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Existence and Multiplicity of Solutions for Schrödinger-Kirchhoff-Type Equations Involving the Fractional $p(x,\cdot)$-Laplacian Without the (AR) Condition

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ABSTRACT: The purpose of this paper is to investigate the existence and multiplicity of weak solutions for a Kirchhoff-type problems driven by the non-local integro-differential operator of elliptic type

$$\begin{cases} M\big(\sigma_{p(x,y)}(u)\big)\mathcal{L}_K^{p(x,\cdot)}(u) = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases}$$

where

$$\sigma_{p(x,y)}(u) = \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)} K(x,y)\,dx\,dy,$$

$\mathcal{L}_K^{p(x,\cdot)}$ is a non-local operator with singular kernel K, Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, M is a continuous function and f is a Carathéodory function. Under suitable assumptions on $f(x,u)$ without (AR) condition, the existence and multiplicity solutions for the problem is obtained by using the Mountain Pass Theorem and the Fountain Theorem.

Key Words: Fractional $p(x,\cdot)$-Laplacian, nonlocal and integro-differential operator, Mountain Pass Theorem, Cerami condition, Fountain Theorem.

Contents

1. Introduction

Recently, great attention has been paid to the study of problems involving fractional and non-local operators. This type of problems comes to real world with many different applications in a quite natural way, such as, population dynamics, phase transition phenomena, ultra-relativistic limits of quantum mechanics, material science, water waves, anomalous diffusion, minimal surface and game theory, as they are the typical outcome of stochastically stabilization of Lévey processes, see [9,17,36,39] and the

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references therein.
In this paper we deal with the following Kirchhoff-type problem

$$\begin{cases} M\big(\sigma_{p(x,y)}(u)\big)\mathcal{L}_K^{p(x,\cdot)}(u) = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where

$$\sigma_{p(x,y)}(u) = \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{p(x,y)} K(x,y)\,dx\,dy, \tag{1.2}$$

$\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary $\partial\Omega$, $\mathcal{Q} := \mathbb{R}^{2N}\backslash(\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N\backslash\Omega$, $p : \overline{\mathcal{Q}} \to (1,+\infty)$ is bounded continuous function, $N \geq 3$, $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and $\mathcal{L}_K^{p(x,\cdot)}$ is a non-local operator defined as follows

$$\begin{aligned} \mathcal{L}_K^{p(x,\cdot)}(u) &= p.v. \int_{\mathbb{R}^N} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)\,dy \\ &= \lim_{\epsilon\to 0} \int_{\mathbb{R}^N\backslash B_\epsilon(x)} |u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))K(x,y)\,dy, \end{aligned}$$

for all $x \in \mathbb{R}^N$, where $p.v.$ is a commonly used abbreviation in the principal value sense, $B_\epsilon(x) = \{y \in \mathbb{R}^N : |x-y| < \epsilon\}$ and the kernel $K : \mathbb{R}^N \times \mathbb{R}^N \to (0,+\infty)$ is a measurable function with the following property

$$\begin{cases} \gamma K \in L^1(\mathbb{R}^N \times \mathbb{R}^N) & \text{where } \gamma(x,y) = \min\{1, |x-y|^{p(x,y)}\}; \\ \text{there exists } k_0 > 0 \text{ such that} & \\ K(x,y) \geq k_0|x-y|^{-(N+sp(x,y))} & \text{with } s \in (0,1), \text{ for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \text{ and } x \neq y; \\ K(x,y) = K(y,x) & \text{for any } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \end{cases} \tag{1.3}$$

A typical example for K is given by the singular kernel $K(x,y) = |x-y|^{-(N+sp(x,y))}$. In this case, problem (1.1) becomes

$$\begin{cases} M\Big(\int_{\mathcal{Q}} \frac{1}{p(x,y)} \frac{|u(x)-u(y)|^{p(x,y)}}{|x-y|^{N+sp(x,y)}}\,dx\,dy\Big)\big(-\Delta_{p(x,\cdot)}\big)^s u(x) = f(x,u) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.4}$$

where $(-\Delta_{p(x,\cdot)})^s$ is the fractional $p(x,\cdot)$-Laplacian operator which (up to normalization factors) may be defined as

$$(-\Delta_{p(x,\cdot)})^s u(x) = p.v. \int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^{p(x,y)-2}(u(x)-u(y))}{|x-y|^{N+sp(x,y)}}\,dy, \quad \text{for all} \quad x \in \mathbb{R}^N,$$

see [14,26,34] and the references therein for further details on the fractional $p(x,\cdot)$-Laplacian operator. Note that (1.1) is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - M\Big(\int_\Omega |\nabla u|^2\,dx\Big)\Delta u = f(x,u), \tag{1.5}$$

which was proposed by Kirchhoff in 1883 as a generalization of the well-known D'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = f(x,u),$$

for free vibrations of elastic string, see [35]. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension. It is worth pointing out that problem (1.5) received much attention only after Lion [38] proposed and abstract framework to the problem. Equation (1.5) models several physical systems, where u describes a process which depends on the average of itself. Nonlocal effects also finds its applications in biological systems.

As is well-known problems involving $p(\cdot)$-Laplacian, defined as $(-\Delta)_{p(x)}u := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, $x \in \mathbb{R}^N$, where $p : \mathbb{R}^N \to [1,\infty)$ is continuous function, have been developed very markedly in last decade due to the fact that they have arisen in the mathematical modeling of various physical processes, as in nonlinear elasticity theory, electro-rheological fluids [46,47,48], thermo-rheological fluids [8] and image processing [1,21,37], etc. Elliptic equations involving variable exponents have attracted an increasing attention and many results, especially concerning the existence, multiplicity, uniqueness and regularity of solutions have been obtained by several authors. Some interesting results can be found, for example, in [2,3,4,5,7,20,22,24,25,29,30,31,32,33,40,41,44,45,50] and references therein.

In recent years, a great attention has been focused on the study of the fractional $p(x,\cdot)$-Laplacian on fractional Sobolev spaces. In [34], Kaufmann et al. and [26] first proposed and introduced the fractional Sobolev spaces with variable exponent $W^{s,q(x),p(x,y)}(\Omega)$ and proved compact embedding theorems of these spaces into variable exponent Lebesgue spaces. They also established the existence of solutions for problems involving the non-local fractional $p(x,\cdot)$-Laplacian. In [14], Bahrouni et al. proved some qualitative properties of the fractional Sobolev space $W^{s,q(x),p(x,y)}(\Omega)$ and obtained existence result for fractional problems.

In [10,13], applying the Mountain Pass Theorem and Ekeland's variational principle, Azroul et al. established the existence of solutions of (1.1) with $f(x,t) = \lambda|u(x)|^{r(x)-2}r(x)$ in fractional $p(x,\cdot)$-Laplacian case. In [11], the existence of infinite solutions for a class of fractional $p(x,\cdot)$-Kirchhoff-type problems in $\mathbb{R}^N$ was examined by the Fountain Theorem and the Symmetric Mountain Pass Theorem. We refer also to [6,23,42,43] for related problems. In [12] Azroul et al. extended the fractional Sobolev spaces with variable exponents $W^{s,p(x,y)}$ to include the general fractional case $W_K^{s,p(x,y)}$ and obtained some existence results by considering the well-known Ambrosetti-Rabinowitz condition (AR) and using the Mountain Pass Theorem and the Minty-Browder Theorem for a nonlocal $p(x,\cdot)$-Kirchhoff type problem. Motivated by the works above, we investigate problem (1.1) without the (AR) condition and prove some existence and multiplicity results.

For this, we suppose that the Kirchhoff function $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying the following conditions:

($\mathbf{H_0}$) there exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$;

($\mathbf{H_1}$) there exists $\mu \in (0,1)$ such that $\widehat{M}(t) \geq (1-\mu)\, M(t)\, t$ for all $t \geq 0$, where $\widehat{M}(t) = \int_0^t M(\tau)\, d\tau$;

($\mathbf{H_2}$) M is differentiable and decreasing function on $\mathbb{R}^+$.

Also, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

($\mathbf{f_0}$) there exist $c_1 > 0$ and $1 < q(x) < p_s^*(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)}$ with $s \in (0,1)$ and $\overline{p}(x) = p(x,x)$ such that $|f(x,t) \leq c_1(1 + |t|^{q(x)-1})$ for all $(x,t) \in \Omega \times \mathbb{R}$;

($\mathbf{f_1}$) $\lim_{|t|\to+\infty} \frac{F(x,t)}{|t|^{\frac{p^+}{1-\mu}}} = +\infty$, uniformly for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)\, ds$;

($\mathbf{f_2}$) there exists $\theta \geq 1$ such that $\theta G(x,t) \geq G(x,s't)$ for $(x,t) \in \Omega \times \mathbb{R}$ and $s' \in [0,1]$, where $G(x,t) = f(x,t)\,t - \frac{p^+}{1-\mu} F(x,t)$ and p^+ is defined in (2.2);

($\mathbf{f_3}$) $\lim_{t\to 0} \frac{F(x,t)}{|t|^{p^+}} = 0$, uniformly for a.e. $x \in \Omega$;

($\mathbf{f_4}$) $f(x,-t) = -f(x,t)$ for all $x \in \Omega$ and $t \in \mathbb{R}$;

($\mathbf{f_5}$) there exists $C_* > 0$ such that $G(x,t) \leq G(x,t) + C_*$ for each $x \in \Omega$, $0 < t < t'$ or $t' < t < 0$ where $G(x,t)$ is the function defined in ($\mathbf{f_2}$).

After this overview on the assumptions on the nonlinearity f, we would note that problem (1.1) is variational in nature and the energy functional associated with it is given by the functional $\mathcal{J}_K(u) : X_0 \to \mathbb{R}$ (X_0 is defined in section 2) defined as

$$\mathcal{J}_K(u) = \widehat{M}\big(\sigma_{p(x,y)}(u)\big) - \int_\Omega F(x,t)\,dx, \tag{1.6}$$

where $\sigma_{p(x,y)}(u)$, $\widehat{M}(t)$ and $F(x,t)$ are the functions defined in (1.2), $(\mathbf{H_1})$ and $(\mathbf{f_1})$, respectively.
Now, we can state our main results.

Theorem 1.1 *Let Ω be a Lipschitz bounded domain in $\mathbb{R}^N$ and $s \in (0,1)$, let $p : \overline{\mathcal{Q}} \to (1,+\infty)$ be a continuous function satisfying (2.1) and (2.2) with $sp^+ < N$. Assume that the assumptions $(\mathbf{H_0})$-$(\mathbf{H_2})$ hold and f verifying $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_3})$ and*

- **(a)** $(\mathbf{f_2})$ *or*
- **(b)** $(\mathbf{f_5})$
  if $q^- > p^+$, then problem (1.1) has a nontrivial weak solution.

Theorem 1.2 *Assume that $(\mathbf{H_0})$-$(\mathbf{H_2})$ and $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_3})$, $(\mathbf{f_4})$ are satisfied. Moreover, we assume*

- **(a)** $(\mathbf{f_2})$ *or*
- **(b)** $(\mathbf{f_5})$
  holds true, if $q^- > p^+$, then problem (1.1) has a sequence of weak solutions $\{\pm u_k\}$ such that $\mathcal{J}_k(\pm u_k) \to +\infty$ as $k \to +\infty$.

Throughout this paper, for simplicity, we use c and c_i to denote a generic non-negative or positive constant (the exact value may change from line to line).
The rest of this paper is organized as follow. In Section 2, the definitions and some notations of variable exponent Lebesgue spaces and fractional Sobolev space with variable exponent are given. In Section 3, we will discuss the compactness properties of the energy functional associated with the problem under consideration. In Section 4, the proof of Theorem 1.1 is investigated. Finally, Section 5 will be devoted to the proof of Theorem 1.2.

2. Variational framework and preliminaries results

In this section, we review some definitions and basic properties of the generalized Lebesgue space $L^{p(\cdot)}(\Omega)$ and generalized fractional Sobolev space with variable exponent. Also we give preliminary results which will be used in the sequel.

2.1. Variable exponent Lebesgue space

In this subsection, we recall some useful properties of variable exponent Lebesgue space. For more details we refer the reader to [27,28,31] and the references therein.
Set

$$C_+(\overline{\Omega}) = \{h(x);\ h(x) \in C(\overline{\Omega}),\ h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For all $h(x) \in C_+(\overline{\Omega})$, we define

$$h^+ = \max\{h(x);\ x \in \overline{\Omega}\}, \quad h^- = \min\{h(x);\ x \in \overline{\Omega}\},$$

such that

$$1 < h^- \le h(x) < h^+ < +\infty. \tag{2.1}$$

For any $q \in C_+(\overline{\Omega})$, we define the *variable exponent Lebesgue space* $L^{q(x)}(\Omega)$ by

$$L^{q(x)}(\Omega) = \Big\{u : \Omega \to \mathbb{R};\ u \text{ is a measurable such that } \int_\Omega |u(x)|^{q(x)}\,dx < \infty\Big\}.$$

The *Luxemburg norm* on this space is given by

$$\|u\|_{L^{q(x)}(\Omega)} = \inf\Big\{\mu > 0;\ \int_\Omega \Big|\frac{u(x)}{\mu}\Big|^{q(x)}\,dx \le 1\Big\}.$$

$(L^{q(x)}(\Omega), \|\cdot\|_{L^{q(x)}(\Omega)})$ is called a generalized Lebesgue space.

Proposition 2.1 (See [27]) **(i)** *The space $(L^{q(x)}(\Omega), \|\cdot\|_{L^{q(x)}(\Omega)})$ is a separable, uniformly convex Banach space and its dual space is $L^{q'(x)}(\Omega)$, where $\frac{1}{q(x)} + \frac{1}{q'(x)} = 1$. For any $u \in L^{q(x)}(\Omega)$ and $v \in L^{q'(x)}(\Omega)$, we have*

$$\Big|\int_\Omega uv\,dx\Big| \le \Big(\frac{1}{q^-} + \frac{1}{q'^-}\Big)\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{q'(x)}(\Omega)} \le 2\|u\|_{L^{q(x)}(\Omega)}\|v\|_{L^{q'(x)}(\Omega)}.$$

(ii)*If $q_1(x),\ q_2(x) \in C_+(\overline{\Omega})$ such that $q_1(x) \le q_2(x)$, for all $x \in \overline{\Omega}$, then we have a continuous embedding $L^{q_2(x)}(\Omega) \hookrightarrow L^{q_1(x)}(\Omega)$.*

An important role in manipulating the generalized Lebesgue space is played by the $q(x)$-modular of the $L^{q(x)}(\Omega)$ space, which is the mapping

$$\rho_{q(x)} : L^{q(x)}(\Omega) \to \mathbb{R}$$

defined by

$$\rho_{q(x)}(u) = \int_\Omega |u|^{q(x)}\,dx.$$

Proposition 2.2 (See [31]) *If $\{u_n\} \subset L^{q(x)}$, $u \in L^{q(x)}(\Omega)$ and $q^+ < \infty$, then the following relations hold true:*

(1) $\|u\|_{L^{q(x)}(\Omega)} < 1$ *(respectively$= 1; > 1$)* $\Longleftrightarrow \rho_{q(x)}(u) < 1$ *(respectively$= 1; > 1$);*

(2) *For $u \neq 0$, $\|u\|_{L^{q(x)}(\Omega)} = \lambda \Longleftrightarrow \rho_{q(x)}(\frac{u}{\lambda}) = 1$;*

(3) *If $\|u\|_{L^{q(x)}(\Omega)} > 1$, then $\|u\|^{q^-}_{L^{q(x)}(\Omega)} \le \rho_{q(x)}(u) \le \|u\|^{q^+}_{L^{q(x)}(\Omega)}$;*

(4) *If $\|u\|_{L^{q(x)}(\Omega)} < 1$, then $\|u\|^{q^+}_{L^{q(x)}(\Omega)} \le \rho_{q(x)}(u) \le \|u\|^{q^-}_{L^{q(x)}(\Omega)}$;*

(5) $\|u_n - u\|_{L^{q(x)}(\Omega)} \to 0$ *(respectively $\to \infty$)* $\Longleftrightarrow \rho_{q(x)}(u_n - u) \to 0$ *(respectively $\to \infty$),*

Proposition 2.3 (See [28]) *Let p and q be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x)q(x) \le \infty$, for a.e. $x \in \mathbb{R}^N$. Let $u \in L^{q(\cdot)}(\mathbb{R}^N)$, $u \neq 0$. Then*

(1) *If $\|u\|_{L^{p(x)q(x)}(\Omega)} \le 1$, then $\|u\|^{p^+}_{L^{p(x)q(x)}(\Omega)} \le \||u|^{p(x)}\|_{L^{q(x)}(\Omega)} \le \|u\|^{p^-}_{L^{p(x)q(x)}(\Omega)}$;*

(2) *If $\|u\|_{L^{p(x)q(x)}(\Omega)} \ge 1$, then $\|u\|^{p^-}_{L^{p(x)q(x)}(\Omega)} \le \||u|^{p(x)}\|_{L^{q(x)}(\Omega)} \le \|u\|^{p^+}_{L^{p(x)q(x)}(\Omega)}$, In particular, if $p(x) = p$ is constant, then $\||u|^p\|_{L^{q(x)}(\Omega)} = \|u\|^p_{L^{pq(x)}(\Omega)}$.*

2.2. General fractional Sobolev space with variable exponent

In this subsection, we recall the definition and some results on general fractional Sobolev spaces with variable exponent, see [14,34] and the references therein.
Let Ω be a Lipschitz open bounded set in $\mathbb{R}^N$ and let $p : \overline{\Omega} \times \overline{\Omega} \to (1, +\infty)$ be a continuous bounded function. We assume that

$$1 < p^- = \min_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) \le p(x,y) \le p^+ = \max_{(x,y)\in\overline{\Omega}\times\overline{\Omega}} p(x,y) < +\infty, \tag{2.2}$$

and

$$p \quad \text{is symmetric, that is} \quad p(x,y) = p(y,x) \ \text{ for all } \ (x,y) \in \overline{\Omega}\times\overline{\Omega}. \tag{2.3}$$

We set

$$\overline{p}(x) = p(x,x) \quad \text{for any} \quad x \in \overline{\Omega}.$$

For $s \in (0,1)$, the fractional Sobolev space with variable exponent via the *Gagliardo approach* as follows

$$X := \Big\{ u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable, such that } u|_\Omega \in L^{\overline{p}(x)}(\Omega) \text{ with } \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)\,dx\,dy < +\infty, \text{ for some } \lambda > 0 \Big\},$$

where $\mathcal{Q} := \mathbb{R}^{2N} \backslash (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ with $\mathcal{C}\Omega = \mathbb{R}^N \backslash \Omega$ and $\overline{p} : \overline{\mathcal{Q}} \to (1,+\infty)$ satisfies (2.2) and (2.3) on $\overline{\mathcal{Q}}$. Moreover, X is endowed with the norm

$$\|u\|_X = \|u\|_{L^{\overline{p}(x)}(\Omega)} + [u]_{K,p(x,y)},$$

where

$$[u]_{K,p(x,y)} = \inf \Big\{ \lambda > 0; \int_{\mathcal{Q}} \frac{|u(x)-u(y)|^{p(x,y)}}{\lambda^{p(x,y)}} K(x,y)\,dx\,dy \le 1 \Big\},$$

then we have $(X, \|\cdot\|_X)$ is a separable reflexive Banach space, see [12].
For any $u \in X$, we define the functional

$$\rho_{K,p(\cdot,\cdot)}(u) = \int_{\mathcal{Q}} |u(x)-u(y)|^{p(x,y)} K(x,y)\,dx\,dy + \int_\Omega |u(x)|^{\overline{p}(x)}\,dx.$$

Then $\rho_{K,p(\cdot,\cdot)}$ is a *convex modular* on X and the norm associated with $\rho_{K,p(\cdot,\cdot)}$ is given by

$$\|u\|_{\rho_{K,p(\cdot,\cdot)}} = \inf\{\lambda > 0;\ \rho_{K,p(\cdot,\cdot)}(\frac{u}{\lambda}) \le 1\},$$

which is equivalent to the norm $\|\cdot\|_X$.
We shall work in the closed linear subspace

$$X_0 = \{u \in X;\ u(x) = 0 \text{ a.e. in } \mathbb{R}^N \backslash \Omega\}.$$

For any $u \in X_0$, we define the functional

$$\rho^0_{K,p(\cdot,\cdot)}(u) = \int_{\mathcal{Q}} |u(x)-u(y)|^{p(x,y)} K(x,y)\,dx\,dy,$$

then the norm associated with the *convex modular* $\rho^0_{K,p(\cdot,\cdot)}$ is given by

$$\|u\|_{X_0} = [u]_{K,p(x,y)} = \inf \Big\{ \lambda > 0;\ \rho^0_{K,p(\cdot,\cdot)}(\frac{u}{\lambda}) \le 1 \Big\}.$$

We know $(X_0, \|\cdot\|_{X_0})$ is a separable, reflexive and uniformly Banach space, see [12].

The modular $\rho^0_{K,p(\cdot,\cdot)}$ checks the following result, which is similar to Proposition 2.2.

Proposition 2.4 *[12] Let $p : \overline{\mathcal{Q}} \to (1,+\infty)$ be a continuous variable exponent. For any $u \in X_0, \{u_n\} \subset X_0$, we have*

(1) $\|u\|_{X_0} < 1$ *(respectively$= 1; > 1$)* $\Longleftrightarrow \rho^0_{K,p(\cdot,\cdot)}(u) < 1$ *(respectively$= 1; > 1$);*

(2) *For $u \neq 0$,* $\|u\|_{X_0} = \lambda \Longleftrightarrow \rho^0_{K,p(\cdot,\cdot)}(\frac{u}{\lambda}) = 1$*;*

(3) *If $\|u\|_{X_0} > 1$, then $\|u\|^{p^-}_{X_0} \le \rho^0_{K,p(\cdot,\cdot)}(u) \le \|u\|^{p^+}_{X_0}$;*

(4) *If $\|u\|_{X_0} < 1$, then $\|u\|^{p^+}_{X_0} \le \rho^0_{K,p(\cdot,\cdot)}(u) \le \|u\|^{p^-}_{X_0}$;*

(5) $\|u_n - u\|_{X_0} \to +\infty \Longleftrightarrow \rho^0_{K,p(\cdot,\cdot)}(u_n - u) \to +\infty$,

(6) $\|u_n - u\|_{X_0} \to 0 \Longleftrightarrow \rho^0_{K,p(\cdot,\cdot)}(u_n - u) \to 0 \Longleftrightarrow \{u_n\}$ *converges to* u *in measure and* $\rho^0_{K,p(\cdot,\cdot)}(u_n) \to \rho^0_{K,p(\cdot,\cdot)}(u)$.

Proposition 2.5 *Let* Ω *be a Lipschitz bounded domain in* $\mathbb{R}^N$ *and* $s \in (0,1)$. *Let* $p : \overline{\Omega} \times \overline{\Omega} \to (1,+\infty)$ *be continuous function satisfies* (2.2), (2.3) *with* $sp^+ < N$. *Let* $r : \overline{\Omega} \to (1,+\infty)$ *be a continuous variable exponent such that*

$$1 < r^- = \min_{x\in\overline{\Omega}} r(x) \le r(x) < p^*_s(x) = \frac{N\overline{p}(x)}{N - s\overline{p}(x)} \quad \text{for all} \quad x \in \overline{\Omega}.$$

Then, there exists a constant $c_2 = c(N,s,p,r,\Omega) > 0$ *such that, for any* $u \in X$

$$\|u\|_{L^{r(x)}(\Omega)} \le c_2 \|u\|_X.$$

That is, the space X *is continuously embedded in* $L^{r(x)}(\Omega)$. *Moreover, this embedding is compact. In addition, when one consider function* $u \in X_0$, *it holds that*

$$\|u\|_{L^{r(x)}(\Omega)} \le c_3 \|u\|_{X_0}.$$

Proposition 2.6 *[12] For all* $u, \varphi \in X_0$, *we consider the following functional* $\mathcal{L}^{p(x,\cdot)}_K : X_0 \to X_0^*$ *such that*

$$\langle \mathcal{L}^{p(x,\cdot)}_K(u), \varphi\rangle = \int_{\mathcal{Q}} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x,y)\,dx\,dy,$$

where X_0^* *is the dual space of* X_0 *and* $\langle\cdot,\cdot\rangle$ *denotes the usual duality between* X_0 *and* X_0^*. *Then the following assertions hold:*

(1) $\mathcal{L}^{p(x,\cdot)}_K$ *is a bounded and strictly monotone operator.*

(2) $\mathcal{L}^{p(x,\cdot)}_K$ *is a mapping of type* (S_+), *that is, if* $u_n \rightharpoonup u$ *in* X_0 *and* $\limsup_{n\to+\infty} \mathcal{L}^{p(x,\cdot)}_K(u_n)(u_n - u) \le 0$, *then* $u_n \to u$ *in* X_0.

(3) $\mathcal{L}^{p(x,\cdot)}_K$ *is a homeomorphism.*

2.3. Weak solution and energy functional of the problem

Definition 2.1 *We say that* $u \in X_0$ *is a weak solution of problem* (1.1) *if*

$$M\left(\sigma_{p(x,y)}(u)\right) \int_{\mathcal{Q}} |u(x) - u(y)|^{p(x,y)-2}(u(x) - u(y))(\varphi(x) - \varphi(y))K(x,y)\,dx\,dy - \int_{\Omega} f(x,u)\varphi(x)\,dx = 0,$$

for all $\varphi \in X_0$, *where* $\sigma_{p(x,y)}(u)$ *is defined in* (1.2).
By the assumptions on f *and also due to the embedding properties of* X_0 *into the classical Lebesgue spaces we have* $\mathcal{J}_K \in C^1(X_0)$.

2.4. Preliminary theorems

In this subsection, we give some preliminary theorems which will be used in the proof of our main results.

Definition 2.2 (See [18]) *Let* X_0 *be a Banach space and* $\mathcal{J}_K \in C^1(X_0, \mathbb{R})$. *given* $c \in \mathbb{R}$, *we say that* $\mathcal{J}_K$ *satisfies the Cerami condition (we denote condition* (C_c)*), if*

(*i*) *any bounded sequence* $\{u_n\} \subset X_0$ *such that* $\mathcal{J}_K(u_n) \to c$ *and* $\mathcal{J}'_K(u_n) \to 0$ *has a convergent subsequence;*

(*ii*) *there exist constants* $\delta,\ R,\ \beta>0$ *such that*

$$\|\mathcal{J}_K'(u)\|_{X_0^*}\|u\|_{X_0}\geq\beta \quad \textit{for all} \quad u\in\mathcal{J}_K^{-1}([c-\delta,c+\delta])) \quad \textit{with} \quad \|u\|_{X_0}\geq R.$$

If $\mathcal{J}_K\in C^1(X_0,\mathbb{R})$ *satisfies condition* (C_c) *for any* $c\in\mathbb{R}$, *we say that* $\mathcal{J}_K$ *satisfies condition* (C).

Note that the Cerami condition was introduced by Cerami in [18,19] as a weak version of the Palais-Smale condition. We would remark that if it was shown in [15] that from condition (C_c) it is possible to obtain a deformation lemma, which is fundamental in order to get some min-max theorems. Hence, the Mountain Pass Theorem and the Fountain Theorem holds true also under this compactness assumption.

Proposition 2.7 (Mountain Pass Theorem, see [15]) *Let* X_0 *be a Banach space,* $\mathcal{J}_K\in C^1(X_0,\mathbb{R})$, $e\in X_0$ *and* $r>0$ *be such that* $\|e\|_{X_0}>r$ *and*

$$b:=\inf_{\|u\|_{X_0}}\mathcal{J}_K(u)>\mathcal{J}_K(0)\geq\mathcal{J}_K(e).$$

If $\mathcal{J}_K$ *satisfies the condition* (C_c) *with*

$$c:=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{J}_K(\gamma(t)),$$
$$\Gamma:=\{\gamma\in C([0,1],X_0);\gamma(0)=0,\gamma(1)=e\},$$

then c *is a critical value of* $\mathcal{J}_K$.

Remark 2.1 *Since* X_0 *is a reflexive and separable Banach space, then* X_0^* *is too. Then, there exist (see [51])* $\{e_j\}_{j\in\mathbb{N}}\subset X_0$ *and* $\{e_j^*\}_{j\in\mathbb{N}}\subset X_0^*$ *such that*

$$X_0=\overline{\text{span}\ \{e_j\ :j=1,2,...\}},\qquad X_0^*=\overline{\text{span}\ \{e_j^*\ :j=1,2,...\}},$$

and

$$\langle e_i,e_j^*\rangle=\begin{cases}1 & \text{if} \quad i=j,\\ 0 & \text{if} \quad i\neq j,\end{cases}$$

where $\langle\cdot,\cdot\rangle$ *denote the duality product between* X_0 *and* X_0^*. *We define*

$$X_k=\ \text{span}\ \{e_k\},\qquad Y_k=\bigoplus_{j=1}^{k}X_j,\qquad Z_k=\overline{\bigoplus_{j=k}^{\infty}X_j}.$$

In order to prove Theorem 1.2, we will use the following Fountain Theorem.

Proposition 2.8 (Fountain Theorem, see [49]) *Let* $(X_0,\|\cdot\|_{X_0})$ *be a real reflexive Banach space,* $\mathcal{J}_K\in C^1(X_0,\mathbb{R})$ *is an even functional satisfying the Cerami condition. Moreover, for each* $k=1,2,\cdots$, *there exist* $\rho_k>r_k>0$ *such that*

$(\mathbf{A_1})$ $a_k:=\inf_{\{u\in Z_k,\ \|u\|=r_k\}}\mathcal{J}_K(u)\to+\infty$ *as* $k\to+\infty$,

$(\mathbf{A_2})$ $b_k:=\max_{\{u\in Y_k,\ \|u\|=\rho_k\}}\mathcal{J}_K(u)\leq 0$,

then the functional $\mathcal{J}_K$ *has a sequence of critical values which tends to* $+\infty$.

3. Verification of the compactness condition

In this section we show that the functional $\mathcal{J}_K$ verifies the Cerami condition.

3.1. Nonlinearities satisfying the super-linear condition $(\mathbf{f_2})$

Proposition 3.1 *Under assumptions* $(\mathbf{H_0})-(\mathbf{H_2})$ *and* $(\mathbf{f_0})-(\mathbf{f_2})$, $\mathcal{J}_K$ *satisfies the Cerami condition.*

Proof: First, we show that for all $c\in\mathbb{R}$, $\mathcal{J}_K$ satisfies $(\mathbf{i})$ of Cerami condition. Let $\{u_n\}$ be a bounded sequence such that $\mathcal{J}_K(u_n)\to c$ and $\mathcal{J}_K'(u_n)\to 0$. Since X_0 is a reflexive space, up to a subsequence, still denote by $\{u_n\}$, there exists $u\in X_0$ such that $\langle \mathcal{J}_K'(u_n),u_n-u\rangle\to 0$ as $n\to+\infty$. Thus, we have

$$\begin{aligned}\langle \mathcal{J}_K'(u_n),u_n-u\rangle = {} & M\Big(\sigma_{p(x,y)}(u_n)\Big)\langle \mathcal{L}_K^{p(x,\cdot)}(u_n),u_n-u\rangle-\int_\Omega f(x,u_n)(u_n-u)\,dx\\ & \to 0,\end{aligned}\tag{3.1}$$

as $n\to+\infty$. From $(\mathbf{f_0})$, Proposition 2.1$(\mathbf{i})$ and Proposition 2.3, it follows that

$$\begin{aligned}\Big|\int_\Omega f(x,u_n)(u_n-u)\,dx\Big| \le{} & c_1\||u_n|^{q(x)-1}\|_{L^{\frac{q(x)}{q(x)-1}}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}+c\int_\Omega |u_n-u|\,dx\\ \le{} & c_4\|u_n\|^{q^+-1}_{L^{q(x)}(\Omega)}\|u_n-u\|_{L^{q(x)}(\Omega)}+c_1\int_\Omega |u_n-u|\,dx,\end{aligned}$$

since $q(x)<p_s^*(x)$ for all $x\in\overline{\Omega}$, by Proposition 2.5, X_0 is compactly embedded in $L^{q(x)}(\Omega)$, so we have

$$\int_\Omega f(x,u_n)(u_n-u)\,dx\to 0,\tag{3.2}$$

as $n\to+\infty$. Hence, by (3.1) and (3.2), we get

$$M\Big(\sigma_{p(x,y)}(u_n)\Big)\langle \mathcal{L}_K^{p(x,\cdot)}(u_n),u_n-u\rangle\to 0,\tag{3.3}$$

as $n\to+\infty$. Now, since $\{u_n\}$ is bounded in X_0, we may assume that

$$\sigma_{p(x,y)}(u_n)\to t_1\ge 0,$$

as $n\to+\infty$. If, $t_1=0$, then $\{u_n\}$ converges strongly to $u=0$ in X_0 and the proof is finished. If $t_1>0$, since the function M is continuous, we have

$$M\Big(\sigma_{p(x,y)}(u_n)\Big)\to M(t_1)\ge 0,$$

as $n\to+\infty$. Hence, by $(\mathbf{H_0})$, for n large enough, we have that

$$0<c_5\le M\Big(\sigma_{p(x,y)}(u_n)\Big)\le c_6.\tag{3.4}$$

Combining (3.3) and (3.4), we deduce that

$$\langle \mathcal{L}_K^{p(x,\cdot)}(u_n),u_n-u\rangle\to 0,\tag{3.5}$$

as $n\to+\infty$. Furthermore, since $u_n\rightharpoonup u$ in X_0, we have

$$\langle \mathcal{J}_K'(u),u_n-u\rangle\to 0,$$

as $n\to+\infty$, that is

$$M\Big(\sigma_{p(x,y)}(u)\Big)\langle \mathcal{L}_K^{p(x,\cdot)}(u),u_n-u\rangle-\int_\Omega f(x,u_n)(u_n-u)\,dx\to 0,$$

as $n\to+\infty$, which implies by using the same arguments as before that

$$\langle \mathcal{L}_K^{p(x,\cdot)}(u),u_n-u\rangle\to 0,\tag{3.6}$$

as $n \to +\infty$. Combining (3.5) and (3.6), we deduce

$$\limsup_{n\to+\infty} \langle \mathcal{L}_K^{p(x,\cdot)}(u_n) - \mathcal{L}_K^{p(x,\cdot)}(u), u_n - u\rangle \leq 0.$$

By proposition 2.6 (**2**), $\mathcal{L}_K^{p(x,\cdot)}$ is a mapping of type (S_+), thus $u_n \to u$ in X_0.
Now, we check that $\mathcal{J}_K$ satisfies the assertion (**ii**) of the Cerami condition. We argue by contradiction, there exists $c_7 \in \mathbb{R}$ and $\{u_n\} \subset X_0$ satisfying

$$\mathcal{J}_K(u_n) \to c_7, \quad \|u_n\|_{X_0} \to +\infty, \quad \|\mathcal{J}_K'(u_n)\|_{X_0^*}\|u_n\|_{X_0} \to 0, \tag{3.7}$$

as $n \to +\infty$. From (3.7), we know that $\mathcal{J}_K(u_n) - \dfrac{1-\mu}{p^+}\langle \mathcal{J}_K'(u_n), u_n\rangle \to c_7$, when $n \to +\infty$.
Denote $w_n = \dfrac{u_n}{\|u_n\|_{X_0}}$, then $\|w_n\|_{X_0} = 1$, so $\{w_n\}$ is bounded. Up to a subsequence for some $w \in X_0$, we get

$$\begin{cases} w_n \rightharpoonup w, \text{ in } X_0;\\ w_n \to w, \text{ in } L^{q(x)}(\Omega);\\ w_n(x) \to w(x), \text{ a.e. in } \Omega. \end{cases}$$

If $w \equiv 0$, we can define a sequence $\{t_n\} \subset \mathbb{R}$ such that

$$\mathcal{J}_K(t_n u_n) = \max_{t\in[0,1]} \mathcal{J}_K(t u_n).$$

For any $B > \dfrac{1}{2p^+}$, let $b_n = (2Bp^+)^{\frac{1}{p^-}} w_n$, since $b_n \to 0$ in $L^{q(x)}(\Omega)$ and $|F(x,t)| \leq c_8(1 + |t|^{q(x)})$, by the continuity of the Nemytskii operator, we set that $F(\cdot, b_n) \to 0$ in $L^1(\Omega)$ as $n \to +\infty$; therefore

$$\lim_{n\to+\infty} \int_\Omega F(x, b_n)\, dx = 0. \tag{3.8}$$

Then, for n large enough $\frac{(2Bp^+)^{\frac{1}{p^-}}}{\|u_n\|_{X_0}} \in (0,1)$ and using $(\mathbf{H_0})$ we can write

$$\begin{aligned}
\mathcal{J}_K(t_n u_n) &\geq \mathcal{J}_K(b_n)\\
&= \widehat{M}\Big(\sigma_{p(x,y)}(b_n)\Big) - \int_\Omega F(x,b_n)\, dx\\
&\geq m_0\Big(\sigma_{p(x,y)}(b_n)\Big) - \int_\Omega F(x,b_n)\, dx\\
&\geq \frac{m_0}{p^+}\Big(\rho^0_{K,p(\cdot,\cdot)}(b_n)\Big) - \int_\Omega F(x,b_n)\, dx\\
&\geq 2\,B\,m_0\,\rho^0_{K,p(\cdot,\cdot)}(w_n) - \int_\Omega F(x,b_n)\, dx\\
&\geq 2\,c_9\,B\,m_0\,\|w_n\|_{X_0}^{p^-} - \int_\Omega F(x,b_n)\, dx\\
&\geq 2\,c_9\,B\,m_0 - \int_\Omega F(x,b_n)\, dx.
\end{aligned}$$

That is,

$$\mathcal{J}_K(t_n u_n) \to +\infty, \tag{3.9}$$

as $n \to +\infty$. From $\mathcal{J}_K(0) = 0$ and $\mathcal{J}_K(u_n) \to c$, we know that $t_n \in (0,1)$ and

$$\langle \mathcal{J}_K'(t_n u_n), t_n u_n\rangle = t_n \frac{d}{dt}|_{t=t_n} \mathcal{J}_K(t\,u_n) = 0. \tag{3.10}$$

We claim that

$$\limsup_{n\to+\infty} \mathcal{J}_K(t_n u_n) \leq c_{10}, \tag{3.11}$$

for a suitable positive constant c_{10}. Indeed, from $(\mathbf{H_2})$, $(\mathbf{f_2})$ and using (3.10), we obtain

$$\begin{aligned}
\frac{1}{\theta}\mathcal{J}_K(t_n u_n) &= \frac{1}{\theta}\Big[\mathcal{J}_K(t_n u_n) - \frac{1-\mu}{p^+}\langle \mathcal{J}_K'(t_n u_n), t_n u_n\rangle\Big] \\
&\leq \widehat{M}\Big(\sigma_{p(x,y)}(t_n u_n)\Big) - \frac{1-\mu}{p^+} M\Big(\sigma_{p(x,y)}(t_n u_n)\Big)\rho^0_{K,p(\cdot,\cdot)}(t_n u_n) \\
&+ \frac{1-\mu}{p^+}\int_\Omega \frac{G(x,t_n u_n)}{\theta}\,dx \\
&\leq \widehat{M}\Big(\sigma_{p(x,y)}(u_n)\Big) - \frac{1-\mu}{p^+} M\Big(\sigma_{p(x,y)}(u_n)\Big)\rho^0_{K,p(\cdot,\cdot)}(u_n) \\
&+ \frac{1-\mu}{p^+}\int_\Omega \Big(f(x,u_n)\,u_n - \frac{p^+}{1-\mu}F(x,u_n)\Big)\,dx \\
&= \mathcal{J}_K(u_n) - \frac{1-\mu}{p^+}\langle \mathcal{J}_K'(u_n), u_n\rangle \to c,
\end{aligned}$$

as $n \to +\infty$. This proves (3.11), which contradicts (3.9). Thus, the sequence $\{u_n\}$ has to be bounded in X_0.

Now, suppose that $w \neq 0$. Then, the set $\Omega' := \{x \in \Omega;\ w(x) \neq 0\}$ has positive Lebesgue measure and for a.e. $x \in \Omega'$ we have $|u_n(x)| \to +\infty$ as $n \to +\infty$. Hence, by $(\mathbf{f_1})$ we deduce

$$\frac{F(x,u_n)}{|u_n(x)|^{\frac{p^+}{1-\mu}}}|w_n(x)|^{\frac{p^+}{1-\mu}} \to +\infty, \tag{3.12}$$

as $n \to +\infty$. From $(\mathbf{H_1})$, we can easily obtain that

$$\widehat{M}(t) \leq \frac{\widehat{M}(t_0)}{t_0^{\frac{1}{1-\mu}}} t^{\frac{1}{1-\mu}} \leq c_{11} t^{\frac{1}{1-\mu}}, \tag{3.13}$$

where t_0 is an arbitrary positive constant. Since $\mathcal{J}_K(u_n) \to c$, using (3.13) we have

$$\begin{aligned}
c + o(1) = \mathcal{J}_K(u_n) &= \widehat{M}\Big(\sigma_{p(x,y)}(u_n)\Big) - \int_\Omega F(x,u_n)\,dx \\
&\leq c_{11}\Big(\sigma_{p(x,y)}(u_n)\Big)^{\frac{1}{1-\mu}} - \int_\Omega F(x,u_n)\,dx \\
&\leq \frac{c_{11}}{(p^-)^{\frac{1}{1-\mu}}}\Big(\rho^0_{K,p(\cdot,\cdot)}(u_n)\Big)^{\frac{1}{1-\mu}} - \int_\Omega F(x,u_n)\,dx \\
&\leq \frac{c_{12}}{(p^-)^{\frac{1}{1-\mu}}}\|u_n\|_{X_0}^{\frac{p^+}{1-\mu}} - \int_\Omega F(x,u_n)\,dx,
\end{aligned}$$

thus, considering (3.12) and via the Fatou's Lemma, we deduce

$$\begin{aligned}
\frac{c_{12}}{(p^-)^{\frac{1}{1-\mu}}} - \frac{c+o(1)}{\|u_n\|_{X_0}^{\frac{p^+}{1-\mu}}} &\geq \int_\Omega \frac{F(x,u_n)}{\|u_n\|_{X_0}^{\frac{p^+}{1-\mu}}}\,dx \\
&= \int_{w\neq 0} \frac{F(x,u_n)}{|u_n(x)|^{\frac{p^+}{1-\mu}}}|w_n(x)|^{\frac{p^+}{1-\mu}}\,dx + \int_{w=0} \frac{F(x,u_n)}{|u_n(x)|^{\frac{p^+}{1-\mu}}}|w_n(x)|^{\frac{p^+}{1-\mu}}\,dx \\
&\geq \int_{w\neq 0} \frac{F(x,u_n)}{|u_n(x)|^{\frac{p^+}{1-\mu}}}|w_n(x)|^{\frac{p^+}{1-\mu}}\,dx \to +\infty,
\end{aligned}$$

which is impossible. Thus, the sequence $\{u_n\}$ is bounded in X_0. □

We would remark that along the proof of Proposition 3.1, the assumption $(\mathbf{f_2})$ was used (and was crucial) just for proving the inequality (3.11).

3.2. Nonlinearities verifying the superlinear condition $(\mathbf{f_5})$

Proposition 3.2 *Under assumptions* $(\mathbf{H_0})$-$(\mathbf{H_2})$, $(\mathbf{f_0})$, $(\mathbf{f_1})$ *and* $(\mathbf{f_5})$, $\mathcal{J}_K$ *satisfies the Cerami condition.*

Proof: We can argue exactly as in the proof of Proposition 3.1. Thus, we just have to modify the proof of inequality (3.11).
Here, we will show the validity of (3.11) making use of the assumptions $(\mathbf{H_2})$, $(\mathbf{f_5})$ and (3.10). Then, we can write

$$\begin{aligned}
\mathcal{J}_K(t_nu_n) &= \mathcal{J}_K(t_nu_n) - \frac{1-\mu}{p^+}\langle \mathcal{J}'_K(t_nu_n), t_nu_n\rangle \\
&= \widehat{M}\Big(\sigma_{p(x,y)}(t_nu_n)\Big) - \frac{1-\mu}{p^+}M\Big(\sigma_{p(x,y)}(t_nu_n)\Big)\rho^0_{K,p(\cdot,\cdot)}(t_nu_n) \\
&+ \frac{1-\mu}{p^+}\int_\Omega \Big(f(x,t_nu_n)(t_nu_n) - \frac{p^+}{1-\mu}F(x,t_nu_n)\Big)\,dx \\
&\le \widehat{M}\Big(\sigma_{p(x,y)}(u_n)\Big) - \frac{1-\mu}{p^+}M\Big(\sigma_{p(x,y)}(u_n)\Big)\rho^0_{K,p(\cdot,\cdot)}(u_n) \\
&+ \frac{1-\mu}{p^+}\int_\Omega \Big(G(x,u_n) + C_*\Big)\,dx \\
&= \mathcal{J}_K(u_n) - \frac{1-\mu}{p^+}\langle \mathcal{J}'_K(u_n), u_n\rangle + \frac{(1-\mu)C_*}{p^+}\,|\Omega| \\
&\to c + \frac{(1-\mu)C_*}{p^+}\,|\Omega|,
\end{aligned}$$

as $n \to +\infty$. This proves (3.11). Thus, the proof of Proposition 3.2 is completed. □

4. The proof of Theorem 1.1

In this section we give the proofs of the existence of non-trivial solution for problem (1.1). In both following cases, the strategy consists in applying the Mountain Pass Theorem.

4.1. Proof of Theorem 1.1 under assumption (a)

In order to perform the proof of Theorem 1.1 when condition $(\mathbf{a})$ is assumed, we need to prove the following Lemma.

Lemma 4.1 *Assume that the conditions* $(\mathbf{H_0})$-$(\mathbf{H_2})$ *and* $(\mathbf{f_0})$, $(\mathbf{f_1})$, $(\mathbf{f_3})$ *are satisfied. Then we have the following assertions:*

- **(i)** *There exists* $\phi \in X_0$, $\phi > 0$ *such that* $\mathcal{J}_K(t\phi) \to -\infty$ *as* $t \to +\infty$.
- **(ii)** *There exist* $\rho > 0$ *and* $R > 0$ *such that* $\mathcal{J}_K(u) \ge R$ *for any* $u \in X_0$ *with* $\|u\|_{X_0} = \rho$.

Proof: (**i**) From $(\mathbf{f_1})$, it follows that for any $M > 0$ there exists a constant $c_{13} = c(M)$ depending on M, such that

$$|F(x,t)| \ge M|t|^{\frac{p^+}{1-\mu}} - c_{13}, \tag{4.1}$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. Take $\phi \in X_0$ with $\phi > 0$, from (4.1) and (3.13) we get

$$\begin{aligned}
\mathcal{J}_k(t\phi) &= \widehat{M}\Big(\sigma_{p(x,y)}(t\phi)\Big) - \int_\Omega F(x,t\phi)\,dx \\
&\leq c_{11}\Big(\sigma_{p(x,y)}(t\phi)\Big)^{\frac{1}{1-\mu}} - \int_\Omega F(x,t\phi)\,dx \\
&\leq t^{\frac{p^+}{1-\mu}}\Big[c_{11}\big(\frac{1}{p^-}\rho^0_{K,p(\cdot,\cdot)}(\phi)\big)^{\frac{1}{1-\mu}} - M\int_\Omega |\phi|^{\frac{p^+}{1-\mu}}\,dx\Big] + c_{14}|\Omega|,
\end{aligned}$$

where $t > 1$ and $|\Omega|$ denotes the Lebesgue measure of Ω. If M is large enough such that

$$c_{11}\big(\frac{1}{p^-}\rho^0_{K,p(\cdot,\cdot)}(\phi)\big)^{\frac{1}{1-\mu}} - M\int_\Omega |\phi|^{\frac{p^+}{1-\mu}}\,dx < 0,$$

then we have

$$\lim_{t\to+\infty} \mathcal{J}_K(t\phi) = -\infty,$$

which ends the proof of (**i**).
(**i**) Since the embeddings $X_0 \hookrightarrow L^{p^+}(\Omega)$ and $X_0 \hookrightarrow L^{q(x)}(\Omega)$ are continuous, there exist $c_{15}, c_{16} > 0$ such that

$$\|u\|_{L^{p^+}(\Omega)} \leq c_{15}\|u\|_{X_0} \quad \text{and} \quad \|u\|_{L^{q(x)}(\Omega)} \leq c_{16}\|u\|_{X_0}. \tag{4.2}$$

Let $0 < \epsilon c_{15}^{p^+} < \dfrac{1}{2p^+}$, where c_{15} is given by (4.2). Combining ($\mathbf{f_0}$) and ($\mathbf{f_3}$) we have

$$|F(x,t)| \leq \epsilon|t|^{p^+} + c_{17}|t|^{q(x)}, \tag{4.3}$$

for all $(x,t) \in \Omega \times \mathbb{R}$, where $c_{17} > 0$ is a constant. Let $u \in X_0$ with $\|u\|_{X_0} < 1$ sufficiently small. From ($\mathbf{H_0}$) and (4.2)-(4.3) we have

$$\begin{aligned}
\mathcal{J}_K(u) &= \widehat{M}\Big(\sigma_{p(x,y)}(u)\Big) - \int_\Omega F(x,u)\,dx \\
&\geq \frac{m_0}{p^+}(\rho^0_{K,p(\cdot,\cdot)}(u)) - \epsilon\int_\Omega |t|^{p^+}\,dx - c_{17}\int_\Omega |u|^{q(x)}\,dx \\
&\geq \frac{m_0}{p^+}\|u\|^{p^+}_{X_0} - \epsilon\|u\|^{p^+}_{L^{p^+}(\Omega)} - c_{17}\|u\|^{q^-}_{L^{q(x)}(\Omega)} \\
&\geq \frac{m_0}{p^+}\|u\|^{p^+}_{X_0} - \epsilon c_{15}^{p^+}\|u\|^{p^+}_{X_0} - (c_{17})(c_{16})^{q^-}\|u\|^{q^-}_{X_0} \\
&\geq \Big(\frac{1}{2p^+} - (c_{17})(c_{16})^{q^-}\|u\|^{q^- - p^+}_{X_0}\Big)\|u\|^{p^+}_{X_0},
\end{aligned} \tag{4.4}$$

where c_{16}, c_{17} are given by (4.2) and (4.3). From (4.4) and the fact that $q^- > p^+$, we can choose $R > 0$ and $\rho > 0$ such that $\mathcal{J}_K(u) \geq R > 0$ for all $u \in X_0$ with $\|u\|_{X_0} = \rho$. The proof of Lemma 4.1 is completed. □

By Proposition 3.1, we have that $\mathcal{J}_K$ satisfies the Cerami condition. Now, considering Lemma 4.1, we have all assumptions of Proposition 2.7 are fulfilled. Thus, the proof of Theorem 1.1 under assumption (**a**) is completed.

4.2. Proof of Theorem 1.1 under assumption (b)

The functional $\mathcal{J}_K$ satisfies the Cerami condition by Proposition 3.2 and also the verification of the geometric assumption of the Mountain Pass Theorem (Proposition 2.7) follows as in Lemma 4.1. Hence, the proof of Theorem 1.1 under assumption (**b**) is obtained.

5. The proof of Theorem 1.2

In this section we give the proof of the existence of infinitely solutions of problem (1.1). Our strategy consists in applying the Fountain Theorem of Bartsch (see [16]) to the functional $\mathcal{J}_K$.
The Fountain Theorem provides the existence of an unbounded sequence of critical value for a smooth functional, under suitable compactness condition (say, the Cerami condition) and geometric assumptions on it, which, in our framework, conditions (**i**) and (**ii**) of Proposition 2.8.

5.1. Proof of Theorem 1.2 under assumption (a)

In order to perform the proof of Theorem 1.2 when condition (**a**) is assumed, we first need the following result.

Lemma 5.1 *Let $p^+ < q^- \leq q^+ < p_s^*(x)$ and for any $k \in \mathbb{N}$, let*

$$\beta_k := \sup\{\|u\|_{L^{q(x)}(\Omega)} : \|u\|_{X_0} = 1, u \in Z_k\}.$$

Then $\beta_k \to 0$ as $k \to +\infty$.

Proof: It is obvious that for any $k \in \mathbb{N}$, $0 < \beta_{k+1} \leq \beta_k$, so $\beta_k \to \beta$ as $k \to +\infty$. For each $k = 1, 2, \cdots$ taking $u_k \in Z_k$ such that

$$\|u_k\|_{X_0} = 1 \quad \text{and} \quad 0 \leq \beta_k - \|u_k\|_{L^{q(x)}(\Omega)} < \frac{1}{k}. \tag{5.1}$$

As X_0 is reflexive, $\{u_n\}$ has a weakly convergent subsequence, without loss of generality, suppose $u_k \rightharpoonup u$ in X_0 and

$$\langle e_j^*, u\rangle = \lim_{k\to+\infty} \langle e_j^*, u_k\rangle, \quad j = 1, 2, \cdots .$$

Since each Z_k is a closed subspace of X_0, by Mazur's theorem, we have $u \in Z_k$ for any k. Consequently, we get

$$u \in \cap_{k=1}^{\infty} Z_k = \{0\},$$

and so $\{u_k\}$ converges weakly to 0 in X_0 as $k \to +\infty$. Since $p^+ < q^- \leq q^+ < p_s^*(x)$, the embedding $X_0 \hookrightarrow L^{q(x)}(\Omega)$ is compact, then $\{u_k\}$ converges strongly to 0 in $L^{q(x)}(\Omega)$. Hence, by (5.1) and the fact that β is nonnegative, we have $\beta_k \to 0$ as $k \to +\infty$, and this concludes the proof of Lemma 5.1. □

According to Proposition 3.1 and ($\mathbf{f_4}$), $\mathcal{J}_K$ is even functional and satisfies Cerami condition. We will prove that if k is large enough, then there exist $\rho_k > r_k > 0$ such that the geometric assumption ($\mathbf{A_1}$) and ($\mathbf{A_2}$) of the Fountain Theorem hold.
($\mathbf{A_1}$) For $u \in Z_k$ such that $\|u\|_{X_0} = r_k > 1$ (r_k will be specified below), by conditions ($\mathbf{H_0}$) and ($\mathbf{f_0}$) we have

$$\begin{aligned}
\mathcal{J}_K(u) &= \widehat{M}(\sigma_{p(x,y)}(u)) - \int_\Omega F(x,u)dx \\
&\geq \frac{m_0}{p^+}(\rho^0_{K,p(\cdot,\cdot)}(u) - \int_\Omega c_{18}(|u| + |u|^{q(x)})\, dx \\
&\geq \frac{m_0}{p^+}\|u\|_{X_0}^{p^-} - c_{18}\int_\Omega |u|^{q(x)}\, dx - c_{19}\|u\|_{X_0} \\
&\geq \begin{cases} \frac{m_0}{p^+}\|u\|_{X_0}^{p^-} - c_{18} - c_{19}\|u\|_{X_0} & (\text{if } \|u\|_{L^{q(x)}(\Omega)} \leq 1) \\ \frac{m_0}{p^+}\|u\|_{X_0}^{p^-} - c_{18}(\beta_k\|u\|_{X_0})^{q^+} - c_{19}\|u\|_{X_0} - & (\text{if } \|u\|_{L^{q(x)}(\Omega)} > 1) \end{cases} \\
&\geq \frac{m_0}{p^+}\|u\|_{X_0}^{p^-} - c_{18}(\beta_k\|u\|_{X_0})^{q^+} - c_{19}\|u\|_{X_0} - c_{20} \\
&= r_k^{p^-}(\frac{m_0}{p^+} - c_{18}\,\beta_k^{q^+} r_k^{q^+-p^-}) - c_{19}\, r_k - c_{20}.
\end{aligned}$$

We fix r_k as follows

$$r_k = \Big(\frac{c\,q^+\beta_k^{q^+}}{m_0}\Big)^{\frac{1}{p^- - q^+}},$$

then

$$\mathcal{J}_K(u) \geq m_0\, r_k^{p^-}\,(\frac{1}{p^+} - \frac{1}{q^+}) - c_{19}\, r_k - c_{20} = r_k\Big(m_0\, r_k^{p^- -1}(\frac{1}{p^+} - \frac{1}{q^+}) - c_{19}\Big) - c_{20}. \tag{5.2}$$

Using Lemma 5.1 and the fact $1 < p^- \leq p^+ < q^+$, it follows $r_k \to +\infty$. Consequently, $\mathcal{J}_K(u) \to +\infty$ as $\|u\|_{X_0} \to +\infty$ with $u \in Z_k$. The assertion $(\mathbf{A_1})$ is valid.

$(\mathbf{A_2})$ Since $\dim Y_k < +\infty$, all norms are equivalent in the finite dimensional space, by (3.13) and (4.1), for any $\psi \in Y_k$ with $\|\psi\|_{X_0} = 1$ and $t > 1$, we have

$$\begin{aligned}
\mathcal{J}_K(t\psi) &= \widehat{M}(\sigma_{p(x,y)}(t\psi)) - \int_\Omega F(x,t\psi)\,dx \\
&\leq c_{11} t^{\frac{p^+}{1-\mu}} (\frac{1}{p^-}\rho^0_{K,p(\cdot,\cdot)}(\psi))^{\frac{1}{1-\mu}} - M\, t^{\frac{p^+}{1-\mu}} \int_\Omega |\psi|^{\frac{p^+}{1-\mu}}\,dx + c_{13}|\Omega| \\
&\leq t^{\frac{p^+}{1-\mu}} \Big[c_{11}(\frac{1}{p^-}\rho^0_{K,p(\cdot,\cdot)}(\psi))^{\frac{1}{1-\mu}} - M \int_\Omega |\psi|^{\frac{p^+}{1-\mu}}\,dx\Big] + c_{13}|\Omega|.
\end{aligned} \tag{5.3}$$

It is clear that we can choose $M > 0$ large enough such that

$$c_{11}(\frac{1}{p^-}\rho^0_{K,p(\cdot,\cdot)}(\psi))^{\frac{1}{1-\mu}} - M \int_\Omega |\psi|^{\frac{p^+}{1-\mu}}\,dx < 0.$$

For this choice, it follows from (5.3) that

$$\lim_{t\to+\infty} \mathcal{J}_K(t\psi) = -\infty.$$

Hence, there exists $t_* > r_k > 1$ large enough such that $\mathcal{J}_K(t_*\psi) \leq 0$ and thus, by considering $\rho_k = t_*$ we deduce that

$$b_k := \max_{\{u\in Y_k : \|u\|_{X_0} = \rho_k\}} \mathcal{J}_K(u) \leq 0.$$

Thus, the assertion $(\mathbf{A_2})$ is fulfilled.
The proof of Theorem 1.2 under assumption (**a**) is complete.

5.2. Proof of Theorem 1.2 under assumption (b)

The functional $\mathcal{J}_K$ satisfies the Cerami condition by Proposition 3.2 and using $(\mathbf{f_4})$, we get $\mathcal{J}_K(-u) = \mathcal{J}_K(u)$ for any $u \in X_0$. As for the geometric features of $\mathcal{J}_K$, conditions $(\mathbf{A_1})$ and $(\mathbf{A_2})$ can be proved as in section 5.1. Hence, the proof of Theorem 1.2 under assumption (**b**) is complete.

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