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$\lambda-$Statistically Convergent and $\lambda-$Statistically Bounded Sequences Defined by Modulus Functions

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ABSTRACT: In this research paper, we introduce some concepts of $\lambda f-$density in connection with modulus functions under certain conditions. Furthermore, we establish some relations between the sets of $\lambda f-$statistically convergent and $\lambda f-$statistically bounded sequences.

Key Words: Density, modulus function, $\lambda-$statistical convergence.

Contents

1. Introduction

Over the past few decades, one of the most significant and active areas of research in mathematics has been the study of convergence of sequences. The theory of statistical convergence was developed by Zygmund [27] and originally published in his monograph in Warsaw, which is a generalization of classical convergence. Steinhaus [26] and Fast [12] essentially presented the concept of statistical convergence, and Schoenberg [25] later independently reintroduced it. Many mathematicians have utilized statistical convergence as a tool to tackle several open problems in the fields of sequence spaces, summability theory, and some other applications. Over the past several decades, statistical convergence has been explored in a variety of fields and under a variety of names, including Banach spaces, measure theory, Fourier analysis, number theory, ergodic theory, cone metric space, trigonometric series, time scale, and topological space. To generalize this idea, Mursaleen [17] proposed the notion of $\lambda-$statistical convergence by using the sequence $\lambda = (\lambda_n)$. Some other applications and generalizations on $\lambda-$statistical convergence and statistical convergence are available in [3,8,9,10,13,15,21].

The concept of a modulus function was developed by Nakano [18]. By using the modulus functions some authors have introduced and established several sequence spaces such as Ruckle [22], Maddox [16], Ghosh and Srivastava [14], Altin and Et [2], Savas and Patterson [24], Candan [6], Prakash et al. [20], and some others.

Aizpuru et al. [1] have used an unbounded modulus function to characterize another density concept. As a result, they established a new idea of nonmatrix convergence, which is intermediate between ordinary convergence and statistical convergence and coincides with the statistical convergence of the identity modulus.

2. Definitions and Preliminaries

In this section, we provide some definitions that are required for the study.

Definition 2.1. *[23] Let $H \subset \mathbb{N}$. Then, a number $\delta(H)$ is called a natural density of H and is defined by*

$$\delta(H) = \lim_{n\to\infty} \frac{1}{n} |\{h \leq n : h \in H\}|,$$

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in the case the limit exists, where $|\{h \leq n : h \in H\}|$ *is the number of elements of* H *which are less than or equal to* n*.*

Definition 2.2. *[23] A sequence* (x_k) *of numbers is said to be statistically convergent (or* $S-$*convergent) to the number* l *if*

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : |x_k - l| \geq \varepsilon\}| = 0$$

for each $\varepsilon > 0$*. In this case, we write* $S - \lim x_k = l$ *or* $x_k \to l\,(S)$ *and* S *denotes the set of all* $S-$*convergent sequences.*

In the study, $\lambda = (\lambda_n)$ denotes a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_1 = 1$ and $\lambda_{n+1} \leq \lambda_n + 1$. We write I_n to denote the closed and bounded interval $[n - \lambda_n + 1, n]$. Also, we write Λ to denote the set of all such sequences $\lambda = (\lambda_n)$.

Definition 2.3. *[17] Let* $\lambda = (\lambda_n) \in \Lambda$*. Then, a sequence* (x_k) *of numbers is said to be* $\lambda-$*statistically convergent (or* $S_\lambda-$*convergent) to the number* l *if for every* $\varepsilon > 0$*,*

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k - l| \geq \varepsilon\}| = 0.$$

We write $S_\lambda - \lim x_k = l$ *or* $x_k \to l\,(S_\lambda)$ *in this case.*

In the case $\lambda_n = n$ for each $n \in \mathbb{N}$, $S_\lambda-$convergence reduces to $S-$convergence.

Definition 2.4. *[1] A function* $f : [0, \infty) \to [0, \infty)$ *is called a modulus function (or modulus) if*

1. $f(x) = 0 \Leftrightarrow x = 0$,
2. $f(x + y) \leq f(x) + f(y)$ *for every* $x, y \in [0, \infty)$,
3. f *is increasing,*
4. f *is continuous from the right at* 0.

From these properties, it is clear that a modulus function must be continuous everywhere on $[0, \infty)$. A modulus function may be bounded or unbounded. For instance, $f(x) = x^a$ where $a \in (0, 1]$ is an unbounded modulus, but $f(x) = \frac{x}{x+1}$ is a bounded modulus.

Definition 2.5. *[1] Let* f *be an unbounded modulus function. Then, it is said that the sequence* (x_k) *of numbers is* $f-$*statistically convergent (or* S^f-*convergent) to the number* l *if*

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k - l| \geq \varepsilon\}|) = 0$$

for every $\varepsilon > 0$*. We write* $S^f - \lim x_k = l$ *or* $x_k \to S(l)$ *in this case. Throughout the study,* S^f *denotes the set of all statistically convergent sequences. And, we write* S *instead of* S^f *in case* $f(x) = x$*.*

Definition 2.6. *[5] Let* f *be an unbounded modulus function. Then, it is said that the sequence* (x_k) *of numbers is* $f-$*statistically bounded (or* S^f-*bounded) if there is* $M > 0$ *such that*

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k| \geq M\}|) = 0.$$

Throughout the study, $S^f(b)$ *denotes the set of all* $f-$*statistically bounded sequences.*

3. Main Results

In this section, we give the main results of the paper. We focus on giving the relations between the sets of $\lambda f-$statistically convergent and $\lambda f-$statistically bounded sequences.

Definition 3.1. *Let f be an unbounded modulus, $\lambda = (\lambda_n) \in \Lambda$ and $H \subset \mathbb{N}$. Then, a number $\delta_\lambda^f(H)$ is named a $\lambda f-$density (or δ_λ^f-density) of the set H and is defined by*

$$\delta_\lambda^f(H) = \lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in H\}|),$$

in the case this limit exists.

It should be noted that in the case $f(x) = x$, the concepts of δ_λ^f-density and $\delta_\lambda-$density coincide. And, in the case $f(x) = x$ and $(\lambda_n) = (n)$, the concepts of δ_λ^f-density and $\delta-$density coincide.

Remark 3.2. *It is not necessary for the equality $\delta_\lambda^f(H) + \delta_\lambda^f(\mathbb{N}\backslash H) = 1$ to remain true, in general, even though for a natural density it is always true. The example below illustrates this fact.*

Example 3.3. *Let us take $f(x) = \log(x+1)$, $\lambda = (\lambda_n) = (n)$ and $H = \{2n : n \in \mathbb{N}\}$. Then, $\delta_\lambda^f(H) + \delta_\lambda^f(\mathbb{N}\backslash H) \neq 1$. Indeed, since f is an unbounded modulus and $\frac{\lambda_n}{2} - 1 \leq |\{k \in I_n : k \in H\}| \leq \frac{\lambda_n}{2}$ for each $n \in \mathbb{N}$, we may write*

$$\frac{1}{f(\lambda_n)} f\left(\frac{\lambda_n}{2} - 1\right) \leq \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in H\}|) \leq \frac{1}{f(\lambda_n)} f\left(\frac{\lambda_n}{2}\right)$$

or

$$\frac{1}{\log(n+1)} \log\left(\frac{n}{2}\right) \leq \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in H\}|) \leq \frac{1}{\log(n+1)} \log\left(\frac{n}{2} + 1\right).$$

By taking the limits as $n \to \infty$ in the above inequality, we get that

$$1 \leq \lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in H\}|) \leq 1.$$

Thus, $\delta_\lambda^f(H) = 1$. Furthermore, by using the fact $\frac{\lambda_n+1}{2} - 1 \leq |\{k \in I_n : k \in \mathbb{N}\backslash H\}| \leq \frac{\lambda_n+1}{2}$ for each $n \in \mathbb{N}$, we have $\delta_\lambda^f(\mathbb{N}\backslash H) = 1$. Therefore, $\delta_\lambda^f(H) + \delta_\lambda^f(\mathbb{N}\backslash H) = 2$.

Theorem 3.4. *Let $\lambda \in \Lambda$ and $H \subset \mathbb{N}$. If $\delta_\lambda^f(H) = 0$, then $\delta_\lambda(H) = 0$ for any unbounded modulus f.*

Proof. Suppose $\delta_\lambda^f(H) = 0$, then

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in H\}|) = 0.$$

So, for any $t \in \mathbb{N}$, there is $N \in \mathbb{N}$ such that for all $n \geq N$,

$$f(|\{k \in I_n : k \in H\}|) \leq \frac{1}{t} f(\lambda_n) \leq \frac{1}{t} t f\left(\frac{1}{t}\lambda_n\right) = f\left(\frac{1}{t}\lambda_n\right).$$

Since f is a modulus function, we have

$$|\{k \in I_n : k \in H\}| \leq \frac{1}{t}\lambda_n.$$

Therefore, $\delta_\lambda(H) = 0$. □

Remark 3.5. *The converse of Theorem 3.4 does not have to be true, in general. For example, if we take $f(x) = \log(x+1)$, $(\lambda_n) = (n)$ and $H = \{n^2 : n \in \mathbb{N}\}$, then $\delta_\lambda(H) = 0$ but $\delta_\lambda^f(H) = \frac{1}{2}$.*

Remark 3.6. *The δ_λ^f-density of any finite subset of $\mathbb{N}$ is zero. Indeed, if H is any finite subset of $\mathbb{N}$, then the set $\{k \in I_n : k \in H\}$ will be finite. So that for any unbounded modulus f and for each $\lambda \in \Lambda$, we have*

$$\delta_\lambda^f(H) = \lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : k \in H\}|) = 0.$$

Definition 3.7. *Let f be an unbounded modulus function and $\lambda = (\lambda_n) \in \Lambda$. Then, the sequence (x_k) of numbers is said to be $\lambda f-$statistically convergent (or S_λ^f-convergent) to the number l if for every $\varepsilon > 0$, $\delta_\lambda^f(\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}) = 0$, i.e.,*

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : |x_k - l| \geq \varepsilon\}|) = 0,$$

where $f(\lambda_n)$ denotes the nth term of the sequence $(f(\lambda_n))$, that is, $(f(\lambda_n)) = (f(\lambda_1), f(\lambda_2), f(\lambda_3), ...)$. In this case, we write $S_\lambda^f - \lim x_k = l$ or $x_k \to l\left(S_\lambda^f\right)$.

Throughout the study, the set of all S_λ^f-convergent sequences will be denoted by S_λ^f, that is,

$$S_\lambda^f = \left\{(x_k) : \forall\varepsilon > 0,\ \lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : |x_k - l| \geq \varepsilon\}|) = 0 \text{ for some number } l\right\}.$$

In case $l = 0$, we write $S_{\lambda,0}^f$ to denote the set of all S_λ^f-null sequences. It is obvious to note that every S_λ^f-null sequence is S_λ^f-convergent sequence, that is, $S_{\lambda,0}^f \subset S_\lambda^f$ for every unbounded modulus function f and for each $\lambda \in \Lambda$.

It should be noted that the concepts of S_λ^f-convergence and $S_\lambda-$convergence will be identical in the case $f(x) = x$. The concepts of S_λ^f-convergence and S^f-convergence will be identical in the case $(\lambda_n) = (n)$. Also, the concepts of S_λ^f-convergence and $S-$convergence will be identical in the case $f(x) = x$ and $(\lambda_n) = (n)$.

Theorem 3.8. *Suppose (x_k) and (y_k) are sequences of numbers.*

1. *If $x_k \to x_0\left(S_\lambda^f\right)$, then $zx_k \to zx_0\left(S_\lambda^f\right)$ for any $z \in \mathbb{C}$.*
2. *If $x_k \to x_0\left(S_\lambda^f\right)$ and $y_k \to y_0\left(S_\lambda^f\right)$, then $(x_k + y_k) \to (x_0 + y_0)\left(S_\lambda^f\right)$.*

Proof. 1. In case $z = 0$, it is clear. We assume that $z \neq 0$. Then, for every $\varepsilon > 0$, we may write

$$\frac{1}{f(\lambda_n)} f(|\{k \in I_n : |zx_k - zx_0| \geq \varepsilon\}|) = \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : |x_k - x_0| \geq \frac{\varepsilon}{|z|}\right\}\right|\right).$$

Since $x_k \to x_0\left(S_\lambda^f\right)$, we have

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : |zx_k - zx_0| \geq \varepsilon\}|) = 0.$$

Therefore, $zx_k \to zx_0\left(S_\lambda^f\right)$.

2. Suppose $x_k \to x_0\left(S_\lambda^f\right)$ and $y_k \to y_0\left(S_\lambda^f\right)$. Then,

$$\begin{aligned}&\frac{1}{f(\lambda_n)} f(|\{k \in I_n : |(x_k + y_k) - (x_0 + y_0)| \geq \varepsilon\}|)\\ &\leq \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : |x_k - x_0| \geq \frac{\varepsilon}{2}\right\}\right|\right) + \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : |y_k - y_0| \geq \frac{\varepsilon}{2}\right\}\right|\right)\end{aligned}$$

for every $\varepsilon > 0$. Since $x_k \to x_0 \left(S_\lambda^f\right)$ and $y_k \to y_0 \left(S_\lambda^f\right)$, we get

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : |(x_k + y_k) - (x_0 + y_0)| \geq \varepsilon\right\}\right|\right) = 0.$$

Therefore, $(x_k + y_k) \to (x_0 + y_0) \left(S_\lambda^f\right)$. □

Theorem 3.9. *Every convergent sequence is $\lambda f-$statistically convergent, that is, $c \subset S_\lambda^f$ for every unbounded modulus f and for each $\lambda \in \Lambda$.*

Proof. Suppose $(x_k) \in c$ and $x_k \to l$. Then, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|x_k - l| < \varepsilon \text{ for all } k > N.$$

So that the set $\{k \in I_n : |x_k - l| \geq \varepsilon\}$ is finite. By using Remark 3.6, we get

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : |x_k - l| \geq \varepsilon\right\}\right|\right) = 0.$$

Therefore, $(x_k) \in S_\lambda^f$. □

Remark 3.10. *The converse of the above theorem does not have to be true, in general. This fact can be illustrated in the following example.*

Example 3.11. *Let us take the modulus $f(x) = x^a$, where $0 < a \leq 1$ and $\lambda = (\lambda_n) = (n)$. Define the sequence (x_k) as*

$$x_k = \begin{cases} k & \text{if } k = m^3 \\ 0 & \text{if } k \neq m^3 \end{cases} \quad m \in \mathbb{N}.$$

Then, $x_k \to 0 \left(S_\lambda^f\right)$. However, (x_k) is not convergent.

Theorem 3.12. *Every $\lambda f-$statistically convergent sequence is $\lambda-$statistically convergent to the same limit, that is, $S_\lambda^f \subset S_\lambda$ for any unbounded modulus f and for each $\lambda \in \Lambda$.*

Proof. Suppose $(x_k) \in S_\lambda^f$ and $x_k \to l \left(S_\lambda^f\right)$. For every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f\left(\left|\left\{k \in I_n : |x_k - l| \geq \varepsilon\right\}\right|\right) = 0. \tag{3.1}$$

By using (3.1) and Theorem 3.4, we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \left|\left\{k \in I_n : |x_k - l| \geq \varepsilon\right\}\right| = 0.$$

Therefore, $x_k \to l\,(S_\lambda)$ and thus $S_\lambda^f \subset S_\lambda$. □

Remark 3.13. *The converse of Theorem 3.12 is not true, in general. This fact can be illustrated in the following example.*

Example 3.14. *Let $f(x)=\log(x+1)$ and $\lambda=(\lambda_n)=(n)$. Define the sequence (x_k) as in Example 3.11. Then, for every $\varepsilon>0$, we have*

$$\lim_{n\to\infty}\frac{1}{\lambda_n}\left|\{k\in I_n:|x_k|\geq\varepsilon\}\right|\leq\lim_{n\to\infty}\frac{1}{\lambda_n}\sqrt[3]{\lambda_n}=0.$$

So that $x_k\to 0\,(S_\lambda)$ and thus $(x_k)\in S_\lambda$. However, for every $\varepsilon>0$, we have

$$\lim_{n\to\infty}\frac{1}{f(\lambda_n)}f\left(\left|\{k\in I_n:|x_k|\geq\varepsilon\}\right|\right)\geq\lim_{n\to\infty}\frac{1}{f(\lambda_n)}f\left(\sqrt[3]{\lambda_n}-1\right)=\frac{1}{3}.$$

So that $x_k\nrightarrow 0\left(S_\lambda^f\right)$ and thus $(x_k)\notin S_\lambda^f$.

We get the following result by taking $(\lambda_n)=(n)$ from Theorem 3.12, which is the second part of Theorem 2.16 of [4] in case $\alpha=1$.

Corollary 3.15. *Every $f-$statistically convergent sequence is statistically convergent to the same limit, that is, $S^f\subset S$ for any unbounded modulus f.*

Theorem 3.16. *For any unbounded modulus f and for each $\lambda\in\Lambda$, we have $S_\lambda\subset S_\lambda^f$ if $\lim_{n\to\infty}\inf\frac{f(\lambda_n)}{\lambda_n}>0$.*

Proof. Suppose $(x_k)\in S_\lambda$ and $x_k\to l\,(S_\lambda)$. Then, for every $\varepsilon>0$, we have

$$\begin{aligned}\frac{1}{\lambda_n}\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right| &\geq\frac{1}{\lambda_n}\frac{1}{f(1)}f\left(\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right|\right)\\ &=\frac{f(\lambda_n)}{\lambda_n}\frac{1}{f(1)}\frac{f\left(\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right|\right)}{f(\lambda_n)}.\end{aligned}$$

Since $\lim_{n\to\infty}\inf\frac{f(\lambda_n)}{\lambda_n}>0$, by taking the limits as $n\to\infty$ in the above inequality, we get $x_k\to l\,(S_\lambda)$ implies $x_k\to l\left(S_\lambda^f\right)$. □

We get the following result from Theorem 3.16 by taking $(\lambda_n)=(n)$.

Corollary 3.17. *For any unbounded modulus f, we have $S\subset S^f$ if $\lim_{n\to\infty}\inf\frac{f(n)}{n}>0$.*

From Theorem 3.12 and Theorem 3.16, we get the following result.

Corollary 3.18. *For any unbounded modulus f and for each $\lambda\in\Lambda$, we have $S_\lambda=S_\lambda^f$ if $\lim_{n\to\infty}\inf\frac{f(\lambda_n)}{\lambda_n}>0$.*

Theorem 3.19. *For every unbounded modulus f and for each $\lambda\in\Lambda$, we have $S_\lambda^f\subset S$; although the converse is not true, in general.*

Proof. Since $S_\lambda^f\subset S_\lambda$ by Theorem 3.12 and $S_\lambda\subset S$ by Theorem 2.7 of [7], so that $S_\lambda^f\subset S$. For the converse part, recall the sequence (x_k) in Example 3.2, the sequence is $S-$convergent but it is not S_λ^f-convergent if we take $f(x)=\log(x+1)$ and $(\lambda_n)=(n)$. □

Theorem 3.20. *Let f be an unbounded modulus and $\lambda\in\Lambda$. If $\lim_{n\to\infty}\inf\frac{f(\lambda_n)}{n}>0$, then $S\subset S_\lambda^f$.*

Proof. Suppose $(x_k)\in S$ and $x_k\to l\,(S)$. Then, for every $\varepsilon>0$, we have

$$\{k\leq n:|x_k-l|\geq\varepsilon\}\supset\{k\in I_n:|x_k-l|\geq\varepsilon\}.$$

Since f is a modulus, we may write

$$\begin{aligned}\frac{1}{n}\left|\{k\leq n:|x_k-l|\geq\varepsilon\}\right| &\geq \frac{1}{n}\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right| \\ &\geq \frac{1}{n}\frac{1}{f(1)}f\left(\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right|\right) \\ &= \frac{f(\lambda_n)}{n}\frac{1}{f(1)}\frac{f\left(\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right|\right)}{f(\lambda_n)}.\end{aligned}$$

By taking the limits as $n\to\infty$ in the above inequality, we obtain that $(x_k)\in S$ implies $(x_k)\in S_\lambda^f$ since $\lim_{n\to\infty}\inf\frac{f(\lambda_n)}{n}>0$. □

From Theorem 3.19 and Theorem 3.20, we get the following result.

Corollary 3.21. *Let f be an unbounded modulus and $\lambda\in\Lambda$. If $\lim_{n\to\infty}\inf\frac{f(\lambda_n)}{n}>0$, then $S=S_\lambda^f$.*

Definition 3.22. *Let f be an unbounded modulus and $\lambda=(\lambda_n)\in\Lambda$. Then, the sequence (x_k) of numbers is said to be $\lambda f-$statistically bounded (or S_λ^f-bounded) if $\delta_\lambda^f(\{k\in\mathbb{N}:|x_k|\geq M\})=0$ for some $M>0$, i.e.,*

$$\lim_{n\to\infty}\frac{1}{f(\lambda_n)}f\left(\left|\{k\in I_n:|x_k|\geq M\}\right|\right)=0.$$

Throughout the study, the set of all S_λ^f-bounded sequences will be denoted by $S_\lambda^f(b)$, that is,

$$S_\lambda^f(b)=\left\{(x_k):\lim_{n\to\infty}\frac{1}{f(\lambda_n)}f\left(\left|\{k\in I_n:|x_k|\geq M\}\right|\right)=0\text{ for some }M>0\right\}.$$

In the case $f(x)=x$, $\lambda f-$statistical boundedness reduces to $\lambda-$statistical boundedness, that is, $S_\lambda^f(b)=S_\lambda(b)$. In the case $(\lambda_n)=(n)$, $\lambda f-$statistical boundedness reduces to $f-$statistical boundedness, that is, $S_\lambda^f(b)=S^f(b)$. Also, in the case $f(x)=x$ and $(\lambda_n)=(n)$, $\lambda f-$statistical boundedness reduces to statistical boundedness, that is, $S_\lambda^f(b)=S(b)$.

Theorem 3.23. *Every S_λ^f-convergent sequence is S_λ^f-bounded for any unbounded modulus f and for each $\lambda\in\Lambda$, that is, $S_\lambda^f\subset S_\lambda^f(b)$; although the converse is not true, in general*

Proof. Suppose $(x_k)\in S_\lambda^f$ and $x_k\to l\left(S_\lambda^f\right)$. Since f is a modulus and for every $\varepsilon>0$

$$\{k\in I_n:|x_k-l|\geq\varepsilon\}\supset\{k\in I_n:|x_k|>|l|+\varepsilon\},$$

so that

$$\frac{1}{f(\lambda_n)}f\left(\left|\{k\in I_n:|x_k-l|\geq\varepsilon\}\right|\right)\geq\frac{1}{f(\lambda_n)}f\left(\left|\{k\in I_n:|x_k|>|l|+\varepsilon\}\right|\right).$$

By taking the limits on both sides in the above inequality as $n\to\infty$, we obtain that $(x_k)\in S_\lambda^f$ implies $(x_k)\in S_\lambda^f(b)$. For the converse part, let us take $f(x)=x^a$, $0<a\leq 1$, $\lambda=(\lambda_n)=(n)$ and $(x_k)=(1,2,1,2,...)$, then $(x_k)\in S_\lambda^f(b)$, but $(x_k)\notin S_\lambda^f$. This completes the proof. □

We get the following result by taking $f(x)=x$ from Theorem 3.23.

Corollary 3.24. *Every $S_\lambda-$convergent sequence is $S_\lambda(b)-$bounded for each $\lambda\in\Lambda$, that is, $S_\lambda\subsetneq S_\lambda(b)$.*

We get the following result by taking $(\lambda_n)=(n)$ from Theorem 3.23.

Corollary 3.25. *Every S^f-convergent sequence is $S^f(b)-$bounded for any unbounded modulus f, that is, $S^f\subsetneq S^f(b)$.*

Theorem 3.26. *Every $\lambda f-$statistically bounded sequence is $\lambda-$statistically bounded for any unbounded modulus f and for each $\lambda \in \Lambda$, that is, $S_{\lambda}^{f}(b) \subset S_{\lambda}(b)$; although the converse is not true, in general.*

Proof. Suppose $(x_k) \in S_{\lambda}^{f}(b)$. Then, there is $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : |x_k| \geq M\}|) = 0. \tag{3.2}$$

From (3.2) and Theorem 3.4, we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k| \geq M\}| = 0.$$

Therefore, $(x_k) \in S_{\lambda}^{f}(b)$ implies $(x_k) \in S_{\lambda}(b)$. For the converse part, let us take $f(x) = \log(x+1)$, $\lambda = (\lambda_n) = (n)$ and $(x_k) = (1, 0, 0, 4, 0, 0, 0, 0, 9, ...)$. Then, for any number $M > 0$, we have

$$\{k \in \mathbb{N} : |x_k| > M\} = \{1, 4, 9, ...\}$$

a finite subset of $\mathbb{N}$. Since $\delta_{\lambda}^{f}(\{1, 4, 9, ...\}) = \frac{1}{2} \neq 0$ and $\delta_{\lambda}(\{1, 4, 9, ...\}) = 0$, then $(x_k) \notin S_{\lambda}^{f}(b)$ and $(x_k) \in S_{\lambda}(b)$. As a result, $S_{\lambda}^{f}(b) \subsetneq S_{\lambda}(b)$. □

From Theorem 3.26, we get the following result by taking $(\lambda_n) = (n)$.

Corollary 3.27. *Every $f-$statistically bounded sequence is statistically bounded, that is, $S^{f}(b) \subsetneq S(b)$.*

Theorem 3.28. *For any unbounded modulus f and for each $\lambda \in \Lambda$, we have $S_{\lambda}(b) \subset S_{\lambda}^{f}(b)$ if $\lim_{n\to\infty} \inf \frac{f(\lambda_n)}{\lambda_n} > 0$.*

Proof. Suppose $(x_k) \in S_{\lambda}(b)$. Then, there is $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |x_k| \geq M\}| = 0.$$

Since f is a modulus, we have

$$\begin{aligned} \frac{1}{\lambda_n} |\{k \in I_n : |x_k| \geq M\}| &\geq \frac{1}{\lambda_n} \frac{1}{f(1)} f(|\{k \in I_n : |x_k| \geq M\}|) \\ &\geq \frac{f(\lambda_n)}{\lambda_n} \frac{1}{f(1)} \frac{f(|\{k \in I_n : |x_k| \geq M\}|)}{f(\lambda_n)}. \end{aligned}$$

By taking the limits on both sides in the above inequality as $n \to \infty$, we get that $(x_k) \in S_{\lambda}(b)$ implies $(x_k) \in S_{\lambda}^{f}(b)$. □

From Theorem 3.26 and Theorem 3.28, we obtain the following result.

Corollary 3.29. *For any unbounded modulus f and for each $\lambda \in \Lambda$, we have $S_{\lambda}(b) = S_{\lambda}^{f}(b)$ if $\lim_{n\to\infty} \inf \frac{f(\lambda_n)}{\lambda_n} > 0$.*

Theorem 3.30. *For any unbounded modulus f and for each $\lambda \in \Lambda$, we have $S^{f}(b) \subset S_{\lambda}^{f}(b)$ if $\lim_{n\to\infty} \inf \frac{f(\lambda_n)}{f(n)} > 0$.*

Proof. Suppose $(x_k) \in S^{f}(b)$. Then, there is $M > 0$ such that

$$\lim_{n\to\infty} \frac{1}{f(n)} f(|\{k \leq n : |x_k| \geq M\}|) = 0. \tag{3.3}$$

In general, we have

$$\{k \leq n : |x_k| \geq M\} \supset \{k \in I_n : |x_k| \geq M\}.$$

Since f is a modulus, we may write

$$\begin{aligned} \frac{1}{f(n)} f(|\{k \leq n : |x_k| \geq M\}|) &\geq \frac{1}{f(n)} f(|\{k \in I_n : |x_k| \geq M\}|) \\ &= \frac{f(\lambda_n)}{f(n)} \frac{1}{f(\lambda_n)} f(|\{k \in I_n : |x_k| \geq M\}|). \end{aligned}$$

Since $\lim_{n\to\infty} \inf \frac{f(\lambda_n)}{f(n)} > 0$, taking the limits as $n \to \infty$ in the above inequality and using (3.3), we obtain that $(x_k) \in S^f(b)$ implies $(x_k) \in S_\lambda^f(b)$.

□

We get the following result by taking $f(x) = x$ from Theorem 3.30.

Corollary 3.31. *For each $\lambda \in \Lambda$, we have $S(b) \subset S_\lambda(b)$ if $\lim_{n\to\infty} \inf \frac{\lambda_n}{n} > 0$.*

4. Conclusion

In this paper, we have introduced a new version of density by applying to the notion of modulus functions under some conditions. With the help of this density, new types of statistical convergence and statistical boundedness were introduced. There is a significant opportunity that new discoveries and generalizations can be presented in this field using these concepts. Also, this research paper will be a valuable resource for researchers conducting relevant research in related areas as well as for studies that will be conducted in connected fields in the future.

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