Bol. Soc. Paran. Mat. (3s.) **v. 2024 (42)** : 1–10.
©SPM –ISSN-2175-1188 on line ISSN-0037-8712 in press
SPM: www.spm.uem.br/bspm doi:10.5269/bspm.65739

The Continuous Wavelet Transform for a Laguerre Type Operator on the Half Line

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ABSTRACT: In this paper, we consider a Laguerre differential operator Λ on $[0, \infty)$ by accomplishing harmonic analysis tools with respect to the operator Λ. We study some definitions and properties of Laguerre continuous wavelet transform. We also explore generalized Laguerre Fourier transform and convolution product on $[0, \infty)$ associated with the operator Λ. Also a new continuous wavelet transform associated with Laguerre function is constructed and investigated.

Key Words: Generalized wavelets, Laguerre wavelet, generalized continuous wavelet transform.

Contents

1. Introduction

For a function $f \in L^2(\mathbb{R})$, the wavelet transform with respect to the wavelet $\phi \in L^2(\mathbb{R})$ is defined by

$$(W_\varphi f)(\sigma_2, \sigma_1) = \int_{-\infty}^{\infty} f(t)\overline{\varphi_{\sigma_2,\sigma_1}(t)}dt, \sigma_1 > 0 \tag{1.1}$$

where,

$$\varphi_{\sigma_2,\sigma_1}(t) = \sigma_1^{-1/2}\varphi(\frac{t-\sigma_2}{\sigma_1}). \tag{1.2}$$

Translation τ_{σ_2} is defined by

$$\tau_{\sigma_2}\varphi(t) = \varphi(t - \sigma_2), \sigma_1 \in \mathbb{R}$$

and dilation D_{σ_1} is defined by

$$D_{\sigma_1}\varphi(t) = \sigma_1^{-1/2}\phi(\frac{t}{\sigma_1}), \sigma_1 > 0.$$

We can write

$$\varphi_{\sigma_2,\sigma_1} = \tau_{\sigma_2} D_{\sigma_1}\phi(t). \tag{1.3}$$

From above equations, we can say that wavelet transform of the function f on $\mathbb{R}$ is an integral transform and the dilated and translated ϕ is the kernel.
We can also express wavelet transform as the convolution:

$$(W_\varphi)(\sigma_2, \sigma_1) = (f * g_{o,\sigma_1})(\sigma_2), \tag{1.4}$$

where,

$$g(t) = \overline{\varphi(-t)}.$$

Since there is a special type of convolution for every integral transform, therefore one can define wavelet transform with respect to a integral transform using associated convolution.
The concept of wavelet is a collection of a function derived from a single function called mother wavelet,

2010 *Mathematics Subject Classification:* 42C40, 65R10, 44A35.
Submitted November 07, 2022. Published February 10, 2023

after that by applying the two operators known as translation and dilation we get a new type of continuous wavelet.
Consider the Laguerre polynomial L_m^α of degree m and of order $\alpha > -1$,

$$L_m^\alpha = \frac{x^{-\alpha}e^x}{n!}\left(\frac{d}{dx}\right)^m x^{m+\alpha}e^{-x},$$

satisfies the equations

$$\partial_x\left(xe^{-x}L_m^\alpha(x)\right) + me^{-x}L_m^\alpha(x) = 0, x \in (0,\infty).$$

The goal of this work is to extend the classical theory of wavelets to the Laguerre functions.
We call generalized wavelet each function g in a suitable functional space, satisfying the admissibility condition

$$0 < C_g = \sum_n \frac{|F_\Delta(g)(\lambda)|^2}{(\lambda)} < \infty,$$

where $F_\Lambda(g)(\lambda)$ denotes the generalized Fourier transform related to Laguerre function

$$F_\Lambda(g)(\lambda) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)\phi_{-\lambda}(x)e^{-x}x^\alpha dx.$$

Starting from a generalized wavelet g we construct by translation and dilation a family of generalized wavelets by putting

$$g_{a,b}(x) = \frac{1}{a^{1/2}}T^b g_a(x), a > 0, b \geq 0,$$

where $g_a(x) = g(ax)$ and T^b stand for generalized translation operator.

2. Preliminaries

In this section we states some result and facts related to harmonic analysis associated with the Laguerre function. Here we only cite the properties needed for the discussion. Throughout this section assume $\alpha > -1$.
Define $L_\alpha^p[0,\infty), 1 \leq p \leq \infty$, as the class of measurable functions on $[0,\infty)$ for which $||f||_{p,\alpha} < \infty$, where

$$||f||_{p,\alpha} = \left(\frac{1}{\Gamma(\alpha+1)}\int_0^\infty |f(x)|^p e^{-x}x^\alpha dx\right)^{1/p} < \infty, 1 \leq p \leq \infty,$$

and

$$||f||_\infty = ess_{0\leq x<\infty} sup|f(x)| < \infty.$$

The Fourier-Laguerre transform of order α is defined for a function f on $[0,\infty)$ by

$$F_\alpha(f)(\lambda) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)\zeta_n^\alpha e^{-x}x^\alpha dx \tag{2.1}$$

where $\zeta_n^\alpha(x)$ is a Laguerre function

$$\zeta_n^\alpha(x) = \rho(n)\Gamma(\alpha+1)L_n^\alpha(x) \tag{2.2}$$

$\rho(n) = \frac{n!}{\Gamma(n+\alpha+1)}$ and $L_n^\alpha(x)$ is the Laguerre polynomial of degree n and of $\alpha > -1$

Proposition 2.1.

(i) If both f and $F_\alpha(f)$ are in $[0,\infty)$ then

$$f(x) = \sum_n F_\alpha(f)\zeta_n^\alpha(x)\sigma(n)$$

where

$$\sigma(n) = \frac{1}{\Gamma(\alpha+1)\rho(n)}. \tag{2.3}$$

(ii) For every $f \in L^2_\alpha$ we have

$$\sum_n \sigma(n)|F_\alpha(n)|^2 = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty |f(x)|^2 e^{-x}x^\alpha dx.$$

(iii) The inverse transform is given by

$$F_\alpha^{-1}(g)(y) = \sum_n g(n)\zeta_n^\alpha(y)\sigma(n).$$

The Laguerre translation operators $\tau^x, x > 0$ in [4] are defined by

$$\tau^x = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(z)d(x,y,z)e^{-z}z^\alpha dz, \tag{2.4}$$

where

$$d(x,y,z) = \sum_n \zeta_n^\alpha(x)\zeta_n^\alpha(y)\zeta_n^\alpha(z)\sigma(n), \tag{2.5}$$

$$\int_0^\infty d(x,y,z)\zeta_n^\alpha(z)d\Delta(z) = \zeta_n^\alpha(x)\zeta_n^\alpha(y), \tag{2.6}$$

and

$$d\Lambda(\lambda) = \frac{1}{\Gamma(\alpha+1)}e^{-\lambda}\lambda^\alpha d\lambda. \tag{2.7}$$

The Laguerre convolution product of two functions is defined by the relation

$$f * g(x) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty \tau^x f(y)g(y)e^{-y}y^\alpha dy. \tag{2.8}$$

Proposition 2.2. *Let $p \in [1,\infty]$ and $f \in L^p_\alpha$, Then for all $x \geq 0$, $\tau^x f \in L^p_\alpha$ and*

(i) $||\tau^x f||_{p,\alpha} \leq ||f||_{p,\alpha}$.

(ii) $F_\alpha(\tau^x f) = L_n^\alpha(y)F_\alpha$.

(iii) Let $p, q \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}=1$. If $f \in L^p_\alpha$ and $g \in L^q_\alpha$, then

$$\int_0^\infty \tau^x f(y)g(y)d(x,y,z)e^{-y}y^\alpha dy = \int_0^\infty f(y)\tau^x g(y)d(x,y,z)e^{-y}y^\alpha dy.$$

(iv) Let $p, q, r \in [1,\infty)$ such that $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$. If $f \in L^p_\alpha$ and $g \in L^q_\alpha$ then $f * g \in L^r_\alpha$ and

$$||f * g||_{r,\alpha} \leq ||f||_{p,\alpha}||g||_{q,\alpha}.$$

(v) $F_\alpha(f * g) = F_\alpha(f)F_\alpha(g)$.

Definition 2.3. *A function $g \in L^p_\alpha$ be a Laguerre wavelet.The continuous Laguerre wavelet, if it satisfies the admissibility condition*

$$0 < C_g^\alpha = \sum_n \frac{|F_\alpha(g)(n)|^2}{n} < \infty. \tag{2.9}$$

Definition 2.4. *Let $g \in L^p_\alpha$ be a Laguerre wavelet transform is defined for suitable function f by*

$$S_g^\alpha(f)(b,a) = \frac{1}{\Gamma(\alpha+1)}\int_0^\infty f(x)\overline{g_{b,a}^\alpha(x)}e^{-x}x^\alpha dx, \tag{2.10}$$

where $a > 0, b > 0$,

$$g_{b,a}^\alpha(t) = \frac{1}{a^{1/2}}\tau^b g_a(t) \tag{2.11}$$

and

$$g_a(t) = g(at) = g(t/a). \tag{2.12}$$

Theorem 2.5. *Let $g \in L^p_\alpha[0,\infty)$ be a Laguerre wavelet. Then*

(i) For all $f \in L^2_\alpha[0,\infty)$ we have the Plancherel formula

$$\int_0^\infty |f(x)|^2 d\Lambda(x) = \frac{1}{C_g^\alpha} \int_0^\infty \int_0^\infty |S_g^\alpha(f)(b,a)|^2 \frac{d\Lambda(a)}{a} d\Lambda(b).$$

(ii) For $f \in L^2_\alpha[0,\infty)$ such that $F_\alpha(f) \in L^2_\alpha[0,\infty)$ we have

$$f(x) = \frac{1}{C_g^\alpha} \int_0^\infty \left(\int_0^\infty S_g^\alpha(f)(b,a) g_{b,a}^\alpha(x) d\Lambda(b) \right) \frac{d\Lambda(a)}{a}$$

for almost all $x \geq 0$. Where $d\Lambda(\lambda) = \frac{1}{\Gamma(\alpha+1)} e^{-\lambda} \lambda^\alpha d\lambda$.

3. Harmonic analysis associated with Laguerre function

Note 3.1 From here assume $\alpha > -1$ and $n \in \mathbb{N} \cup \{0\}$. Let M be the map defined by

$$Mf(x) = e^{\frac{-|\lambda| x^2}{2}} f(x).$$

Let $L^p_\alpha, 1 \leq p \leq \infty$, be the class of measurable functions f on $[0,\infty)$ for which $||f||_p = ||M^{-1} f||_p < \infty$.

3.1 Generalized Fourier Transform

For $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$, put

$$\phi_\lambda(x) = e^{\frac{-|\lambda| x^2}{2}} \zeta_m^\alpha(|\lambda| x^2), \tag{3.1}$$

where $\zeta_m^\alpha(x)$ is a Laguerre function.

Definition 3.1. *The generalized Fourier transform is defined for a function $f \in L^1_\alpha$ by*

$$F_\Lambda(f)(\lambda) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x) \phi_{-\lambda}(x) e^{-x} x^\alpha dx, \lambda \geq 0. \tag{3.2}$$

Remark 3.2.

(i) By (3.1) and (3.2) observe that

$$F_\Lambda = F_\alpha \circ M^{-1}, \tag{3.3}$$

where F_α is the Fourier-Laguerre transform given by (1.1).

(ii) If $f \in L^1_\alpha$ then $F_\Lambda(f)$ satisfies $||F_\Lambda(f)||_\infty \leq ||f||_{1,\Lambda}$.

Theorem 3.3. *Let f be a measurable function on $[0,\infty)$, Then for almost all $x \geq 0$.*

$$f(x) = \sum_\lambda F_\Lambda(f)(\lambda) \phi_\lambda(x) \sigma(\lambda),$$

where $\sigma(\lambda)$ is given in (2.3).

Proof. By (3.1), (3.3) and proposition 2.1(ii) we have

$$\begin{aligned}\sum_\lambda F_\Lambda(f)(\lambda)\phi_\lambda(x)\sigma(\lambda) &= e^{\frac{-|\lambda| x^2}{2}} \sum_\lambda F_\Lambda(f)(\lambda) \zeta_n^\alpha(x) \rho(\lambda) \\ &= e^{\frac{-|\lambda| x^2}{2}} \sum_\lambda F_\Lambda(M^{-1} f)(\lambda) \zeta_n^\alpha(x) \rho(\lambda) \\ &= e^{\frac{-|\lambda| x^2}{2}} M^{-1} f(x) \\ &= f(x)\end{aligned}$$

□

Theorem 3.4.

(i) ***The Plancherel formula***

$$\sum_\lambda \rho(\lambda)|F_\Lambda(f)(\lambda)|^2 = \int_0^\infty |f(x)|^2 e^{-x} x^\alpha dx.$$

(ii) The inverse transform is given by

$$F_\Lambda^{-1}(g)(x) = \sum_\lambda \rho(\lambda)|F_\Lambda(M^{-1}f)(\lambda)|^2$$

Proof. By (3.3) and proposition 2.1(iii) we have

$$\begin{aligned}\sum_\lambda \rho(\lambda)|F_\Lambda(f)(\lambda)|^2 &= \sum_\lambda \rho(\lambda)|F_\Lambda(M^{-1}f)(\lambda)|^2 \\ &= \int_0^\infty |M^{-1}f(x)|^2 e^{-x} x^\alpha dx \\ &= \int_0^\infty |f(x)|^2 e^{-x} x^\alpha dx\end{aligned}$$

which ends the proof of (i).
The proof of (ii) is obvious. □

3.2 Generalized translation operators and convolution product

Definition 3.5. *The generalized translation operators $T^x, x \geq 0$, by the relation*

$$T^x f(y) = e^{\frac{-|\lambda|(x^2+y^2)}{2}} \tau_\alpha^x (M^{-1}f)(y), \tag{3.4}$$

where τ_α^x is the Laguerre translation operator.

Definition 3.6. *The generalized convolution product of two functions f and g is defined by*

$$f\sharp g(x) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty T^x f(y) g(y) e^{-y} y^\alpha dy. \tag{3.5}$$

Remark 3.7. *By (3.4)*

$$f\sharp g(x) = M[(M^{-1}f) *_y (M^{-1}g)], \tag{3.6}$$

where $*_y$ is the Laguerre convolution given by (2.8).

Proof.

$$\begin{aligned}f\sharp g(x) &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty T^x f(y) g(y) e^{-y} y^\alpha dy \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty e^{\frac{-|\lambda|(x^2+y^2)}{2}} \tau_\alpha^x (M^{-1}f)(y) g(y) e^{-y} y^\alpha dy \\ &= \frac{e^{\frac{-|\lambda|(x^2)}{2}}}{\Gamma(\alpha+1)} \int_0^\infty e^{\frac{-|\lambda|(y^2)}{2}} \tau_\alpha^x (M^{-1}f)(y) g(y) e^{-y} y^\alpha dy \\ &= e^{\frac{-|\lambda|(x^2)}{2}} [(M^{-1}f) *_y (M^{-1}g)](x) \\ &= M[(M^{-1}f) *_y (M^{-1}g)](x).\end{aligned}$$

□

Proposition 3.8.

(i) $||T^x f||_{p,\alpha} \leq e^{\frac{-|\lambda|(x^2)}{2}} ||f||_{p,\alpha}$

(ii) $F_\Lambda(T^x f) = \phi_\lambda(x) F_\Lambda(f)(\lambda)$.

(iii) Let $p, q, r \in [0.\infty)$ *such that* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$, *then*

$$||f\sharp g||_{r,\alpha} \leq ||f||_{p,\alpha} ||g||_{q,\alpha}.$$

(iv) $F_\Lambda(f\sharp g) = F_\Lambda(f) F_\Lambda(g)$.

Proof. (i) By using proposition 2.2(i) and (3.4) we have

$$\begin{aligned} ||T^x f||_{p,\alpha} &= ||e^{\frac{-|\lambda|(x^2+y^2)}{2}} \tau_\alpha^x (M^{-1} f)||_{p,\alpha} \\ &= e^{\frac{-|\lambda|(x^2)}{2}} ||e^{\frac{-|\lambda|(y^2)}{2}} \tau_\alpha^x (M^{-1} f)||_{p,\alpha} \\ &= e^{\frac{-|\lambda|(x^2)}{2}} ||M \circ \tau_\alpha^x \circ (M^{-1} f)||_{p,\alpha} \\ &= e^{\frac{-|\lambda|(x^2)}{2}} ||\tau_\alpha^x \circ M^{-1} f|| \\ &\leq e^{\frac{-|\lambda|(x^2)}{2}} ||M^{-1} f||_{p,\alpha} \\ &= e^{\frac{-|\lambda|(x^2)}{2}} ||f||_{p,\alpha}. \end{aligned}$$

(ii) By (3.1), (3.3), (3.4) and proposition (ii)

$$\begin{aligned} F_\Lambda(T^x f)(\lambda) &= F_\alpha \circ M^{-1} (T^x f)(\lambda) \\ &= F_\alpha \circ M^{-1} \left(e^{\frac{-|\lambda|(x^2)}{2}} M \circ \tau_\alpha^x \circ M^{-1} f \right)(\lambda) \\ &= e^{\frac{-|\lambda|(x^2)}{2}} F_\alpha \left(\tau_\alpha^x \circ M^{-1} f \right)(\lambda) \\ &= e^{\frac{-|\lambda|(x^2)}{2}} L_n^\alpha(\lambda) F_\alpha(f)(\lambda) \\ &= \phi_\lambda(x) F_\alpha(f)(\lambda). \end{aligned}$$

(iii) By (3.4) and proposition 2.2(iii)

$$\begin{aligned} ||f\sharp g||_{r,\alpha} &= ||M[(M^{-1} f)_y (M^{-1} g)]||_{r,\alpha} \\ &\leq ||M^{-1} f||_{p,\alpha} ||M^{-1} g||_{q,\alpha} \\ &= ||f||_{p,\alpha} ||g||_{q,\alpha}. \end{aligned}$$

(iv) By (3.3), (3.6) and proposition 2.2(v)

$$\begin{aligned} F_\Lambda(f\sharp g) &= F_\alpha[(M^{-1} f) *_y (M^{-1} g)] \\ &= F_\alpha(M^{-1} f) F_\alpha(M^{-1} g) \\ &= F_\Lambda(f) F_\Lambda(g). \end{aligned}$$

□

3.3 Transmutation operators

Definition 3.9. *For a function* f *on half line, define the integral transform by*

$$\chi f(x) = \frac{2\Gamma(\alpha+1)}{\Gamma\pi\Gamma(\alpha+1/2)} e^{\frac{-|\lambda|(x^2)}{2}} \int_0^1 f(tx)(1-t^2)^{\alpha-1/2} dt. \tag{3.7}$$

Remark 3.10.

(i) For $\lambda = 0, \chi$ *is just the Riemann-Liouville integral transform of* α *order by*

$$R_\alpha(f)(y) = a_\alpha \int_0^1 f(ty)(1-t^2)^{\alpha-1/2}dt, \tag{3.8}$$

where $a_\alpha = \frac{2\Gamma(\alpha+1)}{\Gamma\pi\Gamma(\alpha+1/2)}$.

(ii) By (3.8)

$$\chi = M \circ R_\alpha. \tag{3.9}$$

Definition 3.11. *Define the integral transform* ${}^t\chi$ *for a smooth function* f *on half line by*

$$^t\chi f(y) = a_\alpha \int_y^\infty e^{\frac{-|\lambda|x^2}{2}} f(x)(x^2-y^2)^{\alpha-1/2}dx \tag{3.10}$$

Remark 3.12.

(i) For $\lambda = 0, {}^t\chi$ *reduces to the Weyl integral transform of order* α*by*

$$W_\alpha(f)(y) = a_\alpha \int_y^\infty f(x)(x^2-y^2)^{\alpha-1/2}dx, y \geq 0. \tag{3.11}$$

(ii) It is seen that

$$^t\chi = W_\alpha \circ M^{-1}. \tag{3.12}$$

Proposition 3.13.

(i) $||\chi f||_{\infty,\alpha} \leq ||f||_\alpha$.

(ii) $||{}^t\chi f||_1 \leq ||f||_{1,\infty}$.

(iii) ${}^t\chi(f\sharp g) = {}^t\chi f * {}^t\chi g$.

(iv) $\chi({}^t\chi f * g) = f\sharp(\chi g)$.

Proof. (i) By (3.9) and [4] we have

$$||\chi f||_{\infty,\alpha} = ||M \circ R_\alpha(f)||_\infty = ||R_\alpha(f)||_\infty \leq ||f||_\infty.$$

(ii) By (3.12) and [4] we have

$$||{}^t\chi f||_1 \leq ||W_\alpha \circ M^{-1}||_1 \leq ||M^{-1}(f)||_{1,\infty} = ||f||_{1,\infty}.$$

(iii) By (3.6), (3.12) and [4] we have

$$\begin{aligned}
{}^t\chi(f\sharp g) &= W_\alpha \circ M^{-1}(f\sharp g) \\
&= W_\alpha \circ M^{-1}(M[(M^{-1}f) *_y (M^{-1}g)]) \\
&= W_\alpha \circ [(M^{-1}f) *_y (M^{-1}g)] \\
&= (W_\alpha M^{-1}f) * (W_\alpha M^{-1}g) \\
&= {}^t\chi f * {}^t\chi g.
\end{aligned}$$

(iv) By (3.6), (3.9), (3.12) and [4] we have

$$\begin{aligned}
f\sharp(\chi g) &= M[(M^{-1}f) *_y (M^{-1}\chi g)] \\
&= M[(M^{-1}f) *_y (M^{-1}M \circ R_\alpha g)] \\
&= M[(M^{-1}f *_y (R_\alpha g))] \\
&= MR_\alpha[(W_\alpha M^{-1}f) * g] \\
&= \chi({}^t\chi f * g).
\end{aligned}$$

□

4. Generalized wavelets

Definition 4.1. *A generalized wavelet is a function g satisfying the admissibility condition*

$$0 < C_g = \sum_n \frac{|F_\Lambda(g)(\lambda)|^2}{(\lambda)} < \infty. \tag{4.1}$$

Remark 4.2. *By (2.9), (3.3) and (4.1), $g \in L^p_\Lambda$ is a generalized wavelet if and only if, $M^{-1}g$ is a Laguerre wavelet and*

$$C_g = C^\alpha_{M^{-1}g}. \tag{4.2}$$

Note 4.1 For $g \in L^p_\Lambda$ and $(a,b) \in (0,\infty) \times [0,\infty)$ let

$$g_{a,b}(t) = a^{-1/2} T^b g_a(t). \tag{4.3}$$

where $g_a(t) = g(at)$ is given by () and T^b is the generalized translation operator defined by ().

Proposition 4.3. *For all $(a,b) \in (0,\infty) \times [0,\infty)$ we have*

$$g_{a,b}(t) = e^{\frac{-|\lambda|(b^2+t^2)}{2}} (M^{-1}g)^\alpha_{b,a}(t). \tag{4.4}$$

Proof. Using (2.12), (3.4) and (4.3) we have

$$\begin{aligned} g_{a,b}(t) &= \frac{1}{a^{1/2}} T^b g_a(t) \\ &= \frac{1}{a^{1/2}} e^{\frac{-|\lambda|(b^2+t^2)}{2}} \tau^b_\alpha (M^{-1}g)(at) \\ &= \frac{e^{\frac{-|\lambda|(b^2+t^2)}{2}}}{a^{1/2}} \tau^b_\alpha (M^{-1}g)(at) \\ &= e^{\frac{-|\lambda|(b^2+t^2)}{2}} (M^{-1}g)^\alpha_{b,a}(t). \end{aligned}$$

□

Definition 4.4. *Let $g \in L^p_\alpha$ be a generalized wavelet. The generalized continuous wavelet transform is defined by*

$$\phi_g(f)(a,b) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x) \overline{g_{a,b}(x)} e^{-x} x^\alpha dx, \tag{4.5}$$

which can also be written as

$$\phi_g(f)(a,b) = \frac{1}{a^{-1/2}} f \sharp \overline{g_a}(b), \tag{4.6}$$

where $\sharp$ is the generalized convolution product given by (3.5).

Proposition 4.5. *We have*

$$\phi_g(f)(a,b) = e^{\frac{-|\lambda| b^2}{2}} S^\alpha_{M^{-1}g}(M^{-1}f)(a,b). \tag{4.7}$$

Proof.

$$\begin{aligned} \phi_g(f)(a,b) &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x) \overline{g_{a,b}(x)} e^{-x} x^\alpha dx \\ &= \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(x) e^{\frac{-|\lambda|(b^2+x^2)}{2}} (M^{-1}g)^\alpha_{b,a}(x) e^{-x} x^\alpha dx \\ &= \frac{e^{\frac{-|\lambda|(b^2)}{2}}}{\Gamma(\alpha+1)} \int_0^\infty f(x) e^{\frac{-|\lambda|(x^2)}{2}} (M^{-1}g)^\alpha_{b,a}(x) e^{-x} x^\alpha dx \\ &= \frac{e^{\frac{-|\lambda|(b^2)}{2}}}{\Gamma(\alpha+1)} \int_0^\infty (M^{-1}f)^\alpha_{b,a}(x) (M^{-1}g)^\alpha_{b,a}(x) e^{-x} x^\alpha dx \\ &= e^{\frac{-|\lambda| b^2}{2}} S^\alpha_{M^{-1}g}(M^{-1}f)(a,b). \end{aligned}$$

□

Theorem 4.6. ***Plancherel formula***

$$\int_0^\infty |f(x)|^2 d\Lambda(x) = \frac{1}{C_g}\int_0^\infty \int_0^\infty |\phi_g(f)(a,b)|^2 d\Lambda(b)\frac{d\Lambda(a)}{a}.$$

Proof. By (4.2), (4.5) and Theorem 2.1(i) we have

$$\begin{aligned}\int_0^\infty \int_0^\infty |\phi_g(f)(a,b)|^2 d\Lambda(b)\frac{d\Lambda(a)}{a} &= \int_0^\infty \int_0^\infty (e^{\frac{-|\lambda|b^2}{2}})^2 |S^\alpha_{M^{-1}g}(M^{-1}f)(a,b)|^2 e^{-(a+b)} a^\alpha b^\alpha db\frac{da}{a}\\ &= \int_0^\infty \int_0^\infty |S^\alpha_{M^{-1}g}(M^{-1}f)(a,b)|^2 e^{-(|\lambda)|b^2+a+b)} a^\alpha b^\alpha db\frac{da}{a}\\ &= C^\alpha_{M^{-1}g}\int_0^\infty |M^{-1}f(x)|^2 e^{-x}x^\alpha dx\\ &= C_g\int_0^\infty |f(x)|^2 d\Lambda(x).\end{aligned}$$

□

Theorem 4.7. ***Inversion formula***

$$f(x) = \frac{1}{C_g}\int_0^\infty \left(\phi_g(f)(a,b)g_{a,b}(x)e^{-(a+b)}a^\alpha b^\alpha db\right)\frac{da}{a}$$

Proof. By (4.2), (4.3) and (4.5) we have

$$\begin{aligned}&\frac{1}{C_g}\int_0^\infty \left(\phi_g(f)(a,b)g_{a,b}(x)e^{-(a+b)}a^\alpha b^\alpha db\right)\frac{da}{a}\\ &= \frac{1}{C^\alpha_{M^{-1}g}}\int_0^\infty \left(\int_0^\infty e^{\frac{-|\lambda|b^2}{2}} S^\alpha_{M^{-1}g}(M^{-1}f)(a,b)g_{a,b}(x)e^{-(a+b)}a^\alpha b^\alpha db\right)\frac{da}{a}\\ &= \frac{1}{C^\alpha_{M^{-1}g}}\int_0^\infty \left(\int_0^\infty (S^\alpha_{M^{-1}g}(M^{-1}f)(a,b))g_{a,b}(x)e^{-(|\lambda|b^2+a+b)}a^\alpha b^\alpha db\right)\frac{da}{a}\end{aligned}$$

The result follows now from the Theorem 2.1(iii). □

Acknowledgments

This work is supported by CSIR-HRDG grant No. 09/0725(15249)/2022-EMR-I.

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