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A Quasistatic Electro-Elastic Contact Problem with Long Memory and Slip Dependent Coefficient of Friction*

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abstract: In this paper we consider a mathematical model which describes a quasistatic frictional contact problem between a deformable body and an obstacle, say a foundation. We assume that the behavior of the material is described by a linear electro-elastic constitutive law with long memory. The contact is modelled with a version of *Coulomb's* law of dry friction in which the normal stress is prescribed on the contact surface. Moreover, we consider a slip dependent coefficient of friction. We derive a variational formulation for the model, in the form of a coupled system for the displacements and the electric potential. Under a smallness assumption on the coefficient of friction, we prove an existence result of the weak solution of the model. We can show the uniqueness of the solution by adding another condition. The proofs are based on arguments of time-dependent variational inequalities, differential equations and *Banach* fixed point theorem.

Key Words: Electro-elastic material, quasistatic process, frictional contact, Coulomb's law, slip dependent friction, quasi-variational inequality, weak solution, fixed point.

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1. Introduction

Since frictional contact is so important in industry and in everyday life, there is a need to model and predict it accurately. However, the main industrial need is to effectively control the process of frictional contact. Currently, there is a considerable interest in frictional contact problems involving piezo-electric materials, see for instance [2], [6], [9], [10], [12], [19], [20], and [21]. Exellent refernce on analysis and numerical appximation of variational inequalities arising from frictional contact problems are [5] and [17].

A piezoelectric material is one that produces an electric charge when a mechanical stress is applied (the material is squeezed or stretched). Conversely, a mechanical deformation (the material shrinks or expands) is produced when an electric field is applied. This kind of materials appears usually in the industry as switches in radiotronics, electroacoustics or measuring equipments. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials, and those for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. Different models have been developed to describe the interaction between the electric and mechanical fields (see [1], [3], [8], [13]- [15], [24], [25]). General models for elastic materials with piezoelectric effect, called electro-elastic materials, can be found in [1], [8] and [22]. A static frictional contact problem for electric-elastic materials was considered in [2], [11] and a slip-dependent frictional contact problem for electro-elastic materials was studied in [20].

This paper is a contribution to the study of the contact problem for piezoelectric materials. In this work, we consider a mathematical model for frictional contact between a body assumed to be electro-elastic with long memory and an obstacle, say foundation. We model the contact with a version of

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Coulomb's law of dry friction in which the normal stress is prescribed on the contact surface and the coefficient of friction depends on the slip. The novelty in the present paper consists in the fact that the material's behavior is assumed to be electro-elastic with long memory. Note that the elastic contact problem is resolved in [4].

The paper is structured as follows. In Section 2 we present the electro-elastic contact model and provide comments on the contact boundary conditions. In Section 3 we list the assumptions on the data and derive the variational formulation. In section 4, we present our main existence results, where we can show the uniqueness of the solution by adding another condition.

2. Problem statement

We consider the following physical setting. An electro-elastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ with a smooth boundary $\partial\Omega = \Gamma$. The body is submitted to the action of body forces of density f_0 and volume electric charges of density q_0. It is also submitted to mechanical and electric constraints on the boundary. To describe them, we consider a partition of Γ into three measurable parts Γ_1, Γ_2 and Γ_3 on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand., such that $meas(\Gamma_1) > 0$, $meas(\Gamma a) > 0$. We assume that the body is clamped on Γ_1 and surface tractions of density f_2 act on Γ_2. On Γ_3 the body is in frictional contact with an insulator obstacle, the so-called foundation. We also assume that the electrical potential vanishes on Γ_a and a surface electric charge of density q_2 is prescribed on Γ_b. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and we use "." and $\|\ \|$ for the inner product and the Euclidean norm on $\mathbb{R}^d$ and $\mathbb{S}^d$, respectively. Also, below ν represents the unit outward normal on Γ. With these assumptions, the classical model for the process is the following.

Problem 1. *Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to \mathbb{S}^d$, an electric potential field $\varphi : \Omega \times [0,T] \to \mathbb{R}$, and an electric displacement field $D : \Omega \times [0,T] \to \mathbb{R}^d$ such that :*

$$\sigma = \mathcal{F}\varepsilon(u) + \int_0^t K(t-s)\varepsilon(u)ds - \mathcal{E}^*E(\varphi) \qquad in\ \Omega \times (0,T), \tag{2.1}$$

$$D = \mathcal{B}E(\varphi) + \mathcal{E}\varepsilon(u) \qquad in\ \Omega \times (0,T), \tag{2.2}$$

$$Div\sigma + f_0 = 0 \qquad in\ \Omega \times (0,T), \tag{2.3}$$

$$divD = q_0 \qquad in\ \Omega \times (0,T), \tag{2.4}$$

$$u = 0 \qquad on\ \Gamma_1 \times (0,T), \tag{2.5}$$

$$\sigma\nu = f_2 \qquad on\ \Gamma_2 \times (0,T), \tag{2.6}$$

$$\sigma_\nu = \boldsymbol{S} \qquad on\ \Gamma_3 \times (0,T), \tag{2.7}$$

$$\begin{cases} \| \sigma_\tau \| \leq \mu(\| u_\tau \|) \mid \boldsymbol{S} \mid \\ \| \sigma_\tau \| < \mu(\| u_\tau \|) \mid \boldsymbol{S} \mid \Longrightarrow \dot{u}_\tau = 0 \\ \| \sigma_\tau \| = \mu(\| u_\tau \|) \mid \boldsymbol{S} \mid \Rightarrow \exists\lambda \geq 0,\ \sigma_\tau = -\lambda\dot{u}_\tau \end{cases} \qquad on\ \Gamma_3 \times (0,T), \tag{2.8}$$

$$\varphi = 0 \qquad on\ \Gamma_a \times (0,T), \tag{2.9}$$

$$D \cdot \nu = q_2 \qquad on\ \Gamma_b \times (0,T), \tag{2.10}$$

$$u(0) = u_0 \qquad in\ \Omega. \tag{2.11}$$

We now provide some comments on equations and conditions (2.1)–(2.11). Equations (2.1) and (2.2) represent the electro-elastic constitutive law with long memory of the material such that: $\mathcal{F} = (\mathcal{F}_{ijkl})$ is a $4th$ rank tensor, called the elastic tensor and its components $\mathcal{F}_{ijkl}$ are called coefficients of elasticity; $\varepsilon(u)$ denotes the linearized strain tensor; $\int_0^t K(t-s)\varepsilon(u)ds$ is the memory term in which K denotes the tensor of relaxation; the stress $\sigma(t)$ at current instant t depends on the whole history of strains up to this moment of time; $E(\varphi) = -\nabla\varphi$ is the electric field, where φ is the electric potential, $\mathcal{E}$ represents the piezoelectric operator, $\mathcal{E}^*$ is its transposed, $\mathcal{B}$ denotes the electric permittivity operator, and $D = (D_1, \dots, D_d)$ is the electric displacement vector. Details on the constitutive equations of the form (2.1) and (2.2) can be find, for instance, in [1], [2] and in [23]. Next, equations (2.3) and (2.4) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "*Div*" and "*div*" denote the divergence operator for tensor and vector valued functions, respectively. Equations (2.5) and (2.6) represent the displacement and traction boundary conditions. Conditions (2.9) and (2.10) represent the electric boundary conditions. Condition (2.7) states that the normal stress σ_ν is prescribed on the contact surface, since $\mathbf{S}$ is a given data. Such kind conditions arise in the study of some mechanisms and were already used in [7,18]. Condition (2.8) represents the Coulomb's law of dry friction, where σ_τ is the tangential stress, u_τ, $\dot{u}_\tau$ are the tangential displacement and velocity, respectively. The function μ, which assumed to depend on the slip $\| u_\tau \|$, is the coefficient of friction. When the strong inequality holds the surface of the body adheres to the foundation and is in the so-called *stick* state and when equality holds, there is relative sliding, the so-called *slip* state. Here and below in this paper, a dot above a function represents the derivative with respect to the time variable. Finally, (2.11) represent the initial condition where u_0 is given.

3. Variational formulation and preliminaries

In this section, we list the assumptions on the data and derive a variational formulation for the contact problem. To this end we need to introduce some notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$u \cdot \upsilon = u_i \upsilon_i \ , \quad \|\upsilon\| = (\upsilon \cdot \upsilon)^{\frac{1}{2}}, \quad \forall\, u, \upsilon \in \mathbb{R}^d,$$
$$\sigma \cdot \tau = \sigma_{ij} \tau_{ij} \ , \quad \|\tau\| = (\tau \cdot \tau)^{\frac{1}{2}}, \quad \forall\, \sigma, \tau \in \mathbb{S}^d.$$

Here and everywhere in this paper, i, j, k, l run from 1 to d, summation over repeated indices is applied and the index that follows a comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g. $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$.

Everywhere below, we use the classical notation for L^p and *Sobolev* spaces associated to Ω and Γ. Moreover, we use the notation $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ for the following spaces

$$L^2(\Omega)^d = \{\ \upsilon = (\upsilon_i) \mid \upsilon_i \in L^2(\Omega)\ \},\ H^1(\Omega)^d = \{\ \upsilon = (\upsilon_i) \mid \upsilon_i \in H^1(\Omega)\ \},$$
$$\mathcal{H} = \{\ \tau = (\tau_{ij}) \mid \tau_{ij} = \tau_{ji} \in L^2(\Omega)\ \}, \quad \mathcal{H}_1 = \{\ \tau \in \mathcal{H} \mid \tau_{ij,j} \in L^2(\Omega)\ \}.$$

The spaces $L^2(\Omega)^d$, $H^1(\Omega)^d$, $\mathcal{H}$ and $\mathcal{H}_1$ are real *Hilbert* spaces endowed with the canonical inner products given by

$$(u, \upsilon)_{L^2(\Omega)^d} = \int_\Omega u \cdot \upsilon \, dx, \quad (u, \upsilon)_{H^1(\Omega)^d} = \int_\Omega u \cdot \upsilon \, dx + \int_\Omega \nabla u \cdot \nabla \upsilon \, dx,$$

$$(\sigma, \tau)_{\mathcal{H}} = \int_\Omega \sigma \cdot \tau \, dx, \quad (\sigma, \tau)_{\mathcal{H}_1} = \int_\Omega \sigma \cdot \tau \, dx + \int_\Omega Div\, \sigma \cdot Div\, \tau \, dx,$$

and the associated norms $\| \ \|_{L^2(\Omega)^d}$, $\| \ \|_{H^1(\Omega)^d}$, $\| \ \|_{\mathcal{H}}$ and $\| \ \|_{\mathcal{H}_1}$, respectively. Here and below we use the notation

$$\nabla \upsilon = (\upsilon_{i,j}), \quad \varepsilon(\upsilon) := (\varepsilon_{ij}(\upsilon)), \quad \varepsilon_{ij}(\upsilon) := \frac{1}{2}(\upsilon_{i,j} + \upsilon_{j,i}), \quad \forall\, \upsilon \in H^1(\Omega)^d,$$

$$Div\, \tau = (\tau_{ij,j}), \quad \forall\, \tau \in \mathcal{H}_1.$$

For every element $\upsilon \in H^1(\Omega)^d$. We also write υ for the trace of υ on Γ and we denote by υ_ν and υ_τ the normal and tangential components of υ on Γ given by $\upsilon_\nu = \upsilon \cdot \nu$, $\upsilon_\tau = \upsilon - \upsilon_\nu \nu$.

Let now consider the closed subspace of $H^1(\Omega)^d$ defined by

$$V := \{\ \upsilon \in H^1(\Omega)^d :\ \upsilon = 0 \text{ on } \Gamma_1\ \}.$$

Since $meas\,(\Gamma_1) > 0$, the following *Korn's* inequality holds

$$\|\varepsilon(\upsilon)\|_{\mathcal{H}} \geq c_K\, \|\upsilon\|_{H^1(\Omega)^d}, \quad \forall\, \upsilon \in V, \tag{3.1}$$

where $c_K > 0$ is a constant which depends only on Ω and Γ_1. Over the space V we consider the inner product given by

$$(u, v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{3.2}$$

and let $\| \ \|_V$ be the associated norm. It follows from *Korn's* inequality (3.1) that $\| \ \|_{H^1(\Omega)^d}$ and $\| \ \|_V$ are equivalent norms on V and, therefore, $(V, \| \ \|_V)$ is a real *Hilbert* space. Moreover, by the *Sobolev* trace theorem, (3.1) and (3.2), there exists a constant C_0 depending only on the domain Ω, Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leq C_0 \|v\|_V, \quad \forall v \in V. \tag{3.3}$$

We also introduce the following spaces

$$W = \{ \ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a \ \},$$
$$\mathcal{W}_1 = \{ \ D = (D_i) \mid D_i \in L^2(\Omega), \ D_{i,i} \in L^2(\Omega) \ \}.$$

Since $meas\,(\Gamma_a) > 0$, the following *Friedrichs-Poincaré* inequality holds

$$\|\nabla \psi\|_{L^2(\Omega)^d} \geq c_F \, \|\psi\|_{H^1(\Omega)}, \quad \forall \, \psi \in W, \tag{3.4}$$

where $c_F > 0$ is a constant which depends only on Ω and Γ_a. Over the space W, we consider the inner product given by

$$(\varphi, \psi)_W = \int_\Omega \nabla \varphi \cdot \nabla \psi \, dx,$$

and let $\| \ \|_W$ be the associated norm. It follows from (3.4) that $\| \ \|_{H^1(\Omega)}$ and $\| \ \|_W$ are equivalent norms on W and therefore $(W, \| \ \|_W)$ is a real *Hilbert* space. Moreover, by the *Sobolev* trace theorem, there exists a constant $\tilde{c}_0$, depending only on Ω, Γ_a and Γ_3, such that

$$\|\psi\|_{L^2(\Gamma_3)} \leq \tilde{c}_0 \|\psi\|_W, \quad \forall \, \psi \in W. \tag{3.5}$$

The space $\mathcal{W}_1$ is a real Hilbert space with the inner product

$$(D, E)_{\mathcal{W}_1} = \int_\Omega D \cdot E \, dx + \int_\Omega div\, D \cdot div\, E \, dx,$$

and the associated norm $\| \cdot \|_{\mathcal{W}_1}$.

Finally, for every real *Hilbert* space X we use the classical notation for the spaces $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$, $1 \leq p \leq \infty$, $k \geq 1$.

In the study of the **Problem 1**, we consider the following assumptions on the problem data.

The elasticity operator $\mathcal{F}$, the piezoelectric operator $\mathcal{E}$, the electric permittivity operator $\mathcal{B}$ and the coeffcient of friction satisfy

$$\begin{cases} \text{(a) } \mathcal{F} = (\mathcal{F}_{ijkl}) : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d. \\ \text{(b) } \mathcal{F}_{ijkl} = \mathcal{F}_{klij} = \mathcal{F}_{jikl} \in L^\infty(\Omega). \\ \text{(c) There exists } m_{\mathcal{F}} > 0 \text{ such that } \mathcal{F}_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq m_{\mathcal{F}} \|\varepsilon\|^2, \\ \qquad \forall \varepsilon \in \mathbb{S}^d. \end{cases} \tag{3.6}$$

$$\begin{cases} \text{(a) } \mathcal{E} : \Omega \times \mathbb{S}^d \to \mathbb{R}^d. \\ \text{(b) } \mathcal{E}(x, \tau) = (e_{ijk}(x)\tau_{jk}), \quad \forall \tau = (\tau_{ij}) \in \mathbb{S}^d, \quad \forall \, x \in \Omega. \\ \text{(c) } e_{ijk} = e_{ikj} \in L^\infty(\Omega). \end{cases} \tag{3.7}$$

$$\begin{cases} \text{(a) } \mathcal{B} : \Omega \times \mathbb{R}^d \to \mathbb{R}^d. \\ \text{(b) } \mathcal{B}(x, E) = (b_{ij}(x)E_j), \quad \forall E = (E_i) \in \mathbb{R}^d, \quad \forall x \in \Omega. \\ \text{(c) } b_{ij} = b_{ji} \in L^\infty(\Omega). \\ \text{(d) There exists } m_{\mathcal{B}} > 0 \text{ such that } b_{ij}(x)E_iE_j \geq m_{\mathcal{B}} \|E\|^2, \\ \qquad \forall E = (E_i) \in \mathbb{R}^d. \end{cases} \tag{3.8}$$

$$\begin{cases} \text{(a) } \mu : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+. \\ \text{(b) There exists } C_\mu > 0 \text{ such that} \\ \quad |\mu(x, r_1) - \mu(x, r_2)| \leq C_\mu |r_1 - r_2|, \\ \quad \forall r_1, r_2 \in \mathbb{R}_+, \ \forall x \in \Gamma_3. \\ \text{(c) The mapping } x \longmapsto \mu(x, r) \text{ is Lebesgue measurable} \\ \quad \text{on } \Gamma_3, \forall r \in \mathbb{R}_+. \\ \text{(d) The mapping } x \longmapsto \mu(x, 0) \in L^2(\Gamma_3). \end{cases} \tag{3.9}$$

We note here that, to obtain the uniqueness results, we need to replace assumption (3.9) by the following condition where μ does not depend on the slip $\| u_\tau \|$, i.e.

$$\begin{cases} \mu \text{ is given function which satisfies} \\ \mu \in L^2(\Gamma_3) \text{ and } \mu(x) \geq 0. \end{cases} \tag{3.10}$$

From the assumptions (3.7) and (3.8), we deduce that the piezoelectric operator $\mathcal{E}$ and the electric permittivity operator $\mathcal{B}$ are linear, have measurable bounded components denoted e_{ijk} and b_{ij}, respectively, and moreover, $\mathcal{B}$ is symmetric and positive definite.

Recall also that the transposed operator $\mathcal{E}^*$ is given by $\mathcal{E}^* = (e^*_{ijk})$ where $e^*_{ijk} = e_{kij}$, and the following equality holds

$$\mathcal{E}\sigma \cdot v = \sigma \cdot \mathcal{E}^* v \qquad \forall \sigma \in \mathbb{S}^d, \ v \in \mathbb{R}^d. \tag{3.11}$$

We also suppose that the body forces and surface tractions have the regularity

$$f_0 \in W^{1,\infty}(0, T; L^2(\Omega)^d), \quad f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2)^d). \tag{3.12}$$

We assume that the tensor of relaxation K satisfies

$$K \in W^{1,\infty}(0, T; \mathcal{L}(V)), \tag{3.13}$$

where $\mathcal{L}(V)$ is the space of linear continuous operators from V to V.
We assume that the given normal stress satisfies

$$\mathbf{S} \in L^\infty(\Gamma_3), \tag{3.14}$$

and the densities of electric charges satisfy

$$q_0 \in W^{1,\infty}(0, T; L^2(\Omega)), \quad q_2 \in W^{1,\infty}(0, T; L^2(\Gamma_b)). \tag{3.15}$$

The *Riesz* representation theorem implies the existence of two functions $f : [0, T] \to V$ and $q : [0, T] \to W$ such that

$$(f(t), v)_V = \int_\Omega f_0(t) \cdot v \, dx + \int_{\Gamma_2} f_2(t) \cdot v \, da, \tag{3.16}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t) \psi \, dx - \int_{\Gamma_b} q_2(t) \psi \, da, \tag{3.17}$$

for all $v \in V$, $\psi \in W$ and $t \in [0, T]$. We note that conditions (3.12) and (3.15) imply that

$$f \in W^{1,\infty}(0, T; V), \quad q \in W^{1,\infty}(0, T; W). \tag{3.18}$$

Next, we define the friction functional $V \times V \to \mathbb{R}$ by

$$j_{fr}(u, v) = \int_{\Gamma 3} \mu(\|u_\tau\|) \mid \mathbf{S} \mid \|v_\tau\| \, da. \tag{3.19}$$

Finally, we consider the following assumptions on the initials conditions

$$u_0 \in V, \tag{3.20}$$

$$(\mathcal{F}\varepsilon(u_0), \varepsilon(v))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_0, \varepsilon(v))_{\mathcal{H}} + j_{fr}(u_0, v) \geq (f(0), v)_V \quad \forall v \in V, \tag{3.21}$$

$$(\mathcal{B}\nabla\varphi_0, \nabla\psi)_{L^2(\Omega)^d} = (\mathcal{E}\varepsilon(u_0), \nabla\psi)_{L^2(\Omega)^d} + (q(0), \psi)_W \ \forall\psi \in W. \tag{3.22}$$

By a standard procedure based on Green's formula we can derive the following variational formulation of the contact problem (2.1)–(2.11).

Problem 2. *Find a displacement field $u : [0, T] \longrightarrow V$ and an electric potential field $\varphi : [0, T] \to W$ such that :*

$$\begin{aligned}&(\mathcal{F}\varepsilon(u(t)), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathcal{H}} + \left(\int_0^t K(t-s)\varepsilon(u(s))ds, \varepsilon(v) - \varepsilon(\dot{u}(t))\right)_{\mathcal{H}}\\ &+(\mathcal{E}^*\nabla\varphi(t), \varepsilon(v) - \varepsilon(\dot{u}(t)))_{\mathcal{H}} + j_{fr}(u(t), v) - j_{fr}(u(t), \dot{u}(t))\\ &\geq (f(t), v - \dot{u}(t))_V \ , \quad \forall v \in V \ a.e.\ t \in [0\ T]\,,\end{aligned} \tag{3.23}$$

$$\begin{aligned}(\mathcal{B}\nabla\varphi(t), \nabla\psi)_{L^2(\Omega)^d} - (\mathcal{E}\varepsilon(u(t)), \nabla\psi)_{L^2(\Omega)^d} = (q(t), \psi)_W\\ \forall\psi \in W \quad a.e.\ t \in [0\ T]\,,\end{aligned} \tag{3.24}$$

$$u(0) = u_0, \tag{3.25}$$

4. Existence and uniqueness result

Our main result which states the solvability of **Problem 2**, is the following.

Theorem 4.1. *Assume that (3.6)–(3.8), (3.12)–(3.15) and (3.20)–(3.22) hold. Then*

*(i) Under the assumption (3.9), there exists $\mu_0 > 0$ such that if $C_\mu \parallel S \parallel_{L^\infty(\Gamma_3)} \leq \mu_0$ then the **Problem 2** has at least a solution (u, φ) which satisfies*

$$u \in W^{1,\infty}(0, T; V), \tag{4.1}$$
$$\varphi \in W^{1,\infty}(0, T; W). \tag{4.2}$$

*(ii) Under the assumption (3.10), there exists $\mu_0 > 0$ such that if $C_\mu \|S\|_{L^\infty(\Gamma_3)} \leq \mu_0$ then the **Problem 2** has a unique solution (u, φ) which satisfies*

$$u \in W^{1,\infty}(0, T; V), \tag{4.3}$$
$$\varphi \in W^{1,\infty}(0, T; W). \tag{4.4}$$

Moreover, the mapping $(f, u_0) \longrightarrow u$ is Lipschitz continuous from $W^{1,\infty}(0, T; V) \times V$ to $L^\infty(0, T; V)$.

A quadruple of functions $(u,\ \sigma,\ \varphi,\ D)$ which satisfies (2.1), (2.2), (3.23)–(3.25) is called a *weak solution* of the contact **Problem 1**. To precise the regularity of the weak solution we note that the constitutive relations (2.1) and (2.2), the assumptions (3.6)–(3.8) and the regularities (4.3), (4.4) show that $\sigma \in W^{1,\infty}(0, T; \mathcal{H})$, $D \in W^{1,\infty}(0, T; L^2(\Omega)^d)$. By putting $v = \dot{u}(t) \pm \xi$, where $\xi \in C_0^\infty(\Omega)^d$ in (3.23) and $\psi \in C_0^\infty(\Omega)$ in (3.24) we obtain

$$Div\sigma(t) + f_0(t) = 0,\ divD(t) = q_0(t), \quad \forall t \in [0, T].$$

It follows now from the regularities (3.12), (3.15) that $Div\sigma \in W^{1,\infty}(0, T; L^2(\Omega)^d)$ and $divD \in W^{1,\infty}(0, T; L^2(\Omega))$, which shows that

$$\sigma \in W^{1,\infty}(0, T; \mathcal{H}_1), \tag{4.5}$$
$$D \in W^{1,\infty}(0, T; \mathcal{W}_1). \tag{4.6}$$

We conclude that the weak solution (u, σ, φ, D) of the piezoelectric contact **Problem 1** has the regularity implied in (4.3), (4.4), (4.5) and (4.6).

The proof of Theorem 4.1 is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

Let X be a real *Hilbert* space with the inner product $(\cdot,\cdot)_X$ and the associated norm $\|\ \|_X$.

Let $a : X\times X \longrightarrow \mathbb{R}$ be a bilinear form on X, $j : X\times X \longrightarrow \mathbb{R}$, $f : [0\ T] \longrightarrow X$ and $u_0 \in X$. With these data, we consider the following quasivariational problem: find $u : [0\ T] \longrightarrow X$ such that

$$\begin{aligned} & a(u(t), \upsilon - \dot{u}(t)) + j(u(t), \upsilon) - j(u(t), \dot{u}(t)) \geq (f(t), \upsilon - \dot{u}(t))_X \\ & \forall \upsilon \in X, \text{ a.e. } t \in (0\ T), \end{aligned} \tag{4.7}$$

$$u(0) = u_0. \tag{4.8}$$

$$\left\{ \begin{array}{l} a : X\times X \longrightarrow \mathbb{R} \text{ is a bilinear symmetric form and} \\ (a) \text{ there exists } M>0 \text{ such that} \\ \qquad |a(u,\upsilon)| \leq M\,\|u\|_X\,\|\upsilon\|_X\,, \quad \forall u,\upsilon \in X, \\ (b) \text{ there exists } m>0 \text{ such that} \\ \qquad a(\upsilon,\upsilon) \geq m\,\|\upsilon\|_X^2\,, \quad \forall \upsilon \in X. \end{array} \right. \tag{4.9}$$

In order to solve the problem (4.7)–(4.8), we consider the following assumptions.

$$\left\{ \begin{array}{l} \text{For every } \zeta \in X,\ j(\zeta,.) : X \longrightarrow \mathbb{R} \text{ is a positively} \\ \text{homogeneous subadditive functional, i.e.} \\ (a)\ \ j(\zeta,\lambda u) = \lambda j(\zeta,u)\ \forall u \in X,\ \lambda \in \mathbb{R}_+, \\ (b)\ \ j(\zeta,u+\upsilon) \leq j(\zeta,u) + j(\zeta,\upsilon), \quad \forall u,\upsilon \in X. \end{array} \right. \tag{4.10}$$

$$f \in W^{1,\infty}(0,T;X). \tag{4.11}$$

$$u_0 \in X. \tag{4.12}$$

$$a(u_0,\upsilon) + j(u_0,\upsilon) \geq (f(0),\upsilon)_X\,, \quad \forall \upsilon \in X. \tag{4.13}$$

Keeping in mind (4.10), it results that for all $\zeta \in X$, $j(\zeta,.) : X \longrightarrow \mathbb{R}$ is a convex functional. Therefore, there exists the directional derivative j_2' given by

$$j_2'(\zeta,u;\upsilon) = \underset{\lambda \searrow 0}{lim} \frac{1}{\lambda}\left[j(\zeta,u+\lambda\upsilon) - j(\zeta,u)\ \right], \quad \forall \zeta,u,\upsilon \in X. \tag{4.14}$$

We consider now the following additional assumptions on the functional j.

$$\left\{ \begin{array}{l} \text{For every sequence } (u_n) \subset X \text{ with } \|u_n\|_X \longrightarrow \infty, \\ \text{every sequence } (t_n) \subset [0\ 1] \text{ and each } \tilde{u} \in X \text{ one has} \\ \underset{n\to+\infty}{lim\ inf} \left[\dfrac{1}{\|u_n\|_X^2} j_2'(t_n u_n, u_n - \tilde{u}; -u_n) \right] < m. \end{array} \right. \tag{4.15}$$

$$\left\{ \begin{array}{l} \text{For every sequence } (u_n) \subset X \text{ with } \|u_n\|_X \longrightarrow \infty, \text{ every} \\ \text{bounded sequence } (\zeta_n) \subset X \text{ and each } \tilde{u} \in X \text{ one has} \\ \underset{n\to+\infty}{lim\ inf} \left[\dfrac{1}{\|u_n\|_X^2} j_2'(\zeta_n, u_n - \tilde{u}; -u_n) \right] < m. \end{array} \right. \tag{4.16}$$

$$\left\{ \begin{array}{l} \text{For all sequence } (u_n) \subset X \text{ and } (\zeta_n) \subset X \text{ such that} \\ u_n \rightharpoonup u \in X,\ \zeta_n \rightharpoonup \zeta \in X \text{ and for every } \upsilon \in X, \text{ we have} \\ \underset{n\to+\infty}{lim\ sup}\,[j(\zeta_n,\upsilon) - j(\zeta_n,u_n)] \leq j(\zeta,\upsilon) - j(\zeta,u). \end{array} \right. \tag{4.17}$$

$$\left\{ \begin{array}{l} \text{There exists } k_0 \in (0,m) \text{ such that} \\ j(u,\upsilon-u) - j(\upsilon,\upsilon-u) \leq k_0\,\|u-\upsilon\|_X^2\,,\ \forall u,\upsilon \in X. \end{array} \right. \tag{4.18}$$

$$\left\{ \begin{array}{l} \text{There exist two functions } a_1\ : X \longrightarrow \mathbb{R} \text{ and } a_2\ : X \longrightarrow \mathbb{R}, \\ \text{which map bounded sets in } X \text{ into bounded sets in } \mathbb{R} \\ \text{such that } \ |\, j(\zeta,u)\,| \leq a_1(\zeta)\,\|u\|_X^2 + a_2(\zeta), \quad \forall \zeta,u \in X, \\ \text{and } a_1(0_X) < m - k_0. \end{array} \right. \tag{4.19}$$

$$\left\{\begin{array}{l}\text{For every sequence } (\zeta_n)\subset X \text{ with } \zeta_n \rightharpoonup \zeta\in X \text{ and every}\\ \text{bounded sequence } (u_n)\subset X \text{ one has}\\ \lim\limits_{n\to+\infty}[j(\zeta_n,u_n)-j(\zeta,u_n)]=0.\end{array}\right. \tag{4.20}$$

$$\left\{\begin{array}{l}\text{For every } s\in(0\ T] \text{ and every functions}\\ u,\ v\in W^{1,\infty}(0,T;X) \text{ with } u(0)=v(0),\ u(s)\neq v(s),\\ \text{the inequality below holds}\\ \int_0^s[j(u(t),\dot v(t))-j(u(t),\dot u(t))+j(v(t),\dot u(t))\\ -j(v(t),\dot v(t))]dt<\dfrac{m}{2}\left\|u(s)-v(s)\right\|_X^2.\end{array}\right. \tag{4.21}$$

$$\left\{\begin{array}{l}\text{There exists } \alpha\in\left(0\ ,\frac{m}{2}\right) \text{ such that for every } s\in(0\ T]\\ \text{and for every functions } u,\ v\in W^{1,\infty}(0,T;X)\\ \text{with } u(s)\neq v(s), \text{ the inequality below holds}\\ \int_0^s[j(u(t),\dot v(t))-j(u(t),\dot u(t))+j(v(t),\dot u(t))\\ -j(v(t),\dot v(t))]dt<\alpha\left\|u(s)-v(s)\right\|_X^2.\end{array}\right. \tag{4.22}$$

In the study of the evolutionary problem (4.7)–(4.8), we recall the following result.

Theorem 4.2. *Let (4.9)–(4.13) hold.*

(i) *If the assumptions (4.15)–(4.20) are satisfied then there exists at least a solution $u\in W^{1,\infty}(0,T;X)$ to the problem (4.7)–(4.8).*

(ii) *If the assumptions (4.15)–(4.21) are satisfied then there exists a unique solution $u\in W^{1,\infty}(0,T;X)$ to the problem (4.7)–(4.8).*

(iii) *If the assumptions (4.15)–(4.20) and (4.22) are satisfied then there exists a unique solution $u=u(f,u_0)\in W^{1,\infty}(0,T;X)$ to the problem (4.7)–(4.8) and the mapping $(f,u_0)\longrightarrow u$ is Lipschitz continuous from $W^{1,\infty}(0,T;X)\times X$ to $L^\infty(0,T;X)$.*

Theorem 4.2 will be used in this section in order to prove the existence and the uniqueness of the solution to the variational problem associated with our mechanical model; its proof can be found in [16].

We return now to proof of Theorem 4.1. To this end, we assume in the following that (3.6)–(3.8), (3.12)–(3.15) and (3.20)–(3.22) hold; below, "c" is a generic positive constants which may depend on Ω, Γ_1, Γ_3, $\mathcal{F}$, whose value may change from place to place. For the sake of simplicity, we suppress in what follows the explicit dependence on various functions on $x\in\Omega\cup\Gamma_3$.

Using *Riesz's* representation theorem, we can define the following operators $\mathcal{G}:W\longrightarrow W$ and $\mathcal{R}:V\longrightarrow W$ respectively by

$$(\mathcal{G}\varphi(t),\psi)_W=(\mathcal{B}\nabla\varphi(t),\nabla\psi)_{L^2(\Omega)^d},\quad \forall\varphi,\psi\in W, \tag{4.23}$$

$$(\mathcal{R}v,\varphi)_W=(\mathcal{E}\varepsilon(v),\nabla\varphi)_{L^2(\Omega)^d},\quad \forall\varphi\in W,\ v\in V. \tag{4.24}$$

We can show that $\mathcal{G}$ is a linear continuous symmetric positive definite operator. Therefore, $\mathcal{G}$ is an invertible operator on W. We can also prove that $\mathcal{R}$ is a linear continuous operator on V. Let $\mathcal{R}^*$ the adjoint of $\mathcal{R}$. Thus, from (3.11) we can write

$$(\mathcal{R}^*\varphi,v)_V=(\mathcal{E}^*\nabla\varphi,\varepsilon(v))_{\mathcal{H}},\ \forall\varphi\in W,\ v\in V. \tag{4.25}$$

By introducing (4.23)–(4.24) in (3.24) we get

$$(\mathcal{G}\varphi(t),\psi)_W=(\mathcal{R}u(t),\psi)_W+(q(t),\psi)_W,\ \forall\psi\in W,$$

and consequently

$$\mathcal{G}\varphi(t)=\mathcal{R}u(t)+q(t).$$

On the other hand, $\mathcal{G}$ is invertible where the previous equality gives us

$$\varphi(t)=\mathcal{G}^{-1}\mathcal{R}u(t)+\mathcal{G}^{-1}q(t). \tag{4.26}$$

Using (4.25)–(4.26) and (3.23) we obtain

$$\begin{aligned}&(\mathcal{F}\varepsilon(u(t)),\varepsilon(\upsilon)-\varepsilon(\dot{u}(t)))_{\mathcal{H}}+(\textstyle\int_0^t K(t-s)\varepsilon(u)ds,\varepsilon(\upsilon)-\varepsilon(\dot{u}(t)))_{\mathcal{H}}\\&+(\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}u(t),\upsilon-\dot{u}(t))_V+j_{fr}(u(t),\upsilon)-j_{fr}(u(t),\dot{u}(t))\\&\geq(f(t)-\mathcal{R}^*\mathcal{G}^{-1}q(t),\upsilon-\dot{u}(t))_V\qquad\forall\upsilon\in V,\ \text{a.e. } t\in(0\ T).\end{aligned}\tag{4.27}$$

Let now the operator $L: V \to V$ defined by

$$L\upsilon=\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}\upsilon,\ \forall\upsilon\in V.\tag{4.28}$$

Using the properties of the operators $\mathcal{G}$, $\mathcal{R}$ and $\mathcal{R}^*$, we deduce that L is a linear symmetric positive operator on V, Indeed, we have

$$(Lu,\upsilon)_V=(\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}u,\upsilon)_V=(u,\mathcal{R}^*\mathcal{G}^{-1}\mathcal{R}\upsilon)_V=(u,L\upsilon)_V,\quad\forall u,\upsilon\in V,$$

$$(L\upsilon,\upsilon)_V=(\mathcal{G}^{-1}\mathcal{R}\upsilon,\mathcal{R}\upsilon)_W\geq0,\ \forall\upsilon\in V.\tag{4.29}$$

Now, let the bilinear form $a: V\times V\to\mathbb{R}$ such that

$$a(u,\upsilon)=(\mathcal{F}\varepsilon(u(t)),\varepsilon(\upsilon))_{\mathcal{H}}+(Lu,\upsilon)_V,\ \forall u,\upsilon\in V.\tag{4.30}$$

The bilinear form a is continuous and coercive on V. Indeed, we have

$$\mid a(u,\upsilon)\mid\leq(M+\|L\|)\|u\|_V\|\upsilon\|_V,\ \forall u,\upsilon\in V,\tag{4.31}$$

$$a(\upsilon,\upsilon)\geq m\|\upsilon\|_V^2,\ \forall\upsilon\in V,\tag{4.32}$$

and the symmetry of $\mathcal{F}$ and L leads to the symmetry of a.

Let now the function $\mathbf{f}: [0\ T]\to V$ defined by

$$\mathbf{f}(t)=f(t)-\mathcal{R}^*\mathcal{G}^{-1}q(t),\quad\forall t\in[0\ T].\tag{4.33}$$

From (3.18) we obtain

$$\mathbf{f}\in W^{1,\infty}(0,T,V).\tag{4.34}$$

The relations (4.27), (4.30) and (4.33) lead us to consider the following variational problem, in the terms of displacement field.

Problem 3. *Find a displacement field* $u:[0,T]\longrightarrow V$ *such that :*

$$\begin{aligned}&a(u(t),\upsilon-\dot{u}(t))+(\textstyle\int_0^t K(t-s)\varepsilon(u(s))ds,\varepsilon(\upsilon)-\varepsilon(\dot{u}(t)))_{\mathcal{H}}\\&+j_{fr}(u(t),\upsilon)-j_{fr}(u(t),\dot{u}(t))\geq(\mathbf{f}(t),\upsilon-\dot{u}(t))_V,\ \forall\upsilon\in V,\end{aligned}\tag{4.35}$$

$$u(0)=u_0.\tag{4.36}$$

Theorem 4.3. *Assume that (3.6)–(3.8), (3.12)–(3.15) and (3.20)–(3.22) hold. Then*

(i) Under the assumption (3.9) there exists $\mu_0>0$ *such that:*
if $C_\mu\|S\|_{L^\infty(\Gamma_3)}\leq\mu_0$ *then the* ***Problem 3*** *has at least a solution u which satisfies*

$$u\in W^{1,\infty}(0,T;V).\tag{4.37}$$

(ii) Under the assumption (3.10) the ***Problem 3*** *has a unique solution u which satisfies*

$$u\in W^{1,\infty}(0,T;V).\tag{4.38}$$

Moreover, the mapping $(f,u_0)\longrightarrow u$ *is Lipschitz continuous from*
$W^{1,\infty}(0,T;V)\times V$ *to* $L^\infty(0,T;V)$.

We assume in the following that the conditions of Theorem 4.3 hold and we introduce the set

$$\mathcal{Z} := \left\{ \eta \in W^{1,\infty}(0,T;V) : \eta(0) = 0_V \right\}. \tag{4.39}$$

Let $\eta \in \mathcal{Z}$ be given and we consider the following intermediate problem, in the term of displacement field.

Problem 4. *Find the displacement field $u_\eta : [0,T] \to V$ such that :*

$$\begin{aligned} & a(u_\eta(t), \upsilon - \dot{u}_\eta(t)) + j_{fr}(u_\eta(t), \upsilon) \\ & -j_{fr}(u_\eta(t), \dot{u}_\eta(t)) \geq (\mathbf{f}_\eta(t), \upsilon - \dot{u}_\eta(t))_V, \ \forall \upsilon \in V, \end{aligned} \tag{4.40}$$

$$u_\eta(0) = u_0, \tag{4.41}$$

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \eta(t), \quad \forall t \in [0,T]. \tag{4.42}$$

Remark 4.4. *From (4.34) and the regularity of η we deduce that $\mathbf{f}_\eta \in W^{1,\infty}(0,T,V)$.*

Remark 4.5. *From (3.22) and (3.23), we deduce that (4.13) is verified.*

Theorem 4.6. *Assume that (3.6)–(3.8), (3.12)–(3.15) and (3.20)–(3.22) hold. Then*

(i) Under the assumption (3.9) there exists $\mu_0 > 0$ such that:
*if $C_\mu \|S\|_{L^\infty(\Gamma_3)} \leq \mu_0$ then the problem **Problem 4** has at least a solution u_η which satisfies*

$$u_\eta \in W^{1,\infty}(0,T;V). \tag{4.43}$$

*(ii) Under the assumption (3.10) the **Problem 4** has a unique solution u_η which satisfies*

$$u_\eta \in W^{1,\infty}(0,T;V). \tag{4.44}$$

Moreover, the mapping $(f, u_0) \longrightarrow u$ is Lipschitz continuous from $W^{1,\infty}(0,T;V) \times V$ to $L^\infty(0,T;V)$.

We will use the results given by the Theorem 4.2 to give a result of existence and uniquness of solutions of **Problem 4**. We remark that the functional j_{fr}, given by (3.19), satisfies condition (4.10). In addition, we have the following results.

Lemma 4.7. *The functional j_{fr} satisfies the assumptions (4.15) and (4.16).*

Proof. Let $\zeta, u, \tilde{u} \in V$ and let $\lambda \in (0,1]$. Using (3.19), it follows that j_{fr} satisfies

$$j_{fr}(\zeta, u - \tilde{u} - \lambda u) - j_{fr}(\zeta, u - \tilde{u}) \leq \lambda \int_{\Gamma 3} \mu(\| \zeta_\tau \|) \mid S \mid \| \tilde{u}_\tau \| \, da.$$

Using (4.14) we find

$$j_2^{'}(\zeta, u - \tilde{u}\ ; -u) \leq \int_{\Gamma 3} \mu(\| \zeta_\tau \|) \mid S \mid \| \tilde{u}_\tau \| \, da, \ \forall \zeta, u, \tilde{u} \in V. \tag{4.45}$$

Let now consider the sequences $(u_n) \subset V$, $(t_n) \subset [0\ 1]$ and the element $\tilde{u} \in V$. Using (3.3), (3.9), (3.14) and (4.45), we obtain

$$\begin{aligned} j_2^{'}(t_n u_n, u_n - \tilde{u}; -u_n) \quad & \leq \quad \int_{\Gamma 3} (C_\mu \parallel u_{n\tau} \parallel + \mid \mu(0) \mid) \mid \mathbf{S} \mid \| \tilde{u}_\tau \| \, da \\ & \leq \quad \left(C_0 C_\mu \parallel u_n \parallel_V + \mid \mu(0) \mid_{L^2(\Gamma_3)} \right) C_0 \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} \| \tilde{u} \|_V . \end{aligned} \tag{4.46}$$

It follows from the previous inequality that if $\|u_n\|_V \longrightarrow +\infty$, then

$$\underset{n \longrightarrow +\infty}{lim} inf \left[\frac{1}{\|u_n\|_V^2} j_2^{'}(t_n u_n, u_n - \tilde{u}; -u_n) \right] \leq 0,$$

where we deduce that j_{fr} satisfies assumption (4.15).

Let now consider the consequences $(u_n) \subset V$, $(\zeta_n) \subset V$ such that

$$\|u_n\|_V \longrightarrow +\infty, \tag{4.47}$$

$$\|\zeta_n\|_V \leq C, \ \forall n \in \mathbb{N}. \tag{4.48}$$

such that $C > 0$. By using (3.3), (3.9), (3.14) and (4.45) we obtain

$$j_2^{'}(\zeta_n, u_n - \tilde{u}; -u_n) \leq \big(C_0 C_\mu \parallel \zeta_n \|_V + \mid \mu(0) \mid_{L^2(\Gamma_3)}\big)\, C_0 \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} \| \, \tilde{u} \parallel_V, \tag{4.49}$$

for all $\tilde{u} \in V$ and $n \in \mathbb{N}$. Then, using (4.47)–(4.49), we can conclude that

$$\lim_{n \longrightarrow +\infty} inf \left[\frac{1}{\|u_n\|_V^2} j_2^{'}(\zeta_n, u_n - \tilde{u}; -u_n) \right] \leq 0 \ ,$$

where we deduce that j_{fr} satisfies (4.16). □

Lemma 4.8. *The functional j_{fr} satisfies the conditions (4.17) and (4.20).*

Proof. Let $(u_n) \subset V$, $(\zeta_n) \subset V$ be two sequences such that $u_n \rightharpoonup u \in V$ and $\zeta_n \rightharpoonup \zeta \in V$. It follows from the compactness property of the trace map that

$$u_n \longrightarrow u \ \text{ in } L^2(\Gamma_3)^d, \tag{4.50}$$

$$\mu(\| \, \zeta_{n\tau} \, \|) \longrightarrow \mu(\| \, \zeta_\tau \, \|) \ \text{ in } L^2(\Gamma_3). \tag{4.51}$$

We conclude by the last two limits (4.50) and (4.51) that

$$\begin{aligned} &j_{fr}(\zeta_n, \upsilon) \longrightarrow j_{fr}(\zeta, \upsilon), \ \forall \upsilon \in V, \\ &j_{fr}(\zeta_n, u_n) \longrightarrow j_{fr}(\zeta, u), \end{aligned}$$

which implies that

$$\lim_{n \longrightarrow +\infty} sup \, [j_{fr}(\zeta_n, \upsilon) - j_{fr}(\zeta_n, u_n)] \leq j_{fr}(\zeta, \upsilon) - j_{fr}(\zeta, u).$$

Thus, we deduce that j_{fr} satisfies (4.17).

Next, we consider (u_n) a bounded sequence of V, i.e.

$$\|u_n\|_V \leq C, \ \forall n \in \mathbb{N}, \tag{4.52}$$

where $C > 0$. Representation (3.19) yields

$$j_{fr}(\zeta_n, u_n) - j_{fr}(\zeta, u_n) = \int_{\Gamma 3} \mid \mathbf{S} \mid (\mu(\| \, \zeta_{n\tau} \, \|) - \mu(\| \, \zeta_\tau \, \|)) \parallel u_{n\tau} \parallel da.$$

Moreover, using (3.3) and (3.9) we find

$$\mid j_{fr}(\zeta_n, u_n) - j_{fr}(\zeta, u_n) \mid \leq C_0 \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} |\mu(\| \, \zeta_{n\tau} \, \|) - \mu(\| \, \zeta_\tau \, \|)|_{L^2(\Gamma_3)} \parallel u_n \parallel_V,$$

where we deduce that j_{fr} satisfies (4.20), i.e.

$$\lim_{n \longrightarrow +\infty} [j_{fr}(\zeta_n, u_n) - j_{fr}(\zeta, u_n)] = 0.$$

□

Lemma 4.9. *Under the assumption (3.9), the functional j_{fr} satisfies the assumptions (4.18) and (4.19) for all $k_0 \in (0, m)$.*

Proof. Let $u, \upsilon \in V$. Using (3.3), (3.14) and (3.19) we find

$$\begin{aligned} j_{fr}(u, \upsilon - u) - j_{fr}(\upsilon, \upsilon - u) &= \int_{\Gamma 3} \mid \mathbf{S} \mid (\mu(\| u_\tau \|) - \mu(\| \upsilon_\tau \|)) \| u_\tau - \upsilon_\tau \| \, da \\ &\leq C_\mu C_0^2 \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} \| u - \upsilon \|_V^2 . \end{aligned}$$

Choosing $\mu_0 = \dfrac{m}{C_0^2}$ we assume that

$$C_\mu \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} < \mu_0.$$

This implies that there exists $k_0 \in (0, m)$ such that

$$C_\mu C_0^2 \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} < k_0 < m.$$

From above, it follows that j_{fr} satisfies (4.18).

Let now $\zeta, u \in V$. Using again (3.3), (3.9), (3.14) and (3.19) we obtain

$$\begin{aligned} \mid j_{fr}(\zeta, u) \mid &= \mid \int_{\Gamma 3} \mu(\| \zeta_\tau \|) \mid \mathbf{S} \mid \| u_\tau \| \, da \mid \\ &\leq C_0 \mid \mathbf{S} \mid_{L^\infty(\Gamma_3)} \left(C_0 C_\mu \| \zeta \|_V + \mid \mu(0) \mid_{L^2(\Gamma_3)} \right) \| u \|_V . \end{aligned}$$

which implies that condition (4.19) is verified for all $k_0 \in (0, m)$. □

Lemma 4.10. *Under the assumption (3.10) the functional j_{fr} satisfies (4.10) and (4.15)–(4.22).*

Proof. In this case the functional j_{fr} does not depend on the first argument and is given by

$$j_{fr}(\upsilon) = \int_{\Gamma 3} \mu \mid \mathbf{S} \mid \| \upsilon_\tau \| \, da.$$

By using arguments similar to those used in the proof of Lemmas 4.7 – 4.9, it is easy to check that the functional j_{fr} satisfies (4.10) and (4.15)–(4.22). □

Proof of Theorem 4.6. Keeping in mind that the bilinear form a is symmetric, continuous and coercive on V and using (3.20) and Remarks 4.4 – 4.5 we obtain

- The proof of Theorem 4.6(i) follows now from Lemmas 4.7 – 4.9 and Theorem (4.2) (i).
- The proof of Theorem 4.6(ii) follows now from Lemma 4.10 and Theorem 4.2 (ii) and (iii). □

In the next step, we use the displacement field u_η obtained in Theorem 4.6 and we consider the operator $\mathcal{K} : \mathcal{Z} \longrightarrow \mathcal{Z}$ defined by

$$\mathcal{K}\eta(t) := \int_0^t K(t-s)\varepsilon(u_\eta(s))ds. \tag{4.53}$$

We have the following result.

Lemma 4.11. *For any $\eta \in \mathcal{Z}$ there holds $\mathcal{K}\eta \in \mathcal{Z}$ and the operator $\mathcal{K}$ has a unique fixed point $\eta^* \in \mathcal{Z}$.*

Proof. Let $\eta \in \mathcal{Z}$. Using (4.53), (3.13) and the fact that $u_\eta \in W^{1,\infty}(0, T; V)$, it is easy to check that $\mathcal{K}\eta \in \mathcal{Z}$. Moreover, by a standard computation we find that

$$\left(\frac{d}{dt} \mathcal{K}\eta \right)(t) = K(0)\varepsilon(u_\eta(t)) + \int_0^t \dot{K}(t-s)\varepsilon(u_\eta(s))ds. \tag{4.54}$$

Let now η_1, $\eta_2 \in \mathcal{Z}$ and for the sake of simplicity, denote $u_{\eta_1} = u_1$ and $u_{\eta_2} = u_2$, for all $t \in [0\ T]$. Using (4.53), (3.13) and (3.2) we find that

$$\|\mathcal{K}\eta_1(t) - \mathcal{K}\eta_2(t)\|_V \leq c \int_0^t \|u_1(s) - u_2(s)\|_V \, ds. \tag{4.55}$$

Moreover, using (4.54), (3.13) and (3.2) we obtain

$$\begin{aligned} \left\| \left(\frac{d}{dt}\mathcal{K}\eta_1 \right)(t) - \left(\frac{d}{dt}\mathcal{K}\eta_2 \right)(t) \right\|_V &\leq c \|u_1(t) - u_2(t)\|_V \\ &+ c \int_0^t \|u_1(s) - u_2(s)\|_V \, ds. \end{aligned} \tag{4.56}$$

On the other hand, from (4.40) and (4.42) we have

$$\begin{aligned} a(u_1, v - \dot{u}_1) + j_{fr}(u_1, v) - j_{fr}(u_1, \dot{u}_1) &\geq (\mathbf{f} - \eta_1, v - \dot{u}_1)_V, \\ a(u_2, v - \dot{u}_2) + j_{fr}(u_2, v) - j_{fr}(u_2, \dot{u}_2) &\geq (\mathbf{f} - \eta_2, v - \dot{u}_2)_V, \end{aligned}$$

for all $v \in V$. Choose $v = \dot{u}_2$ in the first inequality, $v = \dot{u}_1$ in the second inequality, and sum the results to obtain

$$\begin{aligned} &a(u_1 - u_2, \dot{u}_1 - \dot{u}_2) \leq j_{fr}(u_1, \dot{u}_2) - j_{fr}(u_2, \dot{u}_2) \\ &+ j_{fr}(u_2, \dot{u}_1) - j_{fr}(u_1, \dot{u}_1) - (\eta_1 - \eta_2, \dot{u}_1 - \dot{u}_2)_V. \end{aligned} \tag{4.57}$$

Some algebraic calculations show that

$$\frac{1}{2}\frac{d}{dt} a(u_1 - u_2, u_1 - u_2) \leq -(\eta_1 - \eta_2, \dot{u}_1 - \dot{u}_2)_V.$$

Iintegrating the previous inequality from 0 to t and using (4.41) we obtain

$$\begin{aligned} \frac{1}{2} a(u_1(t) - u_2(t), u_1(t) - u_2(t)) &\leq -(\eta_1(t) - \eta_2(t), u_1(t) - u_2(t))_V \\ &+ \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s))_V ds. \end{aligned}$$

It follows now from (4.32) that

$$\begin{aligned} \frac{m}{2} \|u_1(t) - u_2(t)\|_V^2 &\leq \|\eta_1(t) - \eta_2(t)\|_V \|u_1(t) - u_2(t)\|_V \\ &+ \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_V \|u_1(s) - u_2(s)\|_V \, ds, \end{aligned}$$

where we deduce that

$$\|u_1(t) - u_2(t)\|_V \leq c \left(\|\eta_1(t) - \eta_2(t)\|_V + \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_V \right). \tag{4.58}$$

Now, since

$$\eta_1(t) - \eta_2(t) = \int_0^t (\dot{\eta}_1(s) - \dot{\eta}_2(s)) ds,$$

we deduce that

$$\|\eta_1(t) - \eta_2(t)\|_V \leq \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_V \, ds.$$

Substituting this inequality in (4.58), we obtain

$$\|u_1(t) - u_2(t)\|_V \le c \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_V \, ds. \tag{4.59}$$

By adding the results obtained in (4.55), (4.56) and using (4.59) we obtain

$$\|\mathcal{K}\eta_1(t) - \mathcal{K}\eta_2(t)\|_V + \left\| \left(\frac{d}{dt}\mathcal{K}\eta_1\right)(t) - \left(\frac{d}{dt}\mathcal{K}\eta_2\right)(t) \right\|_V \le c \int_0^t \|\dot{\eta}_1(s) - \dot{\eta}_2(s)\|_V \, ds.$$

Iterating the last inequality, we find

$$\begin{aligned} &\|\mathcal{K}^n\eta_1(t) - \mathcal{K}^n\eta_2(t)\|_V + \left\| \left(\frac{d}{dt}\mathcal{K}^n\eta_1\right)(t) - \left(\frac{d}{dt}\mathcal{K}^n\eta_2\right)(t) \right\|_V \\ &\le c^n \int_0^t \int_0^{s_1} \cdots \int_0^{s_{n-1}} \|\dot{\eta}_1(s_n) - \dot{\eta}_2(s_n)\|_V \, ds_n \ldots ds_1, \end{aligned}$$

where $\mathcal{K}^n$ denotes the n^{th} power of the operator $\mathcal{K}$. The last inequality implies

$$\|\mathcal{K}^n\eta_1 - \mathcal{K}^n\eta_2\|_{W^{1,\infty}(0,T;V)} \le \frac{c^nT^n}{n!} \|\eta_1 - \eta_2\|_{W^{1,\infty}(0,T;V)} .$$

Since $\lim\limits_{n\longrightarrow\infty} \dfrac{c^nT^n}{n!} = 0$, the previous inequality implies that for n large enough, a power $\mathcal{K}^n$ of $\mathcal{K}$ is a contraction in $\mathcal{Z}$. Then, there exists a unique element $\eta^* \in \mathcal{Z}$ such that $\mathcal{K}^m\eta^* = \eta^*$, since $\mathcal{Z}$ is a non-empty closed subset of the *Banach* space $W^{1,\infty}(0,T;V)$. Then, η^* is the unique fixed point of $\mathcal{K}$, i.e $\mathcal{K}\eta^* = \eta^*$ which concludes the proof of Lemma 4.11. □

Proof of Theorem 4.3. Let $\eta^* \in \mathcal{Z}$ be the fixed point of the operator $\mathcal{K}$ and let u be the functions defined in Theorem 4.6 for $\eta = \eta^*$, i.e $u = u_{\eta^*}$. Using (4.53), we deduce that Theorem 4.3 is a consequence of Theorem 4.6. □

We have now all the ingredients needed to prove the Theorem 4.1.

Proof of Theorem 4.1. Let $\eta^* \in \mathcal{Z}$ be the fixed point of the operator $\mathcal{K}$ and let u be the functions defined in Theorem 4.3 for $\eta = \eta^*$. Using (4.36), (4.35), (4.33),(4.30), (4.28), (4.26) and (4.25) we conclude that Theorem 4.1 is a consequence of Theorem 4.3. □

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