Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–23.
ISSN-0037-8712
doi:10.5269/bspm.65823

Blow-up for a viscoelastic plate equation with Balakrishnan-Taylor damping and Nonlinear Source of Polynomial Type

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ABSTRACT: In this paper, we study the initial-boundary value problem for a problem of nonlinear viscoelastic plate wave equations with Balakrishnan–Taylor damping terms. We demonstrate that the nonlinear source of polynomial type is able to force solutions to blow up infinite time even in presence of stronger damping with non positive initial energy combined with a positive initial energy.

Key Words: viscoelastic plate equation; Balakrishnan-Taylor; blow-up.

Contents

1. Introduction

In this paper, we study blow-up of solutions of the following plate problem

$$\begin{aligned} &|u_t(t)|^\gamma u_{tt}(t) + M(t)\Delta^2 u(t) + \Delta^2 u_{tt}(t) \\ &-\textstyle\int_0^t h(t-s)\Delta^2 u(s)\,ds + au_t(t) \\ &= |u(t)|^{p-2} u(t), \quad \text{in } \Omega\times(0,\infty), \end{aligned} \tag{1.1}$$

$$\begin{cases} u(x,t) = \dfrac{\partial u}{\partial v}(x,t) = 0, & \text{in } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) & \text{for } x\in\Omega, \end{cases} \tag{1.2}$$

where Ω is a bounded domain of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$, v is the unit outer normal to $\partial\Omega$, and $p > 4$. The kernel h is satisfying some conditions to be specified later. And $M(t) := \xi_0 + \xi_1 \|\Delta u(t)\|^2_{L^2(\Omega)} + \sigma(\Delta u(t), \Delta u_t(t))_{L^2(\Omega)}$, where u is the plate transverse displacement, x is the spatial coordinate in the direction of the fluid flow, and t is time. The viscoelastic structural damping terms are denoted by ξ_1, σ is the nonlinear stiffness of the membrane, and ξ_0 is an in-plane tensile load. All quantities are physically non-dimensionalized and ξ_0, ξ_1 and σ are fixed positive. For more information on using Balakrishnan-Taylor, see $[4-6]$.

Also, a result of local existence for problem $(1.1)-(1.2)$ for $\xi_1 = \sigma = a = 0$, has been proved in $[2]$, for $\xi_1, \sigma \neq 0$ and $a = 1$, in the same way as [2], we get the same basic results for the local existence of problem $(1.1)-(1.2)$ with a slight change in some calculations that do not affect the basic results.

Lin, X.; Li, F [10] studied the asymptotic energy estimates for nonlinear petrovsky plate model subject to viscoelastic damping. Lagnese [11] studied the a viscoelastic plate equation and showed that the energy decays to zero as time goes to infinity by intorducing a dissipative mechanism on the boundary of the system. Rivera et al.[12] studied the a viscoelastic plate equation proved that the first- and second-order energy, associated with the solutions of the viscoelastic plate equation, decay exponentially provided that the kernel of thememory also decays exponentially. Komornik [13] studied the a viscoelastic plate

Submitted November 13, 2022. Published November 04, 2023
2010 *Mathematics Subject Classification:* 35L35, 35L20, 75D05, 74D10, 93D20

equation investigated the energy decay of a platemodel under weak growth assumptions on the feedback function. For more results in this direction, see $[14-21]$.

Boulaaras, S.; Draifia, A.; Alnegga, M. [3] studied the polynomial decay rate for kirchhoff type in viscoelasticity with logarithmic nonlinearity and not necessarily decreasing kernel

$$\begin{cases} |u_t(t)|^\rho u_{tt}(t) + \left(\xi_0 + \xi_1 \|\Delta u(t)\|^{2\beta}_{L^2(\Omega)}\right)\Delta^2 u(t) + \Delta^2 u_{tt}(t) \\ -\int_0^t h(t-s)\,\Delta^2 u(s)\,ds = u(t)\,|u(t)|^{\gamma-2}\ln|u(t)|^k, \quad \text{in } \Omega\times(0,\infty), \\ u(x,t) = \dfrac{\partial u}{\partial v}(x,t) = 0 \text{ in } \partial\Omega\times(0,\infty), \\ u(x,0) = u^0,\ u_t(x,0) = u^1 \text{ for } x\in\Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$, $\beta \geq 1,$, $\gamma \geq 2$, k, ξ_0 and ξ_1 are positive constants, $\rho \geq 0$ for $N = 1,2$ and $0 \leq \rho \leq \frac{4}{N-2}$ for $N \geq 3$ and h are the relaxation function whose characteristics will be determined in [3].

Al-Gharabli, Guesmia and Messaoudi [2], studied the asymptotic stability results for a viscoelastic plate equation with a logarithmic nonlinearity

$$\begin{cases} |u_t(t)|^\gamma u_{tt}(t) + \Delta^2 u(t) + \Delta^2 u_{tt}(t) - \int_0^t h(t-s)\,\Delta^2 u(s)\,ds \\ = u(t)\ln|u(t)|^k, \quad \text{in } \Omega\times(0,\infty), \\ u(x,t) = \dfrac{\partial u}{\partial v}(x,t) = 0, \quad \text{in } \partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) \quad \text{for } x\in\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^2$, with a smooth boundary $\partial\Omega$, v is the unit outer normal to $\partial\Omega$, $0 < \gamma \leq \dfrac{2}{n-2}$ for $n \geq 3$, $\gamma > 0$ for $n = 1,2$, and $k > 0$. The kernel h of problem (1.3) is satisfying the following hypotheses (see [2]).

However, among the literature, such as the work presented in $[1-21]$, which examines blow-up results of problem $(1.1)-(1.2)$, there is no blow-up result when $E(0) \geq 0$ and $E(0) < 0$. Motivated by the previous research papers, we will study the blow-up results when $E(0) \geq 0$ and $E(0) < 0$ of the model $(1.1)-(1.2)$.

The outline of the paper is as follows. In the second section we present some basic concepts as well as two lemmas to help solve the problem presented in $(1.1)-(1.2)$. In Section 3, we prove our principle result.

2. Preliminaries

In this section, we introduce some functional spaces and establish two lemmas, which will be used for the remaining of the present paper. Let $L^p(\Omega)$ be the weighted Banach space equipped with the norm

$$\|u(t)\|_{L^p(\Omega)} = \left[\int_\Omega |u(t)|^p\,dx\right]^{\frac{1}{p}}.$$

In particular, the Hilbert space of square integral functions having the finite norm

$$\|u(t)\|_{L^2(\Omega)} = \left[\int_\Omega u^2(t)\,dx\right]^{\frac{1}{2}}.$$

Lemma 2.1 *(See [1]) Let $\delta > 0$ and $\beta(t) \in C^2(0,\infty)$ be a nonnegative function satisfying*

$$\beta''(t) - 4(\delta+1)\beta'(t) + 4(\delta+1)\beta(t) \geq 0.$$

If

$$\beta'(0) > r_2\beta(0) + k_0,$$

then

$$\beta'(t) > k_0,$$

for $t > 0$, where k_0 is a constant, $r_2 := 2(\delta+1) - 2\sqrt{(\delta+1)\delta}$ is the smallest root of the equation

$$r^2 - 4(\delta+1)r + 4(\delta+1) = 0.$$

Lemma 2.2 *(See [1]) If $J(t)$ is a non-increasing function on $[t_0,\infty)$, $t_0 \geq 0$ and satisfies the deferential inequality*

$$J'(t)^2 \geq \mu + bJ(t)^{2+\frac{1}{\delta}}, \quad \text{for } t \geq t_0,$$

where $\mu > 0$, $b \in \mathbb{R}$, then there exists a finite time T^* such that

$$\lim_{t\to T^{*-}} J(t) = 0,$$

and the upper bound of T^* is estimated respectively by the following cases:
(i) *If $b < 0$ and $J(t_0) < \min\left\{1, \sqrt{\frac{\mu}{-b}}\right\}$ then*

$$T^* \leq t_0 + \frac{1}{\sqrt{-b}} \ln \frac{\sqrt{\frac{\mu}{-b}}}{\sqrt{\frac{\mu}{-b}} - J(t_0)}.$$

(ii) *If $b = 0$, then*

$$T^* \leq t_0 + \frac{J(t_0)}{\sqrt{\mu}}.$$

(iii) *If $b > 0$, then*

$$T^* \leq \frac{J(t_0)}{\sqrt{\mu}},$$

or

$$T^* \leq t_0 + 2^{\frac{3\delta+1}{2\delta}} \frac{\delta c}{\sqrt{\mu}} \left(1 - [1 + cJ(t_0)]^{\frac{1}{2\delta}}\right),$$

where $c := \left(\frac{b}{\mu}\right)^{\delta/(2+\delta)}$.

3. Blowing up property

In this section we shall discus the Blow-up phenomena of problem (1.1) – (1.2). In order to state our results we make further assumptions on h :

$$h(s) \geq 0,\ h'(s) \leq 0 \text{ and } \int_0^\infty h(s)\,ds < \frac{p(p-2)}{(p-1)^2}\xi_0 < \xi_0,\ \text{for } p > 4. \qquad \mathbf{A}$$

Remark 3.1 The assumptions $\xi_0 - \int_0^\infty h(s)\,ds > 0$ is necessary to guarantee the hyperbolicity of the problem (1.1) – (1.2).

Definition 3.1 *A solution u of $(1.1)-(1.2)$ is called blow-up if there exists a finite time T^* such that*

$$\lim_{t\to T^{*-}}\left(\|\Delta u(t)\|^2_{L^2(\Omega)}\right)^{-1}=0. \tag{3.1}$$

Definite the energy function as

$$\begin{aligned}E(t)\ :\ &=\frac{1}{\gamma+2}\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}+\frac{1}{2}\left(\xi_0-\int_0^t h(s)\,ds\right)\|\Delta u(t)\|^2_{L^2(\Omega)}\\&+\frac{\xi_1}{4}\|\Delta u(t)\|^4_{L^2(\Omega)}+\frac{1}{2}\|\Delta u_t(t)\|^2_{L^2(\Omega)}+\frac{1}{2}(h\Box\Delta u)(t)\\&-\frac{1}{p}\|u(t)\|^p_{L^p(\Omega)},\end{aligned} \tag{3.2}$$

where

$$(h\Box\Delta u)(t):=\int_\Omega\int_0^t g(t-s)\,|\Delta u(x,t)-\Delta u(x,s)|^2\,dsdx.$$

Lemma 3.1 *Let (u,v) be the solution of problem $(1.1)-(1.2)$ then $E(t)$ is a non-increasing function on $[0,t)$ and*

$$\begin{aligned}\frac{d}{dt}[E(t)]\ &=\ \frac{1}{2}(h'\Box\Delta u)(t)-\frac{1}{2}h(t)\|\Delta u(t)\|^2_{L^2(\Omega)}\\&\quad-\|u_t(t)\|^2_{L^2(\Omega)}-\frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|\Delta u(t)\|^2_{L^2(\Omega)}\right\}\right)^2\\&\leq\ 0.\end{aligned} \tag{3.3}$$

Proof: Multiplying (1.1) by $u_t(t)$ and integration over Ω, we have

$$\begin{aligned}&\left(|u_t(t)|^\gamma u_{tt}(t),u_t(t)\right)_{L^2(\Omega)}+\left(M(t)\Delta^2u(t),u_t(t)\right)_{L^2(\Omega)}\\&+\left(\Delta^2u_{tt}(t),u_t(t)\right)_{L^2(\Omega)}-\left(\int_0^t h(t-s)\Delta^2u(s)ds,u_t(t)\right)_{L^2(\Omega)}\\&+\left(u_t(t),u_t(t)\right)_{L^2(\Omega)}\\=\ &\left(|u(t)|^{p-2}u(t),u_t(t)\right)_{L^2(\Omega)}.\end{aligned} \tag{3.4}$$

Using

$$u_{tt}(t)\,u_t(t)=\frac{1}{2}\frac{d}{dt}\left\{u_t^2(t)\right\},$$

we get by a direct calculation

$$\begin{aligned}&\left(|u_t(t)|^\gamma u_{tt}(t),u_t(t)\right)_{L^2(\Omega)}\\=\ &\int_\Omega|u_t(t)|^\gamma\frac{1}{2}\frac{d}{dt}\left\{u_t^2(t)\right\}dx\\=\ &\frac{1}{2}\int_\Omega\frac{d}{dt}\left\{|u_t(t)|^\gamma u_t^2(t)\right\}dx-\frac{1}{2}\int_\Omega\frac{d}{dt}\left\{|u_t(t)|^\gamma\right\}u_t^2(t)\,dx\\=\ &\frac{1}{2}\frac{d}{dt}\left\{\int_\Omega|u_t(t)|^{\gamma+2}dx\right\}-\frac{\gamma}{2}\int_\Omega|u_t(t)|^{\gamma-2}\mathrm{u}_{tt}(t)\,\mathrm{u}_t(t)\,u_t^2(t)\,dx\\=\ &\frac{1}{2}\frac{d}{dt}\left\{\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}\right\}-\frac{\gamma}{2}\left(|u_t(t)|^\gamma u_{tt}(t),u_t(t)\right)_{L^2(\Omega)},\end{aligned}$$

then

$$\left(|u_t(t)|^\gamma u_{tt}(t),u_t(t)\right)_{L^2(\Omega)}=\frac{1}{\gamma+2}\frac{d}{dt}\left\{\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)}\right\}. \tag{3.5}$$

And using integration by parts, we have

$$
\begin{aligned}
& \left(M\left(t\right)\Delta^{2}u\left(t\right),u_{t}\left(t\right)\right)_{L^{2}(\Omega)} \\
= & \left(\xi_{0}+\xi_{1}\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}+\sigma\left(\Delta u\left(t\right),\Delta u_{t}\left(t\right)\right)_{L^{2}(\Omega)}\right)\int_{\Omega}\Delta^{2}u\left(t\right)u_{t}\left(t\right)dx \\
= & \left(\xi_{0}+\xi_{1}\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}+\sigma\left(\Delta u\left(t\right),\Delta u_{t}\left(t\right)\right)_{L^{2}(\Omega)}\right)\int_{\Omega}\Delta u\left(t\right)\Delta u_{t}\left(t\right)dx \\
= & \left(\xi_{0}+\xi_{1}\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}+\sigma\left(\Delta u\left(t\right),\Delta u_{t}\left(t\right)\right)_{L^{2}(\Omega)}\right)\frac{1}{2}\frac{d}{dt}\left\{\int_{\Omega}\left|\Delta u\left(t\right)\right|^{2}dx\right\} \\
= & \frac{\xi_{0}}{2}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\}+\frac{\xi_{1}}{2}\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\} \\
& +\frac{\sigma}{2}\left(\Delta u\left(t\right),\Delta u_{t}\left(t\right)\right)_{L^{2}(\Omega)}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\} \\
= & \frac{\xi_{0}}{2}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\}+\frac{\xi_{1}}{4}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{4}\right\} \\
& +\frac{\sigma}{4}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\}\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\} \\
= & \frac{d}{dt}\left\{\frac{1}{2}\left(\xi_{0}+\frac{\xi_{1}}{2}\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right)\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\} \\
& +\frac{\sigma}{4}\left(\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\}\right)^{2}. \qquad (3.6)
\end{aligned}
$$

Integrating by parts, we have

$$
\begin{aligned}
\left(\Delta^{2}u_{tt}\left(t\right),u_{t}\left(t\right)\right)_{L^{2}(\Omega)} & = \left(\Delta u_{tt}\left(t\right),\Delta u_{t}\left(t\right)\right)_{L^{2}(\Omega)} \\
& = \frac{1}{2}\frac{d}{dt}\left\{\left\Vert \Delta u_{t}\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\}. \qquad (3.7)
\end{aligned}
$$

Also integrating by parts, we get

$$
\begin{aligned}
& -\left(\int_{0}^{t}h(t-s)\Delta^{2}u(s)ds,u_{t}\left(t\right)\right)_{L^{2}(\Omega)} \\
= & -\int_{0}^{t}h(t-s)\left(\Delta u(s),\Delta u_{t}\left(t\right)\right)_{L^{2}(\Omega)}ds \\
= & -\int_{0}^{t}h\left(t-s\right)\left[\int_{\Omega}\Delta u\left(x,s\right)\Delta u_{t}\left(x,t\right)dx\right]ds,
\end{aligned}
$$

using

$$
-\Delta u\left(x,s\right)\Delta u_{t}\left(x,t\right)=\frac{1}{2}\frac{d}{dt}\left\{\left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^{2}\right\}-\frac{1}{2}\frac{d}{dt}\left\{\left|\Delta u\left(x,t\right)\right|^{2}\right\},
$$

then

$$
\begin{aligned}
& -\left(\int_{0}^{t}h(t-s)\Delta^{2}u(s)ds,u_{t}\left(t\right)\right)_{L^{2}(\Omega)} \\
= & \int_{0}^{t}h\left(t-s\right)\int_{\Omega}\left(\frac{1}{2}\frac{d}{dt}\left\{\left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^{2}\right\}\right)dxds \\
& -\int_{0}^{t}h\left(t-s\right)\int_{\Omega}\left(\frac{1}{2}\frac{d}{dt}\left\{\left|\Delta u\left(x,t\right)\right|^{2}\right\}\right)dxds \\
= & \frac{1}{2}\int_{0}^{t}h\left(t-s\right)\left(\frac{d}{dt}\left\{\int_{\Omega}\left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^{2}dx\right\}\right)ds \\
& -\frac{1}{2}\int_{0}^{t}h\left(t-s\right)\left(\frac{d}{dt}\left\{\left\Vert \Delta u\left(t\right)\right\Vert_{L^{2}(\Omega)}^{2}\right\}\right)ds. \qquad (3.8)
\end{aligned}
$$

Using the direct account, we find

$$\begin{aligned}
& \frac{1}{2}\int_0^t h\left(t-s\right)\frac{d}{dt}\left\{\int_\Omega \left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^2 dx\right\} ds \\
= & \frac{1}{2}\int_0^t \frac{d}{dt}\left\{h\left(t-s\right)\left(\int_\Omega \left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^2 dx\right)\right\} ds \\
& -\frac{1}{2}\int_0^t h'\left(t-s\right)\left(\int_\Omega \left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^2 dx\right) ds \\
= & \frac{1}{2}\frac{d}{dt}\left\{\int_0^t h\left(t-s\right)\int_\Omega \left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^2 dxds\right\} \\
& -\frac{1}{2}\int_0^t h'\left(t-s\right)\left(\int_\Omega \left|\Delta u\left(x,s\right)-\Delta u\left(x,t\right)\right|^2 dx\right) ds \\
= & \frac{1}{2}\frac{d}{dt}\left\{\left(h\Box\Delta u\right)\left(t\right)\right\}-\frac{1}{2}\left(h'\Box\Delta u\right)\left(t\right),
\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}
& -\frac{1}{2}\int_0^t h\left(t-s\right)\left(\frac{d}{dt}\left\{\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}\right\}\right) ds \\
= & -\frac{1}{2}\left(\int_0^t h\left(t-s\right)ds\right)\left(\frac{d}{dt}\left\{\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}\right\}\right) \\
= & -\frac{1}{2}\left(\int_0^t h\left(s\right)ds\right)\frac{d}{dt}\left\{\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}\right\} \\
= & -\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h\left(s\right)ds\right)\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}\right\}+\frac{1}{2}h\left(t\right)\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.10}$$

By replacement (3.9) and (3.10) into (3.8), we get

$$\begin{aligned}
& -\left(\int_0^t h(t-s)\Delta^2 u(s)ds, u_t\left(t\right)\right)_{L^2(\Omega)} \\
= & \frac{1}{2}\frac{d}{dt}\left\{\left(h\Box\Delta u\right)\left(t\right)\right\}-\frac{1}{2}\left(h'\Box\Delta u\right)\left(t\right) \\
& -\frac{1}{2}\frac{d}{dt}\left\{\left(\int_0^t h\left(s\right)ds\right)\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}\right\}+\frac{1}{2}h\left(t\right)\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)} \\
= & \frac{d}{dt}\left\{\frac{1}{2}\left(h\Box\Delta u\right)\left(t\right)-\frac{1}{2}\left(\int_0^t h\left(s\right)ds\right)\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}\right\} \\
& -\frac{1}{2}\left(h'\Box\Delta u\right)\left(t\right)+\frac{1}{2}h\left(t\right)\left\|\Delta u\left(t\right)\right\|^2_{L^2(\Omega)}.
\end{aligned} \tag{3.11}$$

And

$$\begin{aligned}
& \left(\left|u\left(t\right)\right|^{p-2}u\left(t\right), u_t\left(t\right)\right)_{L^2(\Omega)} \\
= & \frac{d}{dt}\left\{\frac{1}{p}\left\|u\left(t\right)\right\|^p_{L^p(\Omega)}\right\}.
\end{aligned} \tag{3.12}$$

By replacement $(3.5)-(3.7)$, (3.11) and (3.12) into (3.4), we get

$$\begin{aligned}
& \frac{1}{\gamma+2}\frac{d}{dt}\left\{\|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2}\right\} \\
& +\frac{d}{dt}\left\{\frac{1}{2}\left(\xi_0+\frac{\xi_1}{2}\|\Delta u(t)\|_{L^2(\Omega)}^2\right)\|\Delta u(t)\|_{L^2(\Omega)}^2\right\} \\
& +\frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|\Delta u(t)\|_{L^2(\Omega)}^2\right\}\right)^2+\frac{1}{2}\frac{d}{dt}\left\{\|\Delta u_t(t)\|_{L^2(\Omega)}^2\right\} \\
& +\frac{d}{dt}\left\{\frac{1}{2}(h\Box\Delta u)(t)-\frac{1}{2}\left(\int_0^t h(s)\,ds\right)\|\Delta u(t)\|_{L^2(\Omega)}^2\right\} \\
& -\frac{1}{2}(h'\Box\Delta u)(t)+\frac{1}{2}h(t)\|\Delta u(t)\|_{L^2(\Omega)}^2+\|u_t(t)\|_{L^2(\Omega)}^2 \\
= & \frac{d}{dt}\left\{\frac{1}{p}\|u(t)\|_{L^p(\Omega)}^p\right\}, \qquad (3.13)
\end{aligned}$$

then (3.13), is equivalent to

$$\begin{aligned}
& \frac{d}{dt}\left\{\frac{1}{\gamma+2}\|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2}+\frac{1}{2}\left(\xi_0-\int_0^t h(s)\,ds\right)\|\Delta u(t)\|_{L^2(\Omega)}^2\right. \\
& +\frac{\xi_1}{4}\|\Delta u(t)\|_{L^2(\Omega)}^4+\frac{1}{2}\|\Delta u_t(t)\|_{L^2(\Omega)}^2+\frac{1}{2}(h\Box\Delta u)(t) \\
& \left.-\frac{1}{p}\|u(t)\|_{L^p(\Omega)}^p\right\} \\
= & \frac{1}{2}(h'\Box\Delta u)(t)-\frac{1}{2}h(t)\|\Delta u(t)\|_{L^2(\Omega)}^2 \\
& -\|u_t(t)\|_{L^2(\Omega)}^2-\frac{\sigma}{4}\left(\frac{d}{dt}\left\{\|\Delta u(t)\|_{L^2(\Omega)}^2\right\}\right)^2, \qquad (3.14)
\end{aligned}$$

using (3.2) into (3.14), we get (3.3). Then the proof is complete. □

Remark 3.2 After integration (3.3) over $(0,t)$, we have

$$\begin{aligned}
E(t) = & E(0)+\frac{1}{2}\int_0^t (h'\Box\Delta u)(s)ds-\frac{1}{2}\int_0^t h(s)\|\Delta u(s)\|_{L^2(\Omega)}^2\,ds \\
& -\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2\,ds-\frac{\sigma}{4}\int_0^t\left(\frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^2\right\}\right)^2 ds. \qquad (3.15)
\end{aligned}$$

Now, let u be a solution of $(1.1)-(1.2)$ and define

$$H(t) := H_1(t)+H_2(t)+H_3(t), \qquad (3.16)$$

where

$$\begin{aligned}
H_1(t) : & = \left(|u_t(t)|^\gamma, u^2(t)\right)_{L^2(\Omega)}-\frac{\gamma}{2}\int_0^t\left(\left(|u_s(s)|^2\right)_s, |u_s(s)|^{\gamma-2}u^2(s)\right)_{L^2(\Omega)} ds \\
& +\frac{\gamma}{2}\int_0^T\left(\left|\left(|u_s(s)|^2\right)_s\right|, |u_s(s)|^{p-2}u^2(s)\right)_{L^2(\Omega)} ds \\
& +2(T-t)\int_0^T\left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^\gamma, u(s)u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds,
\end{aligned}$$

and

$$\begin{aligned}
H_2(t) \quad : \quad &= \frac{(\gamma+1)\sigma}{2}\left\{\int_0^t \|\Delta u(s)\|^4_{L^2(\Omega)} ds + (T-t)\|\Delta u(0)\|^4_{L^2(\Omega)}\right\} \\
&+ (\gamma+1)\|\Delta u(t)\|^2_{L^2(\Omega)} + (\gamma+1)\int_0^t \|u(t)\|^2_{L^2(\Omega)} ds \\
&+ (T-t)(\gamma+1)\int_0^T \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds,
\end{aligned}$$

and

$$\begin{aligned}
H_3(t) \quad : \quad &= k_1\int_0^t\int_0^s\int_0^\mu \left(\frac{d}{d\chi}\left\{\|\Delta u(\chi)\|^2_{L^2(\Omega)}\right\}\right)^2 d\chi d\mu ds \\
&+ k_1(T-t)\int_0^T\int_0^s \left(\frac{d}{d\mu}\left\{\|\Delta u(\mu)\|^2_{L^2(\Omega)}\right\}\right)^2 d\mu ds \\
&+ k_2\int_0^t\int_0^s \|\Delta u_\mu(\mu)\|^2_{L^2(\Omega)} d\mu ds + k_2(T-t)\int_0^T \|\Delta u_s(s)\|^2_{L^2(\Omega)} ds \\
&+ k_3\int_0^t\int_0^s\int_0^\mu \|u_\chi(\chi)\|^2_{L^2(\Omega)} d\chi d\mu ds + k_3(T-t)\int_0^T\int_0^s \|u_\mu(\mu)\|^2_{L^2(\Omega)} d\mu ds,
\end{aligned}$$

where $k_1,\ k_2$ and k_3 be determent.

Lemma 3.2 *Assume that* (**A**) *hold, then we have*

$$\begin{aligned}
&H''(t) - \left\{2 + \frac{2(\gamma+1)p}{(\gamma+2)}\right\}\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} \\
\geq\ & -2(\gamma+1)pE(0) + \{k_3 + 2(\gamma+1)p\}\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds \\
&+ \left\{k_1 + \frac{(\gamma+1)p\sigma}{2}\right\}\int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2 ds \\
&+ \{(\gamma+1)(2+p) + k_2\}\|\Delta u_t(t)\|^2_{L^2(\Omega)}. \qquad (3.17)
\end{aligned}$$

Proof: Form (3.16), we have

$$H'(t) = H_1'(t) + H_2'(t) + H_3'(t), \qquad (3.18)$$

by differentiation of $H_1(t),\ H_2(t)$ and $H_3(t)$, we get

$$\begin{aligned}
H_1'(t) \quad &= \quad \gamma\left(|u_t(t)|^{\gamma-2} u_{tt}(t) u_t(t), u^2(t)\right)_{L^2(\Omega)} + 2\left(|u_t(t)|^\gamma, u(t) u_t(t)\right)_{L^2(\Omega)} \\
&- \frac{\gamma}{2}\left(\left(|u_t(t)|^2\right)_t, |u_t(t)|^{\gamma-2} u^2(t)\right)_{L^2(\Omega)} \\
&- 2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^\gamma, u(s) u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
&= \quad 2\left(|u_t(t)|^\gamma, u(t) u_t(t)\right)_{L^2(\Omega)} \\
&- 2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^\gamma, u(s) u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds, \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
H_2'(t) \quad &= \quad \frac{(\gamma+1)\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\} ds \\
&+ 2(\gamma+1)\left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)} + (\gamma+1)\|u(t)\|^2_{L^2(\Omega)} \\
&- (\gamma+1)\int_0^T \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds, \qquad (3.20)
\end{aligned}$$

and

$$
\begin{aligned}
H_3'(t) &= k_1 \int_0^t \int_0^s \left(\frac{d}{d\mu} \left\{ \|\Delta u(\mu)\|^2_{L^2(\Omega)} \right\} \right)^2 d\mu ds \\
&\quad - k_1 \int_0^T \int_0^s \left(\frac{d}{d\mu} \left\{ \|\Delta u(\mu)\|^2_{L^2(\Omega)} \right\} \right)^2 d\mu ds \\
&\quad + k_2 \int_0^t \|\Delta u_s(s)\|^2_{L^2(\Omega)} ds - k_2 \int_0^T \|\Delta u_s(s)\|^2_{L^2(\Omega)} ds \\
&\quad + k_3 \int_0^t \int_0^s \|u_\mu(\mu)\|^2_{L^2(\Omega)} d\mu ds - k_3 \int_0^T \int_0^s \|u_\mu(\mu)\|^2_{L^2(\Omega)} d\mu ds. \qquad (3.21)
\end{aligned}
$$

By replacement $(3.19) - (3.21)$ into (3.18), we get

$$
\begin{aligned}
H'(t) &= 2\left(|u_t(t)|^\gamma, u(t) u_t(t)\right)_{L^2(\Omega)} \\
&\quad - 2 \int_0^T \left| \frac{d}{ds} \left\{ \left| \left(|u_s(s)|^\gamma, u(s) u_s(s)\right)_{L^2(\Omega)} \right| \right\} \right| ds \\
&\quad + \frac{(\gamma+1)\sigma}{2} \int_0^t \frac{d}{ds} \left\{ \|\Delta u(s)\|^4_{L^2(\Omega)} \right\} ds + 2(\gamma+1)\left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)} \\
&\quad + (\gamma+1)\|u(t)\|^2_{L^2(\Omega)} - (\gamma+1) \int_0^T \left| \frac{d}{ds} \left\{ \|u(s)\|^2_{L^2(\Omega)} \right\} \right| ds \\
&\quad + k_1 \int_0^t \int_0^s \left(\frac{d}{d\mu} \left\{ \|\Delta u(\mu)\|^2_{L^2(\Omega)} \right\} \right)^2 d\mu ds \\
&\quad - k_1 \int_0^T \int_0^s \left(\frac{d}{d\mu} \left\{ \|\Delta u(\mu)\|^2_{L^2(\Omega)} \right\} \right)^2 d\mu ds \\
&\quad + k_2 \int_0^t \|\Delta u_s(s)\|^2_{L^2(\Omega)} ds - k_2 \int_0^T \|\Delta u_s(s)\|^2_{L^2(\Omega)} ds \\
&\quad + k_3 \int_0^t \int_0^s \|u_\mu(\mu)\|^2_{L^2(\Omega)} d\mu ds - k_3 \int_0^T \int_0^s \|u_\mu(\mu)\|^2_{L^2(\Omega)} d\mu ds, \qquad (3.22)
\end{aligned}
$$

then (3.22) is equivalent to

$$
\begin{aligned}
H''(t) &= 2(\gamma+1)\left(|u_t(t)|^\gamma u_{tt}(t), u(t)\right)_{L^2(\Omega)} + 2\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} \\
&\quad + \frac{(\gamma+1)\sigma}{2} \frac{d}{dt} \left\{ \|\Delta u(t)\|^4_{L^2(\Omega)} \right\} + 2(\gamma+1) \frac{d}{dt} \left\{ \left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)} \right\} \\
&\quad + 2(\gamma+1)\left(u(t), u_t(t)\right)_{L^2(\Omega)} \\
&\quad + k_1 \int_0^t \left(\frac{d}{ds} \left\{ \|\Delta u(s)\|^2_{L^2(\Omega)} \right\} \right)^2 ds + k_2 \|\Delta u_t(t)\|^2_{L^2(\Omega)} \\
&\quad + k_3 \int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds. \qquad (3.23)
\end{aligned}
$$

By using Esq. (1.1) – (1.2), we get

$$\begin{aligned}
& 2(\gamma+1)\left(|u_t(t)|^{\gamma} u_{tt}(t), u(t)\right)_{L^2(\Omega)} \\
= & -2(\gamma+1)\xi_0 \|\Delta u(t)\|^2_{L^2(\Omega)} - 2(\gamma+1)\xi_1 \|\Delta u(t)\|^4_{L^2(\Omega)} \\
& -\frac{(\gamma+1)\sigma}{2}\frac{d}{dt}\left\{\|\Delta u(t)\|^4_{L^2(\Omega)}\right\} \\
& -2(\gamma+1)\frac{d}{dt}\left\{(\Delta u_t(t), \Delta u(t))_{L^2(\Omega)}\right\} + 2(\gamma+1)\|\Delta u_t(t)\|^2_{L^2(\Omega)} \\
& +2(\gamma+1)\int_\Omega \int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx \\
& -2(\gamma+1)(u(t), u_t(t))_{L^2(\Omega)}. \qquad (3.24)
\end{aligned}$$

By replacement (3.24) into (3.23), we get

$$\begin{aligned}
H''(t) = & -2(\gamma+1)\xi_0 \|\Delta u(t)\|^2_{L^2(\Omega)} - 2(\gamma+1)\xi_1 \|\Delta u(t)\|^4_{L^2(\Omega)} \\
& + \{2(\gamma+1)+k_2\}\|\Delta u_t(t)\|^2_{L^2(\Omega)} \\
& +2(\gamma+1)\int_\Omega \int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx \\
& +2\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} \\
& +k_1\int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2 ds + k_3\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds. \qquad (3.25)
\end{aligned}$$

And multiplying Esq. (3.15) by $2(\gamma+1)p$ summing up in (3.25), we obtain

$$\begin{aligned}
& H''(t) \\
= & -2(\gamma+1)\xi_0 \|\Delta u(t)\|^2_{L^2(\Omega)} - 2(\gamma+1)\xi_1 \|\Delta u(t)\|^4_{L^2(\Omega)} \\
& + \{2(\gamma+1)+k_2\}\|\Delta u_t(t)\|^2_{L^2(\Omega)} \\
& +2(\gamma+1)\int_\Omega \int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx + 2\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} \\
& +k_1\int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2 ds + k_3\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds \\
& +\frac{2(\gamma+1)p}{(\gamma+2)}\|u_t(t)\|^{\gamma+2}_{L^{\gamma+2}(\Omega)} + (\gamma+1)p\left(\xi_0 - \int_0^t h(s)ds\right)\|\Delta u(t)\|^2_{L^2(\Omega)} \\
& +\frac{(\gamma+1)p\xi_1}{2}\|\Delta u(t)\|^4_{L^2(\Omega)} + (\gamma+1)p\|\Delta u_t(t)\|^2_{L^2(\Omega)} + \frac{2(\gamma+1)k}{p}\|u(t)\|^p_{L^p(\Omega)} \\
& +(\gamma+1)p(h\Box\Delta u)(t) - 2(\gamma+1)pE(0) - (\gamma+1)p\int_0^t (h'\Box\Delta u)(s)ds \\
& +(\gamma+1)p\int_0^t h(s)\|\Delta u(s)\|^2_{L^2(\Omega)} ds + 2(\gamma+1)p\int_0^t \|u_s(s)\|^2_{L^2(\Omega)} ds \\
& +\frac{(\gamma+1)p\sigma}{2}\int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2 ds, \qquad (3.26)
\end{aligned}$$

then (3.26) is equivalent to

$$
\begin{aligned}
& H''(t) - \left\{2 + \frac{2(\gamma+1)p}{(\gamma+2)}\right\} \|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} \\
= \;& -2(\gamma+1)pE(0) + \{k_3 + 2(\gamma+1)p\} \int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 ds \\
& + \left\{k_1 + \frac{(\gamma+1)p\sigma}{2}\right\} \int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^2\right\}\right)^2 ds \\
& + \{(\gamma+1)(2+p) + k_2\} \|\Delta u_t(t)\|_{L^2(\Omega)}^2 \\
& + \left\{-2(\gamma+1)\xi_0 + (\gamma+1)p\left(\xi_0 - \int_0^t h(s)\,ds\right)\right\} \|\Delta u(t)\|_{L^2(\Omega)}^2 \\
& + \left\{-2(\gamma+1)\xi_1 + \frac{(\gamma+1)p\xi_1}{2}\right\} \|\Delta u(t)\|_{L^2(\Omega)}^4 \\
& + 2(\gamma+1)\int_\Omega \int_0^t h(t-s)\,\Delta u(x,t)\Delta u(x,s)dsdx + (\gamma+1)p(h\Box\Delta u)(t) \\
& + \frac{2(\gamma+1)k}{p} \|u(t)\|_{L^p(\Omega)}^p - (\gamma+1)p\int_0^t (h'\Box\Delta u)(s)ds \\
& + (\gamma+1)p\int_0^t h(s)\|\Delta u(s)\|_{L^2(\Omega)}^2 ds.
\end{aligned}
$$

And using

$$
\begin{cases}
\dfrac{2(\gamma+1)k}{p} \|u(t)\|_{L^p(\Omega)}^p \geq 0, \\[2ex]
-(\gamma+1)p\int_0^t (h'\Box\Delta u)(s)ds \geq 0, \\[2ex]
(\gamma+1)p\int_0^t h(s)\|\Delta u(s)\|_{L^2(\Omega)}^2 ds \geq 0,
\end{cases}
$$

we get

$$
\begin{aligned}
& H''(t) - \left\{2 + \frac{2(\gamma+1)p}{(\gamma+2)}\right\} \|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} \\
\geq \;& -2(\gamma+1)pE(0) + \{k_3 + 2(\gamma+1)p\} \int_0^t \|u_s(s)\|_{L^2(\Omega)}^2 ds \\
& + \left\{k_1 + \frac{(\gamma+1)p\sigma}{2}\right\} \int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^2\right\}\right)^2 ds \\
& + \{(\gamma+1)(2+p) + k_2\} \|\Delta u_t(t)\|_{L^2(\Omega)}^2 \\
& + \left\{-2(\gamma+1)\xi_0 + (\gamma+1)p\left(\xi_0 - \int_0^t h(s)\,ds\right)\right\} \|\Delta u(t)\|_{L^2(\Omega)}^2 \\
& + \left\{-2(\gamma+1)\xi_1 + \frac{(\gamma+1)p\xi_1}{2}\right\} \|\Delta u(t)\|_{L^2(\Omega)}^4 \\
& + 2(\gamma+1)\int_\Omega \int_0^t h(t-s)\,\Delta u(x,t)\Delta u(x,s)dsdx \\
& + (\gamma+1)p(h\Box\Delta u)(t). \qquad (3.27)
\end{aligned}
$$

Using Young's inequality (for $\varepsilon = p$) , we have

$$\begin{aligned}
& \int_\Omega \int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx \\
= & \int_0^t \int_\Omega h(t-s)\Delta u(t)\left[\Delta u(s) - \Delta u(t) + \Delta u(t)\right] dxds \\
= & \int_0^t \int_\Omega h(t-s)\Delta u(t)\left(\Delta u(s) - \Delta u(t)\right) dxds \\
& + \int_0^t \int_\Omega h(t-s)\left|\Delta u(t)\right|^2 dxds \\
= & \int_0^t \int_\Omega \left[\sqrt{h(t-s)}\Delta u(t)\right]\left[\sqrt{h(t-s)}\left(\Delta u(s) - \Delta u(t)\right)\right] dxds \\
& + \int_0^t h(t-s)\, ds \left(\int_\Omega \left|\Delta u(t)\right|^2 dx\right) \\
\geq & -\frac{1}{2p}\int_0^t \int_\Omega h(t-s)\left|\Delta u(t)\right|^2 dxds - \frac{p}{2}\int_0^t \int_\Omega h(t-s)\left|\Delta u(s) - \Delta u(t)\right|^2 dxds \\
& + \left(\int_0^t h(s)\, ds\right)\left\|\Delta u(t)\right\|_{L^2(\Omega)}^2 \\
= & -\frac{1}{2p}\left(\int_0^t h(t-s)\, ds\right)\left(\int_\Omega \left|\Delta u(t)\right|^2 dx\right) - \frac{p}{2}\left(h\Box\Delta u\right)(t) \\
& + \left(\int_0^t h(s)\, ds\right)\left\|\Delta u(t)\right\|_{L^2(\Omega)}^2 \\
= & \left(1 - \frac{1}{2p}\right)\left(\int_0^t h(s)\, ds\right)\left\|\Delta u(t)\right\|_{L^2(\Omega)}^2 - \frac{p}{2}\left(h\Box\Delta u\right)(t),
\end{aligned}$$

then

$$\begin{aligned}
& 2(\gamma+1)\int_\Omega \int_0^t h(t-s)\Delta u(x,t)\Delta u(x,s)dsdx \\
\geq & (\gamma+1)\left(2 - \frac{1}{p}\right)\left(\int_0^t h(s)\, ds\right)\left\|\Delta u(t)\right\|_{L^2(\Omega)}^2 \\
& -(\gamma+1)\, p\left(h\Box\Delta u\right)(t).
\end{aligned} \tag{3.28}$$

By using (3.28) into (3.27) , we get

$$\begin{aligned}
& H''(t) - \left\{2 + \frac{2(\gamma+1)p}{(\gamma+2)}\right\}\left\|u_t(t)\right\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} \\
\geq & -2(\gamma+1)pE(0) + \left\{k_3 + 2(\gamma+1)p\right\}\int_0^t \left\|u_s(s)\right\|_{L^2(\Omega)}^2 ds \\
& + \left\{k_1 + \frac{(\gamma+1)p\sigma}{2}\right\}\int_0^t \left(\frac{d}{ds}\left\{\left\|\Delta u(s)\right\|_{L^2(\Omega)}^2\right\}\right)^2 ds \\
& + \left\{(\gamma+1)(2+p) + k_2\right\}\left\|\Delta u_t(t)\right\|_{L^2(\Omega)}^2 \\
& + (\gamma+1)\left\{\xi_0(p-2) - \frac{(p-1)^2}{p}\left(\int_0^t h(s)\, ds\right)\right\}\left\|\Delta u(t)\right\|_{L^2(\Omega)}^2 \\
& + \frac{(\gamma+1)\xi_1(p-4)}{2}\left\|\Delta u(t)\right\|_{L^2(\Omega)}^4 .
\end{aligned}$$

using $(\mathbf{A})$, we get

$$\left\{ \xi_0 (p-2) - \frac{(p-1)^2}{p} \left(\int_0^t h(s)\, ds \right) \right\} > 0,$$

and for $p > 4$, we get (3.17). Then the proof is complete. $\square$

Lemma 3.3 *Assume that* $(\mathbf{A})$ *hold and that either one of the following condition is satisfied*
$(\mathbf{i})$ $E(0) < 0$,
$(\mathbf{ii})$ $E(0) = 0$, *and*

$$H'(0) > 2 \left| \left(|u_t(0)|^{\gamma} u_t(0), u(0) \right)_{L^2(\Omega)} \right| + (\gamma+1) \|u(0)\|^2_{L^2(\Omega)}. \tag{3.29}$$

$(\mathbf{iii})$ $E(0) > 0$ *and*

$$\begin{aligned} H'(0) \quad > \quad & r_2 \left[H(0) + \frac{(\gamma+2) k_0}{2(\gamma+1) p} \right] \\ & + 2 \left| \left(|u_t(0)|^{\gamma} u_t(0), u(0) \right)_{L^2(\Omega)} \right| + (\gamma+1) \|u(0)\|^2_{L^2(\Omega)}, \end{aligned} \tag{3.30}$$

where

$$r_2 := \frac{\sqrt{(\gamma+1)p}}{(\gamma+2)} \left\{ \sqrt{(\gamma+1)p} - \sqrt{\gamma(p-2) + (p-4)} \right\},$$

and

$$k_0 := 2(\gamma+1) p E(0) + \frac{2(\gamma+1)^2 p}{(\gamma+2)} \|u(0)\|^2_{L^2(\Omega)}. \tag{3.31}$$

Then

$$H'(t) \geq 2 \left| \left(|u_t(0)|^{\gamma} u_t(0), u(0) \right)_{L^2(\Omega)} \right| + (\gamma+1) \|u(0)\|^2_{L^2(\Omega)},$$

for $t > t_0$, *and*

$$t^* := \max \left\{ 0, \frac{\begin{array}{c} H'(0) - 2 \left| \left(|u_t(0)|^{\gamma} u_t(0), u(0) \right)_{L^2(\Omega)} \right| \\ - (\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \end{array}}{2(\gamma+1) p E(0)} \right\}. \tag{3.32}$$

Where $t_0 = t^*$ *in case* $(\mathbf{i})$ *and* $t_0 = 0$ *in case* $(\mathbf{ii})$ *and* $(\mathbf{iii})$.

Proof: $(\mathbf{i})$ If $E(0) < 0$, by using (3.17), we get

$$H''(t) \geq -2(\gamma+1) p E(0), \tag{3.33}$$

by integration (3.33) for $(0, t)$, we get

$$\int_0^t H''(s)\, ds \geq \int_0^t -2(\gamma+1) p E(0)\, ds,$$

then

$$H'(t) - H'(0) \geq -2(\gamma+1) p E(0)\, t, \tag{3.34}$$

then (3.34) it is written in form

$$\begin{aligned} & H'(t) \\ \geq \quad & 2 \left| \left(|u_t(0)|^{\gamma} u_t(0), u(0) \right)_{L^2(\Omega)} \right| + (\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \\ & + \left\{ -2(\gamma+1) p E(0)\, t + H'(0) - 2 \left| \left(|u_t(0)|^{\gamma} u_t(0), u(0) \right)_{L^2(\Omega)} \right| \right. \\ & \left. - (\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \right\}. \end{aligned} \tag{3.35}$$

Let

$$t \geq \frac{H'(0) - 2\left|\left(|u_t(0)|^{\gamma} u_t(0), u(0)\right)_{L^2(\Omega)}\right| - (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2}{2(\gamma+1)pE(0)}, \tag{3.36}$$

using (3.36) into (3.35) and using $t \geq 0$, then for $t \geq t^*$, où t^* definite in (3.32), we get

$$H'(t) \geq 2\left|\left(|u_t(0)|^{\gamma} u_t(0), u(0)\right)_{L^2(\Omega)}\right| + (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2,$$

(**ii**) If $E(0) = 0$, then by using (3.17), we get for all $t \geq 0$

$$H''(t) \geq 0, \tag{3.37}$$

and by integration (3.37) for $(0,t)$, we get

$$H'(t) - H'(0) \geq 0, \tag{3.38}$$

then (3.38) it is written in form

$$\begin{aligned} & H'(t) \\ \geq\ & 2\left|\left(|u_t(0)|^{\gamma} u_t(0), u(0)\right)_{L^2(\Omega)}\right| + (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2 \\ & + \left\{H'(0) - 2\left|\left(|u_t(0)|^{\gamma} u_t(0), u(0)\right)_{L^2(\Omega)}\right| - (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2\right\}. \end{aligned}$$

For the more if (3.29) hold, then for $t \geq 0$, we get

$$H'(t) \geq 2\left|\left(|u_t(0)|^{\gamma} u_t(0), u(0)\right)_{L^2(\Omega)}\right| + (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2.$$

(**iii**) For the case that $E(0) > 0$, by using (3.22), we get

$$\begin{aligned} & H'(t) \\ \leq\ & 2\left(|u_t(t)|^{\gamma}, u(t)u_t(t)\right)_{L^2(\Omega)} \\ & + \frac{(\gamma+1)\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^4\right\} ds \\ & + 2(\gamma+1)\left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)} + (\gamma+1)\|u(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.39}$$

Using Young's inequality (for $\varepsilon = 1$), we get

$$\begin{aligned} & 2\left(|u_t(t)|^{\gamma}, u(t)u_t(t)\right)_{L^2(\Omega)} \\ \leq\ & \left(|u_t(t)|^{\gamma}, \left[u^2(t) + u_t^2(t)\right]\right)_{L^2(\Omega)} \\ =\ & \left(|u_t(t)|^{\gamma}, u^2(t)\right)_{L^2(\Omega)} + \left(|u_t(t)|^{\gamma}, u_t^2(t)\right)_{L^2(\Omega)} \\ =\ & \left(|u_t(t)|^{\gamma}, u^2(t)\right)_{L^2(\Omega)} + \|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2}. \end{aligned} \tag{3.40}$$

Using Young's inequality (for $\varepsilon = 1$), we get

$$\begin{aligned} & 2(\gamma+1)\left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)} \\ \leq\ & (\gamma+1)\|\Delta u(t)\|_{L^2(\Omega)}^2 + (\gamma+1)\|\Delta u_t(t)\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.41}$$

We first note that

$$\begin{aligned} \|u(t)\|_{L^2(\Omega)}^2 &= \|u(t)\|_{L^2(\Omega)}^2 - \|u_0\|_{L^2(\Omega)}^2 + \|u_0\|_{L^2(\Omega)}^2 \\ &= \int_0^t \frac{d}{ds}\left\{\|u(s)\|_{L^2(\Omega)}^2\right\} ds + \|u_0\|_{L^2(\Omega)}^2 \\ &= 2\int_0^t \left(u(s), u_s(s)\right)_{L^2(\Omega)} ds + \|u_0\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.42}$$

Using Young's inequality (for $\varepsilon = 1$) in (3.42), we have

$$
\begin{aligned}
&(\gamma+1)\|u(t)\|_{L^2(\Omega)}^2 \\
= \;& 2(\gamma+1)\int_0^t\int_\Omega [u(s)][u_s(s)]\,dxds + (\gamma+1)\|u_0\|_{L^2(\Omega)}^2 \\
\leq \;& (\gamma+1)\int_0^t\int_\Omega \left[u^2(s)+u_s^2(s)\right]dxds + (\gamma+1)\|u_0\|_{L^2(\Omega)}^2 \\
= \;& (\gamma+1)\int_0^t \|u(s)\|_{L^2(\Omega)}^2\,ds + (\gamma+1)\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2\,ds \\
&+(\gamma+1)\|u_0\|_{L^2(\Omega)}^2 . \qquad (3.43)
\end{aligned}
$$

Using (3.40), (3.41) and (3.43) into (3.39), we get

$$
\begin{aligned}
H'(t) \leq \;& \left(|u_t(t)|^\gamma, u^2(t)\right)_{L^2(\Omega)} + \|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} \\
&+\frac{(\gamma+1)\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^4\right\}ds \\
&+(\gamma+1)\|\Delta u(t)\|_{L^2(\Omega)}^2 + (\gamma+1)\|\Delta u_t(t)\|_{L^2(\Omega)}^2 \\
&+(\gamma+1)\int_0^t \|u(s)\|_{L^2(\Omega)}^2\,ds + (\gamma+1)\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2\,ds \\
&+(\gamma+1)\|u_0\|_{L^2(\Omega)}^2 . \qquad (3.44)
\end{aligned}
$$

Using (3.16) into (3.44), we get

$$
\begin{aligned}
H'(t) \leq \;& H(t) + \|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} + \frac{(\gamma+1)\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^4\right\}ds \\
&+(\gamma+1)\|\Delta u_t(t)\|_{L^2(\Omega)}^2 + (\gamma+1)\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2\,ds \\
&+(\gamma+1)\|u_0\|_{L^2(\Omega)}^2 - \frac{(\gamma+1)\sigma}{2}\int_0^t \|\Delta u(s)\|_{L^2(\Omega)}^4\,ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\frac{2(\gamma+1)p}{(\gamma+2)}\left\{H'(t)-H(t)\right\} \\
\leq \;& \frac{2(\gamma+1)p}{(\gamma+2)}\|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} + \frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^4\right\}ds \\
&+\frac{2(\gamma+1)^2 p}{(\gamma+2)}\|\Delta u_t(t)\|_{L^2(\Omega)}^2 + \frac{2(\gamma+1)^2 p}{(\gamma+2)}\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2\,ds \\
&+\frac{2(\gamma+1)^2 p}{(\gamma+2)}\|u_0\|_{L^2(\Omega)}^2 - \frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t \|\Delta u(s)\|_{L^2(\Omega)}^4\,ds. \qquad (3.45)
\end{aligned}
$$

And using (3.17), we get

$$
\begin{aligned}
&\frac{2(\gamma+1)p}{(\gamma+2)}\|u_t(t)\|_{L^{\gamma+2}(\Omega)}^{\gamma+2} \\
\leq \;& H''(t) + 2(\gamma+1)pE(0) - \left\{k_3 + 2(\gamma+1)p\right\}\int_0^t \|u_s(s)\|_{L^2(\Omega)}^2\,ds \\
&-\left\{k_1 + \frac{(\gamma+1)p\sigma}{2}\right\}\int_0^t \left(\frac{d}{ds}\left\{\|\Delta u(s)\|_{L^2(\Omega)}^2\right\}\right)^2 ds \\
&-\left\{(\gamma+1)(2+p)+k_2\right\}\|\Delta u_t(t)\|_{L^2(\Omega)}^2 . \qquad (3.46)
\end{aligned}
$$

Using (3.46) into (3.45), we get

$$\begin{aligned}
&\frac{2(\gamma+1)p}{(\gamma+2)}\{H'(t)-H(t)\} \\
\leq\ & H''(t)+2(\gamma+1)pE(0)+\frac{2(\gamma+1)^2 p}{(\gamma+2)}\|u_0\|^2_{L^2(\Omega)} \\
&-\left\{k_3+2(\gamma+1)p-\frac{2(\gamma+1)^2 p}{(\gamma+2)}\right\}\int_0^t \|u_s(s)\|^2_{L^2(\Omega)}\,ds \\
&-\left\{k_1+\frac{(\gamma+1)p\sigma}{2}\right\}\int_0^t\left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2 ds \\
&-\left\{(\gamma+1)(2+p)+k_2-\frac{2(\gamma+1)^2 p}{(\gamma+2)}\right\}\|\Delta u_t(t)\|^2_{L^2(\Omega)} \\
&+\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\} ds \\
&-\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t \|\Delta u(s)\|^4_{L^2(\Omega)}\,ds.
\end{aligned} \tag{3.47}$$

Let

$$\begin{cases}
k_1 := \dfrac{(\gamma+1)^2 p\sigma}{(\gamma+2)}, \\[2ex]
k_2 := \dfrac{2(\gamma+1)^2 p}{(\gamma+2)}, \\[2ex]
k_3 := \dfrac{2(\gamma+1)^2 p}{(\gamma+2)}.
\end{cases} \tag{3.48}$$

Using (3.48) into (3.47), we get

$$\begin{aligned}
&\frac{2(\gamma+1)p}{(\gamma+2)}\{H'(t)-H(t)\} \\
\leq\ & H''(t)+2(\gamma+1)pE(0)+\frac{2(\gamma+1)^2 p}{(\gamma+2)}\|u_0\|^2_{L^2(\Omega)} \\
&-\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t\left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2 ds \\
&+\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\} ds \\
&-\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)}\int_0^t \|\Delta u(s)\|^4_{L^2(\Omega)}\,ds.
\end{aligned} \tag{3.49}$$

Using

$$\begin{aligned}
& -\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)} \int_0^t \left(\frac{d}{ds} \left\{ \|\Delta u(s)\|_{L^2(\Omega)}^2 \right\} \right)^2 ds \\
& +\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)} \int_0^t \frac{d}{ds} \left\{ \|\Delta u(s)\|_{L^2(\Omega)}^4 \right\} ds \\
& -\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)} \int_0^t \|\Delta u(s)\|_{L^2(\Omega)}^4 ds \\
= & -\frac{(\gamma+1)^2 p\sigma}{(\gamma+2)} \left(\|\Delta u(s)\|_{L^2(\Omega)}^2 - \frac{d}{ds} \left\{ \|\Delta u(s)\|_{L^2(\Omega)}^2 \right\} \right)^2 \\
\leq & \ 0,
\end{aligned}$$

into (3.49), we get

$$\begin{aligned}
& \frac{2(\gamma+1)p}{(\gamma+2)} \{H'(t) - H(t)\} \\
\leq & \ H''(t) + 2(\gamma+1)pE(0) + \frac{2(\gamma+1)^2 p}{(\gamma+2)} \|u_0\|_{L^2(\Omega)}^2,
\end{aligned}$$

then

$$H''(t) - \frac{2(\gamma+1)p}{(\gamma+2)} H'(t) + \frac{2(\gamma+1)p}{(\gamma+2)} H(t) + k_0 \geq 0,$$

where k_0 is definite in (3.31).
Let

$$B(t) := H(t) + \frac{(\gamma+2)k_0}{2(\gamma+1)p},$$

then $B'(t) = H'(t)$, $B''(t) = H''(t)$, and $B(t)$ satisfaire

$$B''(t) - \frac{2(\gamma+1)p}{(\gamma+2)} B'(t) + \frac{2(\gamma+1)p}{(\gamma+2)} B(t) \geq 0. \tag{3.50}$$

Using **Lemma 1** in (3.50) and (3.30), then

$$H'(t) \geq 2\left|\left(|u_t(0)|^\gamma u_t(0), u(0)\right)_{L^2(\Omega)}\right| + (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2, \quad t \geq 0.$$

Then the proof of **lem 5** is completed. □

Theorem 3.1 *Assume that* (**A**) *holds and that either one of the following conditions is satisfied*
(**i**) $E(0) < 0$,
(**ii**) $E(0) = 0$ *and* (3.29) *holds,*
(**iii**) $0 < E(0) < \dfrac{\left(\frac{p-4}{4}\right)^2 \left[H'(t_0) - 2\left|\left(|u_t(0)|^\gamma, u(0)u_t(0)\right)_{L^2(\Omega)}\right| - (\gamma+1)\|u(0)\|_{L^2(\Omega)}^2\right]^2 \times J(t_0)^{\frac{1}{\xi}}}{(\gamma+1)((p-2)^2-4)\left(\frac{1}{2} - \frac{1}{p-2}\right)}$, *and* (3.30) *holds. Then the solution u blows–up at finite time* T^* *in the sense of* (3.1).
In case (**i**),

$$T^* \leq t_0 - \frac{J(t_0)}{J'(t_0)}.$$

Furthermore, if $J(t_0) < \min\left\{1, \sqrt{\frac{\mu}{-\beta}}\right\}$, *then we have*

$$T^* \le t_0 + \frac{1}{\sqrt{-\beta}} \ln \frac{\sqrt{\frac{\mu}{-\beta}}}{\sqrt{\frac{\mu}{-\beta}} - J(t_0)}.$$

In case (**ii**)

$$T^* \le t_0 - \frac{J(t_0)}{J'(t_0)},$$

or

$$T^* \le t_0 + \frac{J(t_0)}{J'(t_0)}.$$

In case (**iii**)

$$T^* \le \frac{J(t_0)}{\sqrt{\mu}},$$

or

$$T^* \le t_0 + 3^{\frac{3\xi+1}{2\xi}} \frac{\xi c}{\sqrt{\mu}} \left\{1 - [1 + cJ(t_0)]^{\frac{1}{2\xi}}\right\},$$

where $c := \left(\frac{\beta}{\mu}\right)^{\frac{\xi}{2+\xi}}$, $\xi := \frac{p-4}{4}$, *and* $J(t)$, μ *and* β *are given in* (3.51), (3.62) *and* (3.63) *respectively. Note that in* ***case*** (**i**), $t_0 = t^*$ *is given in* (3.32) *and* $t_0 = 0$ *in* ***case*** (**ii**) *and* (**iii**).

Proof: Let

$$\begin{aligned} J(t) \quad : \quad &= \Big\{ H(t) + 2(T-t) \left| \left(|u_t(0)|^\gamma, u(0) u_t(0) \right)_{L^2(\Omega)} \right| \\ &+ (T-t)(\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \Big\}^{-\xi}. \end{aligned} \tag{3.51}$$

Differentiating $J(t)$ twice, we obtain

$$J'(t) = -\xi J(t)^{1+\frac{1}{\xi}} \left[H'(t) - 2 \left| \left(|u_t(0)|^\gamma, u(0) u_t(0) \right)_{L^2(\Omega)} \right| - (\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \right].$$

Then

$$J''(t) = -\xi J(t)^{1+\frac{2}{\xi}} Q(t), \tag{3.52}$$

where

$$\begin{aligned} Q(t) \quad : \quad &= H''(t) \Big[H(t) + 2(T-t) \left| \left(|u_t(0)|^\gamma, u(0) u_t(0) \right)_{L^2(\Omega)} \right| \\ &+ (T-t)(\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \Big] \\ &- (1+\xi) \Big\{ H'(t) - 2 \left| \left(|u_t(0)|^\gamma, u(0) u_t(0) \right)_{L^2(\Omega)} \right| \\ &- (\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \Big\}^2. \end{aligned} \tag{3.53}$$

From (3.17), we have

$$\begin{aligned} &H''(t) \\ \ge \quad &-2(\gamma+1) pE(0) + \frac{(\gamma+1) p\sigma}{2} \int_0^t \left(\frac{d}{ds} \left\{ \|\Delta u(s)\|^2_{L^2(\Omega)} \right\} \right)^2 ds \\ &+ (\gamma+1) p \|\Delta u_t(t)\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.54}$$

From (3.22), we get

$$\begin{aligned}
& H'(t) - 2\left|\left(|u_t(0)|^{\gamma}, u(0)\, u_t(0)\right)_{L^2(\Omega)}\right| - (\gamma+1)\|u(0)\|^2_{L^2(\Omega)} \\
\leq\ & 2\left(|u_t(t)|^{\gamma}, u(t)\, u_t(t)\right)_{L^2(\Omega)} - 2\left|\left(|u_t(0)|^{\gamma}, u(0)\, u_t(0)\right)_{L^2(\Omega)}\right| \\
& -2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
& +\frac{(\gamma+1)\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\} ds \\
& +2(\gamma+1)\left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)} \\
& +(\gamma+1)\|u(t)\|^2_{L^2(\Omega)} - (\gamma+1)\|u(0)\|^2_{L^2(\Omega)} \\
& -(\gamma+1)\int_0^T \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds,
\end{aligned}$$

using

$$\begin{aligned}
& 2\left(|u_t(t)|^{\gamma}, u(t)\, u_t(t)\right)_{L^2(\Omega)} - 2\left|\left(|u_t(0)|^{\gamma}, u(0)\, u_t(0)\right)_{L^2(\Omega)}\right| \\
& -2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
\leq\ & 2\left|\left(|u_t(t)|^{\gamma}, u(t)\, u_t(t)\right)_{L^2(\Omega)}\right| - 2\left|\left(|u_t(0)|^{\gamma}, u(0)\, u_t(0)\right)_{L^2(\Omega)}\right| \\
& -2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
=\ & 2\int_0^t \frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\} ds \\
& -2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
\leq\ & 2\int_0^t \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
& -2\int_0^T \left|\frac{d}{ds}\left\{\left|\left(|u_s(s)|^{\gamma}, u(s)\, u_s(s)\right)_{L^2(\Omega)}\right|\right\}\right| ds \\
\leq\ & 0,
\end{aligned}$$

and

$$\begin{aligned}
& (\gamma+1)\|u(t)\|^2_{L^2(\Omega)} - (\gamma+1)\|u(0)\|^2_{L^2(\Omega)} \\
& -(\gamma+1)\int_0^T \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds \\
=\ & (\gamma+1)\int_0^t \frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\} ds - (\gamma+1)\int_0^T \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds \\
\leq\ & (\gamma+1)\int_0^t \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds - (\gamma+1)\int_0^T \left|\frac{d}{ds}\left\{\|u(s)\|^2_{L^2(\Omega)}\right\}\right| ds \\
\leq\ & 0,
\end{aligned}$$

we get

$$\begin{aligned}
& H'(t) - 2\left|\left(|u_t(0)|^{\gamma}, u(0)\, u_t(0)\right)_{L^2(\Omega)}\right| - (\gamma+1)\|u(0)\|^2_{L^2(\Omega)} \\
\leq\ & \frac{(\gamma+1)\sigma}{2}\int_0^t \frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\} ds + 2(\gamma+1)\left(\Delta u(t), \Delta u_t(t)\right)_{L^2(\Omega)},
\end{aligned}$$

then

$$\begin{aligned}
&\left\{H'(t)-2\left|\left(|u_t(0)|^{\gamma},u(0)\,u_t(0)\right)_{L^2(\Omega)}\right|-(\gamma+1)\|u(0)\|^2_{L^2(\Omega)}\right\}^2\\
\leq\ &\left\{\frac{(\gamma+1)\sigma}{2}\int_0^t\frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\}ds+2(\gamma+1)\left(\Delta u(t),\Delta u_t(t)\right)_{L^2(\Omega)}\right\}^2\\
=\ &4\left\{\frac{(\gamma+1)\sigma}{4}\int_0^t\frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\}ds+(\gamma+1)\left(\Delta u(t),\Delta u_t(t)\right)_{L^2(\Omega)}\right\}^2,
\end{aligned}\tag{3.55}$$

by multiplying (3.55) by $-(1+\xi)$, we get

$$\begin{aligned}
&-(1+\xi)\left\{H'(t)-2\left|\left(|u_t(0)|^{\gamma},u(0)\,u_t(0)\right)_{L^2(\Omega)}\right|-(\gamma+1)\|u(0)\|^2_{L^2(\Omega)}\right\}^2\\
\geq\ &-4(1+\xi)\left\{\frac{(\gamma+1)\sigma}{4}\int_0^t\frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\}ds\right.\\
&\left.+(\gamma+1)\left(\Delta u(t),\Delta u_t(t)\right)_{L^2(\Omega)}\right\}^2.
\end{aligned}\tag{3.56}$$

And using (3.16), we get

$$\begin{aligned}
&H(t)+2(T-t)\left|\left(|u_t(0)|^{\gamma},u(0)\,u_t(0)\right)_{L^2(\Omega)}\right|\\
&+(T-t)(\gamma+1)\|u(0)\|^2_{L^2(\Omega)}\\
\geq\ &\frac{(\gamma+1)\sigma}{2}\int_0^t\|\Delta u(s)\|^4_{L^2(\Omega)}ds+(\gamma+1)\|\Delta u(t)\|^2_{L^2(\Omega)}.
\end{aligned}\tag{3.57}$$

Using (3.54), (3.56) and (3.57) into (3.53), we get

$$\begin{aligned}
Q(t)\ \geq\ &-2(\gamma+1)pE(0)\left[H(t)+2(T-t)\left|\left(|u_t(0)|^{\gamma},u(0)\,u_t(0)\right)_{L^2(\Omega)}\right|\right.\\
&\left.+(T-t)(\gamma+1)\|u(0)\|^2_{L^2(\Omega)}\right]\\
&+p\left\{\frac{(\gamma+1)\sigma}{2}\int_0^t\left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2ds+(\gamma+1)\|\Delta u_t(t)\|^2_{L^2(\Omega)}\right\}\\
&\times\left[\frac{(\gamma+1)\sigma}{2}\int_0^t\|\Delta u(s)\|^4_{L^2(\Omega)}ds+(\gamma+1)\|\Delta u(t)\|^2_{L^2(\Omega)}\right]\\
&-4(1+\xi)\left\{\frac{(\gamma+1)\sigma}{4}\int_0^t\frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\}ds\right.\\
&\left.+(\gamma+1)\left(\Delta u(t),\Delta u_t(t)\right)_{L^2(\Omega)}\right\}^2.
\end{aligned}$$

Let us designate by

$$\begin{aligned}
\mathbf{A}\ &:=\frac{(\gamma+1)\sigma}{2}\int_0^t\|\Delta u(s)\|^4_{L^2(\Omega)}ds+(\gamma+1)\|\Delta u(t)\|^2_{L^2(\Omega)},\\
\mathbf{B}\ &:=\frac{(\gamma+1)\sigma}{4}\int_0^t\frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\}ds+(\gamma+1)\left(\Delta u(t),\Delta u_t(t)\right)_{L^2(\Omega)},\\
\mathbf{C}\ &:=\frac{(\gamma+1)\sigma}{2}\int_0^t\left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2ds+(\gamma+1)\|\Delta u_t(t)\|^2_{L^2(\Omega)}.
\end{aligned}$$

Thus, we get

$$Q(t)\geq-2(\gamma+1)pE(0)\,J(t)^{-\frac{1}{\xi}}+p\left\{\mathbf{AC}-\mathbf{B}^2\right\}.\tag{3.58}$$

Now we observe that, for all $(\rho,\eta)\in\mathbb{R}^2$ and $t>0$,

$$\begin{aligned}
&\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\\
=\;&\frac{(\gamma+1)\,\sigma}{2}\rho^2\int_0^t\|\Delta u(s)\|^4_{L^2(\Omega)}\,ds+(\gamma+1)\,\rho^2\,\|\Delta u(t)\|^2_{L^2(\Omega)}\\
&+\frac{(\gamma+1)\,\sigma}{2}\rho\eta\int_0^t\frac{d}{ds}\left\{\|\Delta u(s)\|^4_{L^2(\Omega)}\right\}ds+2\,(\gamma+1)\,\rho\eta\,(\Delta u(t)\,,\Delta u_t(t))_{L^2(\Omega)}\\
&+\frac{(\gamma+1)\,\sigma}{2}\eta^2\int_0^t\left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2ds+(\gamma+1)\,\eta^2\,\|\Delta u_t(t)\|^2_{L^2(\Omega)}\,.
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\\
=\;&\frac{(\gamma+1)\,\sigma}{2}\int_0^t\left\{\rho^2\,\|\Delta u(s)\|^4_{L^2(\Omega)}+2\left(\rho\,\|\Delta u(s)\|^2_{L^2(\Omega)}\times\eta\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)\right.\\
&\left.+\eta^2\left(\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right)^2\right\}ds\\
&+(\gamma+1)\int_\Omega\left\{\rho^2\,|\Delta u(t)|^2+2\rho\Delta u(t)\,\eta\Delta u_t(t)+\eta^2\,|\Delta u_t(t)|^2\right\}dx.
\end{aligned}$$

This identify camber written in the form

$$\begin{aligned}
&\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\\
=\;&\frac{(\gamma+1)\,\sigma}{2}\int_0^t\left\{\rho\,\|\Delta u(s)\|^2_{L^2(\Omega)}+\eta\frac{d}{ds}\left\{\|\Delta u(s)\|^2_{L^2(\Omega)}\right\}\right\}^2ds\\
&+(\gamma+1)\int_\Omega|\rho\Delta u(t)+\eta\Delta u_t(t)|^2\,dx,
\end{aligned}$$

it is easy to see that

$$\mathbf{A}\rho^2+2\mathbf{B}\rho\eta+\mathbf{C}\eta^2\geq 0,$$

and

$$\mathbf{A}\geq\mathbf{0},$$

then

$$\mathbf{B}^2-\mathbf{AC}\leq 0.$$

Hence, we obtain from (3.58) that

$$Q(t)\geq -2\,(\gamma+1)\,pE(0)\,J(t)^{-\frac{1}{\xi}}\,,\quad t\geq t_0. \tag{3.59}$$

Therefore by (3.52) and (3.59), we get

$$J''(t)\leq(\gamma+1)\left(\frac{(p-2)^2}{2}-2\right)E(0)\,J(t)^{1+\frac{1}{\xi}}\,,\quad t\geq t_0. \tag{3.60}$$

Note that by **Lemma 5** $J'(t)<0$ for $t\geq t_0$. Multiplying (3.60) by $J'(t)$, we get

$$J''(t)\,J'(t)\geq(\gamma+1)\left(\frac{(p-2)^2}{2}-2\right)E(0)\,J'(t)\,J(t)^{1+\frac{1}{\xi}}\,. \tag{3.61}$$

Integrating (3.61) from t_0 to t, we get

$$J'(t)^2\geq\mu+\beta J(t)^{2+\frac{1}{\xi}}\,,$$

where

$$
\begin{aligned}
\mu \quad : \quad &= \Bigg\{ \left(\frac{p-4}{4}\right)^2 \Big[H'(t_0) - 2 \left| \left(|u_t(0)|^{\gamma}, u(0)\, u_t(0) \right)_{L^2(\Omega)} \right| \\
&\quad - (\gamma+1) \|u(0)\|^2_{L^2(\Omega)} \Big]^2 \\
&\quad - (\gamma+1)\left((p-2)^2 - 4\right) \left(\frac{1}{2} - \frac{1}{p-2}\right) E(0)\, J(t_0)^{-\frac{1}{\gamma}} \Bigg\} \times J(t_0)^{2+\frac{2}{\gamma}} \\
> \quad & 0, \qquad (3.62)
\end{aligned}
$$

and

$$\beta := (\gamma+1)\left((p-2)^2 - 4\right)\left(\frac{1}{2} - \frac{1}{p-2}\right) E(0). \qquad (3.63)$$

Then by **Lemma 2** the proof of theorem is completed.
Hence there exist a finite time T^* such that $\lim_{t\to T^{*-}} \{J(t)\} = 0$ and the upper bounds of T^* are estimation respectively according to the sign of $E(0)$ (see **Lemma 2**). □

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