\varepsilon|^{p-2} z_\varepsilon, z_\varepsilon >_{B_\varepsilon} =$$

$$-\frac{1}{2}\frac{d}{dt_1}\int_\Omega z_\varepsilon^2 - \frac{1}{\varepsilon^\alpha}\int_{B_\varepsilon} |\nabla z_\varepsilon|^2 - \int_{B_\varepsilon} u_\varepsilon |z_\varepsilon|^p = 0.$$

And by integration from 0 to t_ε,

$$\frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 + \int_{]0,t_\varepsilon[\times B_\varepsilon} u_\varepsilon |z_\varepsilon|^p = -\frac{1}{2}\int_{]0,t_\varepsilon[}\frac{d}{dt_1}\int_\Omega z_\varepsilon^2 =$$

$$\frac{1}{2}(-\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)} + \|z_{0,\varepsilon}\|^2_{L^2(\Omega)}) \le -\frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)} + C.$$

Then, by Holder's inequality, and since the control u_ε has a negative sign, we have;

$$-c\int_{]0,t_\varepsilon[} \|z_\varepsilon\|^p_{L^p(B_\varepsilon)} \le -\| - u_\varepsilon\|_\infty \int_{]0,t_\varepsilon[} \|z_\varepsilon\|^p_{L^p(B_\varepsilon)} \le \int_{]0,t_\varepsilon[\times B_\varepsilon} u_\varepsilon |z_\varepsilon|^p.$$

Thus,

$$\frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 \le c\int_{]0,t_\varepsilon[} \|z_\varepsilon\|^p_{L^p(B_\varepsilon)} - \frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)} + C.$$

By minoration (use that H_0^1 is continuously embedded in L^p (see lemma 5.3)) and minoring $-\frac{1}{2}\|z_\varepsilon(t_\varepsilon,x)\|^2_{L^2(\Omega)}$ by 0.

$$\frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 - cC\int_{]0,t_\varepsilon[} \|z_\varepsilon\|^2_{1,2} \le \frac{1}{\varepsilon^\alpha}\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 - c\int_{]0,t_\varepsilon[} \|z_\varepsilon\|^p_{L^p(B_\varepsilon)} \le C.$$

Multiply by ε^α;

$$(1 - cC\varepsilon^\alpha)\int_{]0,t_\varepsilon[\times B_\varepsilon} |\nabla z_\varepsilon|^2 \le C\varepsilon^\alpha.$$

For an $\varepsilon \le \frac{1}{2}$, we get $(1 - \frac{cC}{2^\alpha}) \le (1 - cC\varepsilon^\alpha)$, then $T = t_\varepsilon$ and by making T tend to ∞, so

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(B_\varepsilon)} \le C\varepsilon^\alpha.$$

Then z_ε is bounded in $L^2(0,\infty,H_0^1(B_\varepsilon))$ and since H_0^1 is continuously embedded in L^p then we have boundedness in $L^2(0,\infty,H_0^1(B_\varepsilon)\cap L^p(B_\varepsilon))$.

On the other hand in Ω_ε,

$$< \dot{z_\varepsilon}(t,x), z_\varepsilon >_{\Omega_\varepsilon} + < \Delta z_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} = 0 \quad \text{in} \quad]0,t_\varepsilon[\times\Omega_\varepsilon.$$

Also by a change of variable from t to $t_\varepsilon - t$ with $t_\varepsilon - t \in]0,t_\varepsilon[$, we obtain;

$$- < \dot{z_\varepsilon}, z_\varepsilon >_{\Omega_\varepsilon} + < \Delta z_\varepsilon, z_\varepsilon >_{\Omega_\varepsilon} = 0 \quad \text{in} \quad]0,t_\varepsilon[\times\Omega_\varepsilon.$$

Which gives by integration on $]0, t_\varepsilon[$,

$$\int_{]0,t_\varepsilon[\times\Omega_\varepsilon} |\nabla z_\varepsilon|^2 = -\frac{1}{2}\|z_\varepsilon(t_\varepsilon, x)\|^2_{L^2(\Omega_\varepsilon)} + C,$$

then, let's reduce $-\frac{1}{2}\|z_\varepsilon(t_\varepsilon, x)\|^2_{L^2(\Omega_\varepsilon)}$ by 0 and tends $T \to +\infty$, we get;

$$\int_{]0,\infty[\times\Omega_\varepsilon} |\nabla z_\varepsilon|^2 \leq C$$

$$\int_{]0,\infty[} \|\nabla z_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \leq C.$$

Then z_ε is bounded in $L^2(0, \infty, H_0^1(\Omega_\varepsilon))$ and since H_0^1 is continuously embedded in L^p then we have boundedness in $L^2(0, \infty, H_0^1(\Omega_\varepsilon) \cap L^p(\Omega_\varepsilon))$.

□

Hence the boundness on $L^2(0, \infty, H_0^1(\Omega) \cap L^p(\Omega))$, and since $L^2(0, \infty; V)$ is a reflexive space, then there exists a sub-sequence of $(z_\varepsilon)_{\varepsilon>0}$, always denoted by $(z_\varepsilon)_{\varepsilon>0}$, such that $z_\varepsilon \rightharpoonup z^*$ in $L^2(0, \infty; V)$. Hence the strong convergence in $L^2(0, \infty; L^2(\Omega))$ (see remark 5.1).

And as $|z_\varepsilon|^{p-2} z_\varepsilon$ is bounded in $L^\infty(0, \infty; L^{p'}(B_\varepsilon))$.
Indeed;

$$\begin{aligned} \sup\ \text{ess}_t \| \, |z_\varepsilon|^{p-2} z_\varepsilon\|^{p'}_{L^{p'}(B_\varepsilon)} &= \sup\ \text{ess}_t \int_{B_\varepsilon} \left(|z_\varepsilon(x,t)|^{p-1}\right)^{p'} dx \\ &= \sup\ \text{ess}_t \int_{B_\varepsilon} |z_\varepsilon(x,t)|^{(p-1)p'} dx \\ &= \sup\ \text{ess}_t \|z_\varepsilon(t)\|^p_{L^p(B_\varepsilon)} < \infty. \end{aligned}$$

We can assume for a sub-sequence that:

$$|z_\varepsilon|^{p-2} z_\varepsilon \overset{*}{\rightharpoonup} |z^*|^{p-2} z^* \text{ in } L^\infty\left(0, \infty; L^{p'}(B_\varepsilon)\right).$$

3. Limit Behavior

Note that the problem $(\mathscr{P})$ is equivalent to the minimization problem

$$\inf_{z\in L^2(0,\infty;V)} \left\{ \frac{1}{2}\int_{]0,\infty[\times\Omega_\varepsilon} |\nabla z|^2 + \frac{1}{2\varepsilon^\alpha}\int_{]0,\infty[\times B_\varepsilon} |\nabla z|^2 + \frac{1}{p}\int_{]0,\infty[\times B_\varepsilon} u|z|^p \right\} \qquad (\mathscr{P}_1)$$

Remark 3.1 *According to theorem 7.10 [11] we have the existence of a solution. Moreover z is given by the formula*

$$\boldsymbol{z(t) = \mathrm{S}_\mathrm{A}(t) z_0 + \int_0^t \mathrm{S}_\mathrm{A}(t-s) u(s) L z(s) \mathrm{d}s}$$

where $S_A(t)$ denotes the semigroup associated to A.

Lemma 3.1 *The operator m^ε is linear and bounded of $L^2(0, \infty; L^2(B_\varepsilon))$(respectively $L^2(0, \infty; H_0^1(B_\varepsilon))$) in $L^2(0, \infty; L^2(\Sigma))$ (respectively $L^2(0, \infty; H_0^1(\Sigma))$), moreover, for all $z \in L^2(0, \infty; H_0^1(B_\varepsilon))$, we have*

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C\varepsilon \int_0^\infty \int_{B_\varepsilon} |\nabla z|^2. \qquad (3.1)$$

Proof: We have

$$\int_\Sigma |m^\varepsilon z|^2\, dx_1 dx_2 = \int_\Sigma \left(\frac{1}{2\varepsilon\varphi_\varepsilon}\right)^2 \left|\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} z dx_3\right|^2 dx_1 dx_2.$$

Since $0 < a_1 \le \varphi_\varepsilon \le a_2$, and according to the Hölder inequality,

$$\int_\Sigma |m^\varepsilon z|^2\, dx_1 dx_2 \le \int_\Sigma \frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |z|^2 dx_3\right) dx_1 dx_2 \le \frac{1}{2\varepsilon a_1}\int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |z|^2 dx_3\right) dx_1 dx_2. \tag{3.2}$$

Since $z \in L^2\left(]0,\infty[\times B_\varepsilon\right)$ and (3.2), it follows that $m^\varepsilon z \in L^2(]0,\infty[\times\Sigma)$. Let $z \in \overline{\mathcal{D}}\left(]0,\infty[\times B_\varepsilon\right)$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)(t,x_1,x_2) =& \frac{1}{2}\frac{\partial}{\partial x_\lambda}\left(\int_{-1}^{1} z\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right) dx_3\right)\\
=& \frac{1}{2}\left(\int_{-1}^{1} \frac{\partial z}{\partial x_\lambda}\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right) + \varepsilon x_3 \frac{\partial \varphi_\varepsilon}{\partial x_\lambda}\frac{\partial z}{\partial x_3}\left(t,x_1,x_2,x_3\varepsilon\varphi_\varepsilon\right) dx_3\right)\\
=& \frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial z}{\partial x_\lambda} + \left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3} dx_3\right).
\end{aligned}$$

So that,

$$\begin{aligned}
\int_\Sigma \left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^2 =& \int_\Sigma \left|\frac{1}{2\varepsilon\varphi_\varepsilon}\left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \frac{\partial z}{\partial x_\lambda} + \left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3} dx_3\right)\right|^2\\
\le& \left(\frac{1}{2\varepsilon a_1}\right)^2 \int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|\frac{\partial z}{\partial x_\lambda} + \left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right)\left(\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}\right)\frac{\partial z}{\partial x_3}\right|^2 dx_3\right).
\end{aligned}$$

However, $\frac{\partial\varphi}{\partial x_\lambda} \in \mathcal{C}(\Sigma)\cap L^\infty(\Sigma)$, then $\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x_\lambda}$ is bounded, and therefore

$$\int_\Sigma \left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^2 \le \frac{C}{\varepsilon}\int_{B_\varepsilon}\left(\left|\frac{\partial z}{\partial x_\lambda}\right|^2 + \left|\frac{\partial z}{\partial x_3}\right|^2\right) dx_3 \le \frac{C}{\varepsilon}\int_{B_\varepsilon} |\nabla z|^2.$$

By density arguments, for any $z \in L^2(0,\infty; H_0^1\left(B_\varepsilon\right))$, we have

$$\int_0^\infty \int_\Sigma \left|\frac{\partial}{\partial x_\lambda}\left(m^\varepsilon z\right)\right|^2 \le \frac{C}{\varepsilon}\int_0^\infty \int_{B_\varepsilon} |\nabla z|^2.$$

Let $z \in \overline{\mathcal{D}}\left(]0,\infty[\times B_\varepsilon\right)$, so that

$$\left\|m^\varepsilon z - z_{|\Sigma}\right\|_{L^2(\Sigma)}^2 = \int_\Sigma \left|\left(\frac{1}{2\varepsilon\varphi_\varepsilon}\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} z\left(t,x_1,x_2,x_3\right) dx_3\right) - z\left(t,x_1,x_2,0\right)\right|^2 dx_1 dx_2.$$

Using the Hölder inequality,

$$\begin{aligned}
\left\|m^\varepsilon z - z_{|\Sigma}\right\|_{L^2(\Sigma)}^2 \le& \frac{1}{2\varepsilon a_1}\int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|z\left(t,x_1,x_2,x_3\right) - z\left(t,x_1,x_2,0\right)\right|^2 dx_3\right) dx_1 dx_2\\
\le& \frac{C}{\varepsilon}\int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|\int_0^{x_3} \frac{\partial z}{\partial x_3}\left(t,x_1,x_2,w\right) dw\right|^2 dx_3\right) dx_1 dx_2\\
\le& \frac{C}{\varepsilon}\int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} |x_3| \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|\frac{\partial z}{\partial x_3}\left(t,x_1,x_2,w\right)\right|^2 dw\right) dx_3\right) dx_1 dx_2\\
\le& C\varepsilon \int_\Sigma \left(\int_{-\varepsilon\varphi_\varepsilon}^{\varepsilon\varphi_\varepsilon} \left|\frac{\partial z}{\partial x_3}\right|^2 dx_3\right) dx_1 dx_2\\
\le& C\varepsilon \int_{B_\varepsilon} |\nabla z|^2 dx.
\end{aligned}$$

By density arguments, we have for all $z \in L^2(0,\infty; H_0^1(B_\varepsilon))$

$$\left\| m^\varepsilon z - z_{|\Sigma} \right\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C\varepsilon \int_0^\infty \int_{B_\varepsilon} |\nabla z|^2 dxdt.$$

Hence the result. □

Lemma 3.2 *Let $(z_\varepsilon)_{\varepsilon>0} \subset L^2(0,\infty; V)$ which satisfies (2.2) and (2.3). Then*

$$\|\nabla' (m^\varepsilon z_\varepsilon)\|^2_{(L^2(]0,\infty[\times\Sigma))^2} \leq C\varepsilon^{\alpha-1}. \tag{3.3}$$

In addition, $m^\varepsilon z_\varepsilon$ have a bounded sub-sequence in $L^2(]0,\infty[\times\Sigma)$.

Proof: According to a result of lemma 3.1, we have

$$\int_0^\infty \left\| \frac{\partial (m^\varepsilon z_\varepsilon)}{\partial x_\lambda} \right\|^2_{L^2(\Sigma)^2} \leq C\varepsilon^{-1} \int_0^\infty \int_{B_\varepsilon} |\nabla z_\varepsilon|^2 dx.$$

According to (2.2), one has

$$\int_0^\infty \left\| \frac{\partial (m^\varepsilon z_\varepsilon)}{\partial x_\lambda} \right\|^2_{L^2(\Sigma)^2} \leq C\varepsilon^{\alpha-1}.$$

Then from lemma 3.1, we get

$$\left\| m^\varepsilon z - z_{|\Sigma} \right\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C\varepsilon \int_0^\infty \int_{B_\varepsilon} |\nabla z|^2 \leq C\varepsilon^{\alpha+1}.$$

z_ε is bounded in $L^2(0,\infty; H_0^1(\Omega))$, it follows that there exists $z^* \in L^2(0,\infty; H_0^1(\Omega))$ and a sub-sequence z_ε, always noted z_ε, such as $z_\varepsilon \rightharpoonup z^*$ in $L^2(0,\infty; H_0^1(\Omega))$, then $z_{\varepsilon|\Sigma}$ is a bounded sequence in $L^2(]0,\infty[\times\Sigma)$.

Since,

$$\|m^\varepsilon z_\varepsilon\|_{L^2(]0,\infty[\times\Sigma)} \leq \left\| m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma} \right\|_{L^2(]0,\infty[\times\Sigma)} + \|z_{\varepsilon|\Sigma}\|_{L^2(]0,\infty[\times\Sigma)},$$

then there exists C such that $\|m^\varepsilon z_\varepsilon\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C$.

□

Proposition 3.1 *$(z_\varepsilon)_\varepsilon$, has a weakly convergent sub-sequence to an element z^* in $L^2(0,\infty; H_0^1(\Omega))$ satisfactory,*
(1) *If $\alpha = 1$, $z^*|_\Sigma \in L^2(0,\infty; H_0^1(\Sigma))$.*
(2) *If $\alpha > 1$, $z^*|_\Sigma = C$.*

Proof: According to lemma 2.1, the sequence z_ε is bounded in $L^2(0,\infty; H_0^1(\Omega))$, it follows that there is an element $z^* \in L^2(0,\infty; H_0^1(\Omega))$ and a sub-sequence of z_ε, always designated by z_ε such as $z_\varepsilon \rightharpoonup z^*$ in $L^2(0,\infty; H_0^1(\Omega))$. We have

$$\left\| m^\varepsilon z_\varepsilon - z_{\varepsilon|\Sigma} \right\|^2_{L^2(]0,\infty[\times\Sigma)} \leq C\varepsilon^{\alpha+1} \text{ and } z_{\varepsilon|\Sigma} \rightharpoonup z^*_{|\Sigma} \text{ in } L^2(]0,\infty[\times\Sigma).$$

For $\alpha = 1$, according to the evaluation (3.3), the sequence $\nabla' m^\varepsilon z_\varepsilon$ has a sub-sequence, always denoted by $\nabla' m^\varepsilon z_\varepsilon$ weakly convergent to an element z_2 in $L^2(0,\infty; L^2(\Sigma))^2$, as $m^\varepsilon z_\varepsilon \rightharpoonup z^*_{|\Sigma}$ in $L^2(0,\infty; H_0^1(\Sigma))$ and that $\nabla' z^*_{|\Sigma} = z_2$. Hence $z^*_{|\Sigma} \in L^2(0,\infty; H_0^1(\Sigma))$.
For $\alpha > 1$, one shows, as in the case $\alpha = 1$ and taking $z_2 = 0$, that $z^*_{|\Sigma} = C$.
Hence the results.

□

The limit behavior of the problem $(\mathscr{P}_1)$, will be derived with the epi-convergence method.
Let.

$$F^\varepsilon(z_\varepsilon) = \frac{1}{2} \int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \frac{1}{2\varepsilon^\alpha} \int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \frac{1}{p} \int_{B_\varepsilon^\infty} u_\varepsilon |z_\varepsilon|^p$$

Theorem 3.1 *According to the values of α, there exists a functional F^{α} defined on $L^2(0,\infty;H_0^1(\Omega))$ with a value in $\mathbb{R}\cup\{+\infty\}$ such that $\tau_f - \lim_e F^{\varepsilon} = F^{\alpha}$ in $L^2(0,\infty;H_0^1(\Omega))$, where the functional F^{α} is given by*

(1) If $0 \leq \alpha < 1$:

$$F^{\alpha}(z) = \frac{1}{2}\int_{]0,\infty[\times\Omega} |\nabla z|^2, \quad \forall z \in L^2(0,\infty;H_0^1(\Omega)).$$

(2) If $\alpha \geq 1$:

$$F^{\alpha}(z) = \frac{1}{2}\int_{]0,\infty[\times\Omega} |\nabla z|^2 + m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma} \left|\nabla' z_{|\Sigma}\right|^2, \qquad \forall z \in \mathbb{G} \subset L^2(0,\infty;H_0^1(\Omega)).$$

Proof:
(a) We will determine the upper epi-limit:

From a density result, let $z \in \mathbb{G} \subset L^2(0,\infty;V)$, there is a sequence (z_n) in $\mathbb{D}$ such as

$$z_n \to z \text{ in } \mathbb{G} \text{ , as } n \to +\infty.$$

So that $z_n \to z$ in $L^2(0,\infty;V)$.

Let θ be a smooth function verifying $\theta(x_3) = 1$ if $|x_3| \leq 1, \theta(x_3) = 0$ if $|x_3| \geq 2$ and $|\theta'(x_3)| \leq 2, \quad \forall x \in \mathbb{R}$.

We define

$$\theta_\varepsilon(x) = \theta\left(\frac{x_3}{\varepsilon\varphi_\varepsilon}\right).$$

And $\qquad z_{\varepsilon,n} = \theta_\varepsilon(x) z_{n|\Sigma} + (1-\theta_\varepsilon(x))\, z_n$.
It is easy to show that $z_{\varepsilon,n} \in L^2(0,\infty;V)$ and $z_{\varepsilon,n} \to z_n$ in $\mathbb{G}$, when $\varepsilon \to 0$. Since

$$F^{\varepsilon}(z_{\varepsilon,n}) = \frac{1}{2}\int_{\Omega_\varepsilon^\infty} |\nabla z_{\varepsilon,n}|^2 + \frac{1}{2\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_{\varepsilon,n}|^2 + \frac{1}{p}\int_{B_\varepsilon^\infty} u_{\varepsilon,n}|z_{\varepsilon,n}|^p.$$

So that

$$\begin{aligned} F^{\varepsilon}(z_{\varepsilon,n}) = & \frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2 + \frac{1}{2}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2 + \\ & \frac{1}{2\varepsilon^\alpha}\int_{]0,\infty[\times B_\varepsilon} |\nabla z_{\varepsilon,n}|^2 + \int_{B_\varepsilon^\infty} u_{\varepsilon,n}|z_{\varepsilon,n}|^p \\ = & \frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_\varepsilon)} |\nabla z_n|^2 + \frac{1}{2}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2 + \\ & \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left|\nabla' z_{n|\Sigma}\right|^2 + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon u_n |z_{n|\Sigma}|^p. \end{aligned}$$

Since φ_ε is bounded, we can easily verify that

$$\lim_{\varepsilon\to 0}\left\{\frac{1}{2}\int_{]0,\infty[\times(\varepsilon\varphi_\varepsilon<|x_3|<2\varepsilon\varphi_\varepsilon)} |\nabla z_{\varepsilon,n}|^2\right\} = 0.$$

(1) If $\alpha \leq 1$: Since $\varphi_\varepsilon \overset{*}{\rightharpoonup} m(\varphi)$ in $L^\infty(\Sigma)$ and $\varepsilon^{1-\alpha} \to \eta(\alpha)$, it follows that

$$\lim_{\varepsilon\to 0} \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left|\nabla' z_{n|\Sigma}\right|^2 = m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma} \left|\nabla' z_{n|\Sigma}\right|^2.$$

By passing to the upper limit, we have

$$\lim_{\varepsilon\to 0}\sup F^{\varepsilon}\left(z_{\varepsilon,n}\right)=\lim_{\varepsilon\to 0}\sup\left(\frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_{\varepsilon})}\left|\nabla z_n\right|^2+\varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma}\varphi_{\varepsilon}\left|\nabla' z_{n|\Sigma}\right|^2+ 2\varepsilon\int_{]0,\infty[\times\Sigma}\varphi_{\varepsilon}u_n|z_{n|\Sigma}|^p\right).$$

$$\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}\left|\nabla z_n\right|^2+m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{n|\Sigma}\right|^2.$$

(2) If $\alpha>1$: By passing to the upper limit, we have

$$\lim_{\varepsilon\to 0}\sup F^{\varepsilon}\left(z_{\varepsilon,n}\right)=\lim_{\varepsilon\to 0}\sup\left(\frac{1}{2}\int_{]0,\infty[\times(|x_3|>2\varepsilon\varphi_{\varepsilon})}\left|\nabla z_n\right|^2+2\varepsilon\int_{]0,\infty[\times\Sigma}\varphi_{\varepsilon}u_n|z_{n|\Sigma}|^p\right)$$
$$\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z_n|^2.$$

Since $z_n\to z$ in $\mathbb{G}$, when $n\to+\infty$. According to the classical result, the diagonalization lemma [[15], Lemma 1.15], there is a function $n(\varepsilon):\mathbb{R}^+\to\mathbb{N}$ increasing to $+\infty$ when $\varepsilon\to 0$, such as $z_{\varepsilon,n(\varepsilon)}\to z$ in $\mathbb{G}$, when $\varepsilon\to 0$. While n approaches $+\infty$;

(1) If $\alpha\neq 1$:

$$\lim_{\varepsilon\to 0}\sup F^{\varepsilon}\left(z_{\varepsilon,n(\varepsilon)}\right)\leq\lim_{n\to+\infty}\sup\lim_{\varepsilon\to 0}\sup F^{\varepsilon}\left(z_{\varepsilon,n}\right)$$
$$\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2.$$

(2) If $\alpha=1$:

$$\lim_{\varepsilon\to 0}\sup F^{\varepsilon}\left(z_{\varepsilon,n(\varepsilon)}\right)\leq\lim_{n\to+\infty}\sup\lim_{\varepsilon\to 0}\sup F^{\varepsilon}\left(z_{\varepsilon,n}\right)$$
$$\leq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2+m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{|\Sigma}\right|^2.$$

(b) We will determine the lower epi-limit.
Let $z\in\mathbb{G}$ and (z_{ε}) a sequence in $L^2(0,\infty;H_0^1(\Omega))$ such as $z_{\varepsilon}\rightharpoonup z$ in $L^2(0,\infty;H_0^1(\Omega))$, so that

$$\chi_{\Omega_{\varepsilon}^{\infty}}\nabla z_{\varepsilon}\rightharpoonup\nabla z\quad\text{ in } L^2(0,\infty,L^2(\Omega))^3. \tag{3.4}$$

(1) If $\alpha\neq 1$: Since

$$F^{\varepsilon}\left(z_{\varepsilon}\right)\geq\frac{1}{2}\int_{\Omega_{\varepsilon}^{\infty}}\left|\nabla z_{\varepsilon}\right|^2+\int_{B_{\varepsilon}^{\infty}}u_{\varepsilon}|z_{\varepsilon}|^p.$$

According to (3.4) and by passage to the lower limit, one obtains

$$\liminf_{\varepsilon\to 0}F^{\varepsilon}\left(z_{\varepsilon}\right)\geq\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2.$$

(2) If $\alpha=1$: If $\liminf_{\varepsilon\to 0}F^{\varepsilon}\left(z_{\varepsilon}\right)=+\infty$, there is nothing to prove, because

$$\frac{1}{2}\int_{]0,\infty[\times\Omega}|\nabla z|^2+m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma}\left|\nabla' z_{|\Sigma}\right|^2\leq+\infty.$$

Otherwise, $\liminf_{\varepsilon\to 0}F^{\varepsilon}\left(z_{\varepsilon}\right)<+\infty$, there is a sub-sequence of $F^{\varepsilon}\left(z_{\varepsilon}\right)$ still designated by $F^{\varepsilon}\left(z_{\varepsilon}\right)$ and a constant $C>0$, such as $F^{\varepsilon}\left(z_{\varepsilon}\right)\leq C$, which implies that

$$\frac{1}{2\varepsilon^{\alpha}}\int_{B_{\varepsilon}^{\infty}}\left|\nabla z_{\varepsilon}\right|^2+\int_{B_{\varepsilon}^{\infty}}u_{\varepsilon}|z_{\varepsilon}|^p\leq C. \tag{3.5}$$

So z_ε satisfies the hypothesis of the lemma 3.2, and according to this last one, $\nabla' m^\varepsilon z_\varepsilon$ is bounded in $L^2(0,\infty;L^2(\Sigma))^2$, so there is an element $z_1 \in L^2(0,\infty;L^2(\Sigma))^2$ and a sub-sequence of $\nabla' m^\varepsilon z_\varepsilon$, always designated by $\nabla' m^\varepsilon z_\varepsilon$, such as $\nabla' m^\varepsilon z_\varepsilon \rightharpoonup z_1$ in $L^2(0,\infty;L^2(\Sigma))^2$, since $z_{\varepsilon|\Sigma} \rightharpoonup z_{|\Sigma}$ in $L^2(]0,\infty[\times\Sigma)$, and thanks to (3.1) and (3.5), one has $m^\varepsilon z_\varepsilon \rightharpoonup z_{|\Sigma}$ in $L^2(]0,\infty[\times\Sigma)$, then $m^\varepsilon z_\varepsilon \rightharpoonup z_{|\Sigma}$ in $L^2(0,\infty;H_0^1(\Sigma))$, so $z_1 = \nabla' z_{|\Sigma}$, so that $\nabla' m^\varepsilon z_\varepsilon \rightharpoonup \nabla' z_{|\Sigma}$ in $L^2(0,\infty;L^2(\Sigma))^2$,

$$\begin{aligned} F^\varepsilon(z_\varepsilon) &\geq \frac{1}{2}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \frac{1}{2\varepsilon^\alpha}\int_{B_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \int_{B_\varepsilon^\infty} u_\varepsilon |z_\varepsilon|^p \\ &\geq \frac{1}{2}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left|\nabla' m^\varepsilon z_\varepsilon\right|^2 + 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon u_\varepsilon |m^\varepsilon z_\varepsilon|^p. \end{aligned}$$

Using the subdifferential inequality, we have

$$\begin{aligned} F^\varepsilon(z_\varepsilon) \geq & \frac{1}{2}\int_{\Omega_\varepsilon^\infty} |\nabla z_\varepsilon|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left|\nabla' z_{|\Sigma}\right|^2 + \varepsilon^{1-\alpha}\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon \left|\nabla' z_{|\Sigma}\right| \nabla' z_{|\Sigma}\left(\nabla' m^\varepsilon z_\varepsilon - \nabla' z_{|\Sigma}\right) \\ &+ 2\varepsilon\int_{]0,\infty[\times\Sigma} \varphi_\varepsilon u_\varepsilon |m^\varepsilon z_\varepsilon|^p. \end{aligned}$$

Thanks to the lemma 5.2, we have $\varphi_\varepsilon \to m(\varphi)$ in $L^2(\Sigma)$, so according to (3.4) and by passing to the lower limit, we obtain

$$\liminf_{\varepsilon\to 0} F^\varepsilon(z_\varepsilon) \geq \frac{1}{2}\int_{]0,\infty\times\Omega} |\nabla z|^2 + m(\varphi)\eta(\alpha)\int_{]0,\infty[\times\Sigma} \left|\nabla' z_{|\Sigma}\right|^2.$$

Hence the result.

□

Proposition 3.2 *According to the values of parameter α, there exists (z^*, u^*) satisfying*

$$\begin{gathered} z_\varepsilon \rightharpoonup z^* \text{ in } L^2(0,\infty;V) \\ u_\varepsilon \overset{*}{\rightharpoonup} u^* \text{ in } L^\infty(]0,\infty[) \\ F^\alpha(z^*) = \inf_{v\in\mathbb{G}}\{F^\alpha(v)\}. \end{gathered}$$

Proof:

Initially (z_ε) is bounded in $L^2(0,\infty;V)$, so it has a $\tau-$ cluster point z^* in $L^2(0,\infty;V)$. And thanks to a classical result of epi-convergence (see theorem 5.3), we have z^* is a solution of the problem

$$\inf_{v\in L^2(0,\infty;V)}\{F^\alpha(v)\}. \qquad (\mathscr{P}_{lim})$$

□

Conclusion: theory part

In this paper, we have worked on a class of semi-linear evolution systems with an operator generating a compact C_0 semigroup. We have shown that this approach is strongly stabilizable by a well-defined control for the equivalent approximating problem on a three-dimensional bung of a nano structure. We also learned the limiting behavior of this type of problem, finding that the effect of the nanolayer does not exist and the control vanishes, i.e., the nanolayer behaves as a part of Ω, and the limiting problem becomes an autonomous problem. In future work, we extend the proposed scheme to the nonlinear problem with interface conditions in a nanostructure.

4. Numerical Tests

We have shown that for a sufficiently small value of ε, the solution z_ε of the problem (2.1) approaches the solution z^* of the limit problem. In this section, we are interested in the numerical treatment; we will focus on the impact of the control on the B_ε^∞ domain, with

$$\begin{array}{c|c} T = 1e+8 & Lz_\varepsilon = z_\varepsilon |z_\varepsilon|^{p-2} \\ \Omega = \{(x,y)| \ \ x \in]0,1[, y \in]-1,1[\} & u_\varepsilon(t) = -\frac{c\langle L(z_\varepsilon), z_\varepsilon\rangle}{1+\langle L(z_\varepsilon), z_\varepsilon\rangle^2} \\ B_\varepsilon =]0,1[\times]-\varphi_\varepsilon(x), \varphi_\varepsilon(x)[& \varphi_\varepsilon(x) = \varepsilon\left(1.6 + \sin\left(\pi \frac{x}{\varepsilon}\right)\right). \end{array}$$

Using the language FreeFem++ (see [12]), with the finite element method using the newtons method, and Gnuplot to visualize the results, with $p = 2.75$ and $\varepsilon = 1e-06$, one will have the results shown in figures.

Figure 2: shows that the solution of the approximation problem converges to that of the limit problem,

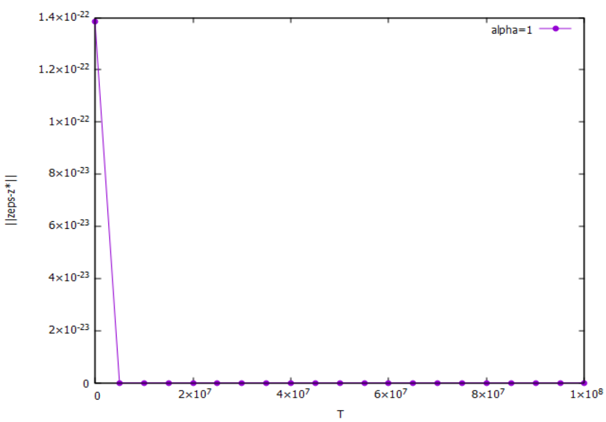
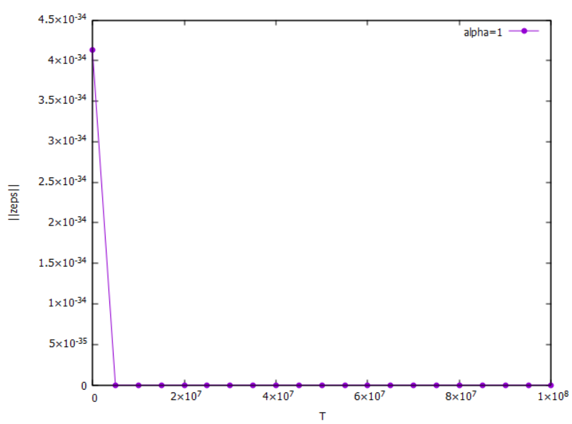


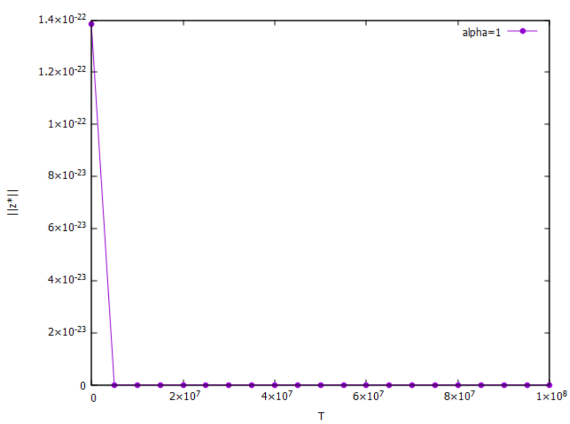
Figure 2:

Initially, u^* does not stabilize the state on all Ω, which is normal since the control is defined just on B_ε, so the control will stabilize the state just on a subregion, so we are only interested in $\alpha = 1$.

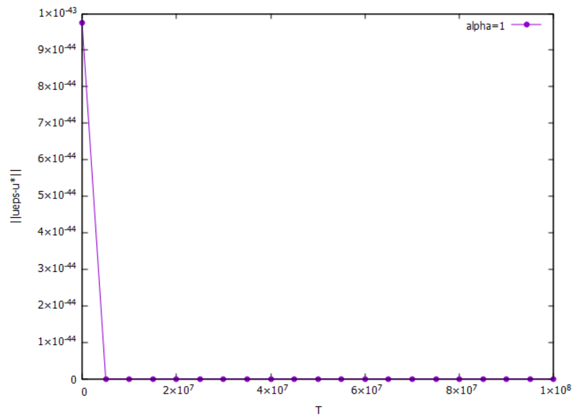
The two figures (a) and (b) show that u_ε stabilizes the state z_ε, and u^* stabilizes the state z^* on the nanolayer.



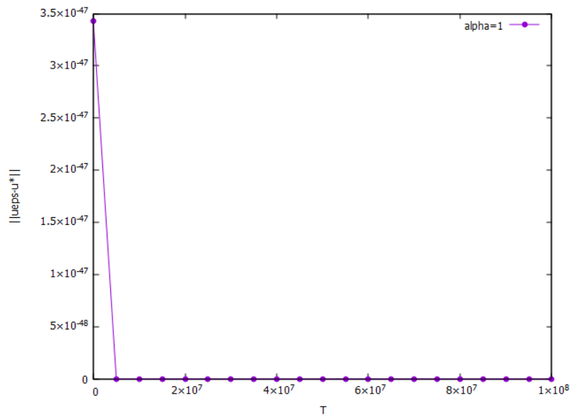
(a)



(b)



(c) $\varepsilon = 1e - 05$



(d) $\varepsilon = 1e - 06$

These two figures show that u_ε converges to u^*, and we see that for minimal ε values, the speed of convergence increases.

Conclusion: Practical part

The interface Σ does not affect the problem, as we work on a classical problem, and for $\alpha = 1$, we found that it stabilized, which is the goal, which shows that the model is suitable for control specialists on the nanolayer.

5. Appendix

We give three Banach spaces X_0, X and X_1 with X_0 and X_1 reflexive and

$$(T.Ev) \qquad X_0 \hookrightarrow\hookrightarrow X \hookrightarrow X_1$$

For two real $1 < p_0, p_1 < \infty$ and $T > 0$, we define the space W by :

$$W = \left\{ v \in L^{p_0}(0,T;X_0) : \frac{\partial v}{\partial t} \in L^{p_1}(0,T;X_1) \right\}$$

Lemma 5.1 *(of compactness) [13]. Suppose that the hypothesis $(T.Ev)$ is satisfied and that $1 < p_0, p_1 < \infty$, then*

$$W \hookrightarrow\hookrightarrow L^{p_0}(0,T;X).$$

Remark 5.1 *for the case of our problem we have;*

$$W = \left\{ z \in L^2\left(0,\infty;H_0^1(\Omega)\right) : \frac{\partial z}{\partial t} \in L^2\left(0,\infty;H^{-1}(\Omega) + L^{p'}(\Omega)\right) \right\}$$

And the condition

$$(T.Ev) \qquad H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) + L^{p'}(\Omega)$$

is verified.
So

$$W \hookrightarrow\hookrightarrow L^2(0,\infty;L^2(\Omega)).$$

Theorem 5.1 *Rellich-kondrachoff [13]. The space $H^1(\mathcal{Q})$ is compactly embedded in $L^2(\mathcal{Q})$.*

Theorem 5.2 *1. The eigenvalues of $-\Delta$ form an infinite sequence of real numbers such that*

$$0 < \lambda_1(\Omega) \le \lambda_2(\Omega) \le \cdots \to +\infty$$

2. Each eigenvalue has finite multiplicity.
3. (See [10]) Let Ω be any open of $\mathbb{R}^d$. The operator $A = -\Delta$, defined by

$$D(A) = \left\{u \in H_0^1(\Omega), \Delta u \in L^2(\Omega)\right\}, Au = -\Delta u$$

is the infinitesimal generator of a semigroup S of contractions on $L^2(\Omega)$.

Lemma 5.2 *(see [14], Appendix). Let $\varphi \in L^\infty(\Sigma)$, a Y -periodic, $Y =]0,1[\times]0,1[$. Let*

$$\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right), \text{ for a small enough } \varepsilon > 0.$$

So

$$\varphi_\varepsilon \to m(\varphi) \text{ in } L^s(\Sigma) \quad \text{for } 1 \le s < \infty,$$
$$\varphi_\varepsilon \rightharpoonup^* m(\varphi) \text{ in } L^\infty(\Sigma).$$

Theorem 5.3 *(see [15], Theorem 1.10). Suppose that*
(1) F^ε admits a minimizer on X,
(2) the sequence $(\bar{z}_\varepsilon)$ is τ -relatively compact,
(3) The sequence F^ε epi-converges to F in this topology τ.
Then each cluster point $\bar{z}$ of the sequence $(\bar{z}_\varepsilon)$ minimizes F on $\mathbb{X}$ and

$$\lim_{\varepsilon' \to 0} F^{\varepsilon'}(\bar{z}_{\varepsilon'}) = F(\bar{z})$$

if $(\bar{z}_{\varepsilon'})_{\varepsilon'}$ denotes the subsequence of $(\bar{z}_\varepsilon)_\varepsilon$ that converges to $\bar{z}$.

Theorem 5.4 *[9]. If L is a locally lipschitzian function and $r(z) : \Omega\times]0,\infty[\to \mathbb{R}^+$ is a function of class C^k, $k \in \mathbb{N}^*$ which verifies*

$$\forall a > 0; \exists r_a > 0; \forall z \in \mathcal{B}(0,a), r(z) \ge r_a.$$

If $(S(t))_{t\ge 0}$ is compact and $\{z \in H \mid \langle L(S(t)z), S(t)z\rangle = 0\} = \{0\}$, then the system (2.1) controlled by the feedback: $u(t) = -r(z)\langle L(z), z\rangle$ is strongly stabilizable.

Remark 5.2 *[9]. If the system (2.1) is such that the command u must satisfy the saturation constraint, i.e. $\|u\|_\infty = \sup_{s\ge 0}|u(s)| \le c$, for $c > 0$, a judicious choice of the weight function $r(z)$ allows to obtain such a constrained return law $u(z)$.*

For this, let us take for example:

$$r(z) = \frac{c}{1 + \langle L(z), z\rangle^2} \quad \forall c > 0.$$

Lemma 5.3 *[16]. V is a separable Banach space for the norm of V, and for $p \le \frac{2N}{N-2}$, we have $V = H_0^1(\Omega)$. i.e. H^1 is continuously embedded in L^p.*

Proof: Initially, V is a Banach space (obvious) and by the Sobolev extension theorem we have

$$\left\{\begin{array}{c} H_0^1(\Omega) \hookrightarrow L^q(\Omega) \\ \frac{1}{q} = \frac{1}{2} - \frac{1}{N} \quad \text{if} \quad N \ge 3. \end{array}\right.$$

So if $p \le \frac{2N}{N-2}$, then $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$, which gives $V = H_0^1(\Omega)$. Let us now show that V is separable.

Consider the application,

$$\Psi : V \to L^p(\Omega) \times \prod_{i=1}^N L^2(\Omega)$$
$$v \mapsto \left(v, \frac{\partial v}{\partial x_i}, \ldots, \frac{\partial v}{\partial x_N}\right)$$

V can be identified with $\Psi(V)$ which is a closed vector space of $L^p(\Omega) \times \prod_{i=1}^N L^2(\Omega)$ which is separable and uniformly convex.
So $V(\sim \Psi(V))$ is also separable and uniformly convex.

□

References

1. Zerrik, E. Ouzahra M. Regional stabilization for infinite-dimensional systems, Int. J. Control, 2003, vol 76, number 1, pp. 73-81.
2. Amourous, M., El Jai, A., and Zerrik, E., 1994, Regional observability of distributed systems. International Journal of Systems Sciences, 25, 301-313.
3. El Jai, A., Simon, M. C., Zerrik, E., and Pritchard, A. J., 1995, Regional controllability of distributed parameter systems. International Journal of Control, 62, 1351-1365.
4. Triggiani, R., 1975, On the stabilizability problem in Banach space. Journal of Mathematical Analysis and Applications, 52, 383-403.
5. Balakrishnan, A. V., 1981, Strong stability and the steady state Riccati equation. Applied Mathematics and Optimization, 7, 335-345.
6. J.M. Ball, M. Slemrod, Feedback stabilization of distributed semilinear control systems, App. Math. Optim. 5 (1979) 169-179.
7. A. Bounabat, A. Bouslouss, H. Hammouri, Stabilisation faible d'un systeme bilineaire dissipatif en dimension infinie, Extracta Mathematicae 7 (2) (1993) 95-98.
8. A. Bounabat, J.P. Gauthier, Weak stabilizability of infinite dimensional nonlinear systems, Appl. Math. Lett. 4 (1) (1991) 95-98.
9. H. Bounit, H. Hammouri: Feedback stabilization for a class of distributed semilinear control systems, Nonlinear Analysis. 37 (1999), 953-969.
10. K. J. ENGEL et R. NAGEL - One parameter semigroups for linear evolution equations, Graduate Texts in Mathematics 194, Springer-Verlag New York, 2000.
11. Brezis H., Analyse fonctionnelle, Théorie et Applications, Masson (1992).
12. Hecht F., Pironneau O., Le Hyaric A., Ohtsuka K., FreeFem++ Manual, downloadable at http://www.Freefem.org.
13. J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod, Gautier Villars Paris 1969.
14. A. Ait Moussa and J. Messaho, Limit behavior of an oscillating thin layer, in Electronic Journal of Differential Equations, Conference, vol. 14, pp. 21-33, Oujda International Conference on Nonlinear Analysis, Oujda, Morocco, 2006.
15. Attouch H., Variational Convergence for Functions and Operators, Pitman (London) 1984.
16. S. I. SOBOLEV, Applications de l'analyse fonctionnelle aux équations de la physique mathématique. Léningrad, 1950 .
17. S. Axler and K.A. - A Short Course on Operator Semigroups, (C) 2006 Springer Science+Business Media, LLC.
18. C.M. Dafermos, M. Slemrod, Asymptotic behavior of nonlinear contractions semigroups, J. Funct. Analy. 13 (1973) 97 - 106.
19. A. PAZY - Semigroups of linear operators and applications to partial differential equations, 1ère éd., Applied Mathematical Sciences 44, Springer-Verlag New York, 1983.
20. Zerrik E. Ouzahra M. and Ztot K. Regional stabilization for infinite bilinear systems, IEE Cont. Theory. Appl. 2004, volume: 151, issue : 1, pp. 109-116.
21. Zerrik, E. Ouzahra M. An output stabilization problem for infinite-dimensional systems, Systems and Control Letters, 2004, proposé.
22. Zerrik, E. and Ouzahra, M. Output stabilization for distributed semilinear systems. IET. Control theory and applications. 2007, vol 1, Issue 3, pp. 838-843.

Tarik BOULAHROUZ,

Mohammed FILALI,

Jamal MESSAHO,

Najib TSOULI,

Laboratory of Applied Mathematics of the Oriental,

Faculty of Sciences of Oujda, Mohamed First University,

Oujda, Morocco.

E-mail address: tarik.boulahrouz@usmba.ac.ma, m.filali@ump.ac.ma, j.messaho@gmail.com, n.tsouli@ump.ac.ma