Bol. Soc. Paran. Mat.
©SPM –ISSN-2175-1188 on line
SPM: www.spm.uem.br/bspm

(3s.) **v. 2024 (42)** : 1–6.
ISSN-0037-8712 in press
doi:10.5269/bspm.65843

Results for Self-Inversive Rational Functions

Idrees Qasim

Abstract: In this paper, we find some relations between maximum modulus of a rational function $r(z)$ satisfying $r(z) = B(z)r(1/z)$ and the maximum modulus of its derivative. We also find analogue of Cohn's Theorem for rational functions.

Key Words: Rational Functions, Poles, self-inversive, self-reciprocal, Polar Derivative.

Contents

1. Introduction

Let $\mathcal{P}_n$ denote the space of complex polynomials $p(z) := \sum_{j=0}^{n} \alpha_j z^j$ of degree $n \geq 1$. Let $T := \{z : |z| = 1\}$, $D_- := \{z : |z| < 1\}$ and $D_+ := \{z : |z| > 1\}$. For $z_j \in \mathbb{C}$ with $j = 1, 2, \ldots, n$, we write

$$w(z) = \prod_{j=1}^{n} (z - z_j), \tag{1.1}$$

and

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{z_j} z}{z - z_j} \right).$$

$B(z)$ is known as finite Blaschke product.
Let $p(z)$ be a polynomial of degree at most n with complex variable z. We consider the following space of rational functions

$$\mathcal{R}_n := \mathcal{R}_n(z_1, z_2, \ldots, z_n) := \left\{ \frac{p(z)}{w(z)} \right\}.$$

Throughout this paper, we shall assume that all the poles $z_1, z_2, \ldots, z_n$ are in D_+ unless otherwise stated. For the case when all the poles are in D_-, we can obtain analogous results with suitable modification of our method.

Definition of conjugate transpose

1. For $p(z) := \sum_{j=0}^{n} \alpha_j z^j$, the conjugate transpose (reciprocal) p^* of p is defined by

$$p^*(z) = z^n \overline{p\left(\frac{1}{\overline{z}}\right)}.$$

2010 *Mathematics Subject Classification:* 30A10, 30C15, 30D15.
Submitted November 15, 2022. Published January 17, 2023

2. For $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$, the conjugate transpose r^* of r is defined by

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)} = \frac{p^*(z)}{w(z)}.$$

3. $p \in \mathcal{P}_n$ is said to be self-inversive if $p^*(z) = \lambda p(z)$ with $|\lambda| = 1$. Similarly, $r \in \mathcal{R}_n$ is said to be self-inversive if $r^*(z) = \lambda r(z)$ with $|\lambda| = 1$. Note that $r(z)$ is self-inversive if and only if $p(z)$ is self inversive.

4. $p \in \mathcal{P}_n$ is said to be self-reciprocal if $p(z) = z^n p(1/z)$. Also, $r \in \mathcal{R}_n$ is said to be self-reciprocal if $r(z) = B(z)r(1/z)$.

In 1927, Bernstein [3] proved the following result.

If $p \in \mathcal{P}_n$, then

$$\max_{z\in T} |p'(z)| \le n \max_{z\in T} |p(z)|, \tag{1.2}$$

where the equality holds for polynomials having all zeros at the origin.

In 1969, Malik [5] improved inequality (1.2) and proved the following:

If $p \in \mathcal{P}_n$, then for $z \in T$

$$|p'(z)| + |Q'(z)| \le n \max_{z\in T} |p(z)|, \tag{1.3}$$

where $Q(z) = z^n \overline{p\left(\frac{1}{\bar{z}}\right)}$.

As an easy consequence of inequality (1.3), we have the following result which improves inequality (1.2) for self-inversive polynomials.

Theorem A. If $p \in \mathcal{P}_n$ is self-inversive, then for $z \in T$,

$$\max_{z\in T} |p'(z)| \le \frac{n}{2} \max_{z\in T} |p(z)|. \tag{1.4}$$

For a complex number α and for $p \in \mathcal{P}_n$, let

$$D_\alpha p(z) := np(z) + (\alpha - z)p'(z).$$

$D_\alpha p(z)$ is a polynomial of degree at most $n-1$ and is known as polar derivative of p with respect to α. It generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty} \frac{D_\alpha p(z)}{\alpha} = p'(z).$$

Aziz and Shah [2] extended inequality (1.2) to the polar derivative of a polynomial and proved the following result.

Theorem B. If $p \in \mathcal{P}_n$, then for every α with $\alpha \in T \cup D_+$ and $z \in T$,

$$|D_\alpha p(z)| \le n|\alpha| \max_{z\in T} |p(z)|. \tag{1.5}$$

Li, Mohapatra and Rodriguez [6] extended inequality (1.2) and (1.4) to rational functions with prescribed poles and proved the following results.

Theorem C. If $r \in \mathcal{R}_n$, then for $z \in T$

$$|r'(z)| \leq |B'(z)| \max_{z \in T} |r(z)|. \tag{1.6}$$

Equality holds for $r(z) = uB(z)$, where $u \in T$.

Theorem D. If $r(z) = \dfrac{p(z)}{w(z)} \in \mathcal{R}_n$ and $r(z)$ is self-inversive, then

$$\max_{z \in T} |r'(z)| \leq \frac{|B'(z)|}{2} \max_{z \in T} |r(z)|.$$

Regarding the number of zeros of a self-inversive polynomial inside a unit circle, we have the following well-known result [4].
Theorem E(Cohn's Theorem). Let $g(z)$ be a self inversive polynomial, then $g(z)$ has the same number of zeros inside the unit circle as does the polynomial $c[g'(z)]^*$.

In this paper, we give improvement of inequality (1.6) for self-reciprocal rational functions. Inequality for polar derivative of a polynomial is deduced which improves inequality (1.5) for the class of polynomials $p(z)$ satisfying $p(z) = z^n p(1/z)$. Moreover, the analogue of Cohn's Theorem for rational functions is also discussed.

2. Main Results

The first result gives the improvement of inequality (1.6) for self-reciprocal rational functions.
Theorem 1. If $r(z) = p(z)/w(z) \in \mathcal{R}_n$, where $p(z) = \sum_{j=0}^{n} (a_j + ib_j) z^j,\ a_j \geq 0,\ b_j \geq 0,\ z_j > 1\ \forall j$ be a self-reciprocal rational function, then

$$\max_{z \in T} |r'(z)| \leq \frac{|B'(z)|}{\sqrt{2}} \max_{z \in T} |r(z)|, \tag{2.1}$$

where equality holds for $r(z) = B(z) + 2i\sqrt{B(z)} + 1$.

For $|\alpha| > 1$, applying Theorem 1 to rational functions $p(z)/(z-\alpha)^n$ and noting that $(p(z)(z-\alpha)^n)' = -D_\alpha p(z)/(z-\alpha)^{n+1}$, we get the following improvement of inequality (1.5) for polynomials $p(z)$ satisfying $p(z) = z^n p(1/z)$.

Corollary 1. If $p(z) = \sum_{j=0}^{n} (a_j + ib_j) z^j,\ a_j \geq 0,\ b_j \geq 0,\ \forall\ j$ be a self-reciprocal polynomial, then, for $|\alpha| \geq 1$,

$$|D_\alpha p(z)| \leq \frac{n(|\alpha| + 1)}{\sqrt{2}} \max_{z \in T} |p(z)|. \tag{2.2}$$

When we look at the analogous of Cohn's theorem for rational functions of the form $r(z) = p(z)/w(z)$, we see that since $r'(z) = [w(z)p'(z) - w'(z)p(z)]/(w(z))^2$, therefore, the zeros of $w(z)$ also play a role. However, one would expect that analogous of Cohn's Theorem might be true, if we restrict zeros of $w(z)$ in a region. The feasible regions where we can restrict zeros of $w(z)$ are either $|z| < 1$ or $|z| > 1$. But both the cases does not work as is clear from the following two examples:

$$r(z) = \frac{z^2 - 3z + 1}{2z - 1},\ r(z) = \frac{iz^2 + 2z - i}{z - 2}.$$

The following result gives the indirect analogue of Cohn's Theorem for rational functions

Theorem 2. If $r(z)=p(z)/w(z)$ is a self-inversive rational function of degree n, having s zeros inside $|z|<1$ and n poles in $|z|>1$. If degree of $p(z)=$ degree of $w(z)$ and for $z\in T$

$$|p^*(z)(w'(z))^*|<|w^*(z)(p'(z))^*|$$

then $[r'(z)]^*$ has exactly $s+n+1$ zeros inside $|z|<1$.

3. Lemmas

For the proofs of these theorems we need the following lemma due to Li, Mahapatra and Rodrigues [6].

Lemma 1. For $z_j\in\mathbb{C}$ with $|z_j|>1$,

$$\frac{zB'(z)}{B(z)}=|B'(z)| \text{ for } z\in T.$$

4. Proofs of the Theorems

Proof of Theorem 1. Since

$$r(z)=\frac{\sum_{j=0}^{n}(a_j+ib_j)z^j}{w(z)}=\frac{\sum_{j=0}^{n}a_jz^j}{w(z)}+i\frac{\sum_{j=0}^{n}b_jz^j}{w(z)},$$

therefore, we can write

$$r(z)=r_1(z)+ir_2(z),$$

where $r_1(z)$ and $r_2(z)$ are rational functions of degree less than for equal to n. Also, $r(z)=B(z)r\left(1/z\right)$, therefore,

$$r_1(z)=B(z)r_1\left(\frac{1}{z}\right)=B(z)\overline{r_1\left(\frac{1}{\overline{z}}\right)},$$

and

$$r_2(z)=B(z)r_2\left(\frac{1}{z}\right)=B(z)\overline{r_2\left(\frac{1}{\overline{z}}\right)}.$$

We claim that

$$\max_{z\in T}|r_1'(z)|\leq\frac{|B'(z)|}{2}|r_1(1)|, \tag{4.1}$$

and

$$\max_{z\in T}|r_2'(z)|\leq\frac{|B'(z)|}{2}|r_2(1)|. \tag{4.2}$$

To prove our claim, let

$$F(z)=\alpha B(z)+\overline{\alpha}+r_1(z),$$

where α is a complex number with $|\alpha|=1$. Then

$$\begin{aligned}B(z)\overline{F\left(\frac{1}{\overline{z}}\right)}&=\overline{\alpha}+\alpha B(z)+B(z)\overline{r_1\left(\frac{1}{\overline{z}}\right)}\\&=\overline{\alpha}+\alpha B(z)+r_1(z)=F(z).\end{aligned}$$

This shows that $F(z)$ is a self-inversive rational function of degree n and therefore, by Theorem D, we have for $z\in T$

$$|F'(z)|\leq\frac{|B'(z)|}{2}\max_{z\in T}|F(z)|.$$

Equivalently,

$$\begin{aligned}|\alpha B'(z)+r_1'(z)| &\le \frac{|B'(z)|}{2}\max_{z\in T}|\alpha B(z)+\overline{\alpha}+r_1(z)|\\ &\le \frac{|B'(z)|}{2}\left[2+\max_{z\in T}|r_1(z)|\right]. \end{aligned} \tag{4.3}$$

Choosing argument of α such that for $z\in T$,

$$|\alpha B'(z)+r_1'(z)|=|\alpha||B'(z)|+|r_1'(z)|.$$

Letting $|\alpha|\to 1$ and using this in inequality (4.3) we have for $z\in T$,

$$\begin{aligned}|B'(z)|+|r_1'(z)| &\le |B'(z)|+\frac{|B'(z)|}{2}\max_{z\in T}|r_1(z)|\\ \Rightarrow |r_1'(z)| &\le \frac{|B'(z)|}{2}\max_{z\in T}|r_1(z)|.\end{aligned}$$

This proves inequality (4.1). Similarly, inequality (4.2) follows.

Let $|r'(z)|$ becomes maximum at $z=e^{i\xi}$, $0\le\xi<2\pi$ on T, then

$$\begin{aligned}\max_{z\in T}|r'(z)| &= |r'(e^{i\xi})|\\ &= |r_1'(e^{i\xi})+\iota r_2'(e^{i\xi})|\\ &\le |r_1'(e^{i\xi})|+|r_2'(e^{i\xi})|\\ &\le \frac{|B'(z)|}{2}\left(|r_1(1)|+|r_2(1)|\right)\\ &= \frac{|B'(z)|}{2}\left(\frac{p_1(1)+p_2(1)}{|w(1)|}\right). \end{aligned}\tag{4.4}$$

Since $2\left[(p_1(1))^2+(p_2(1))^2\right]\ge\left[p_1(1)+p_2(1)\right]^2$, therefore, from inequality (4.4) we have

$$\begin{aligned}\max_{z\in T}|r'(z)| &\le \frac{|B'(z)|}{2}\frac{\sqrt{2\left[(p_1(1))^2+(p_2(1))^2\right]}}{|w(1)|}\\ &= \frac{|B'(z)|}{\sqrt{2}}|r(1)|\\ &= \frac{|B'(z)|}{\sqrt{2}}\max_{z\in T}|r(z)|,\end{aligned}$$

which proves the required result. □

Proof of Theorem 2. We have

$$r'(z)=\frac{w(z)p'(z)-p(z)w'(z)}{[w(z)]^2}$$

Therefore,

$$\begin{aligned}[r'(z)]^* &= \frac{[w(z)p'(z)-p(z)w'(z)]^*}{[w(z)]^2}\\ &= \frac{z^{2n}\left[\overline{w\left(\frac{1}{\overline{z}}\right)p'\left(\frac{1}{\overline{z}}\right)}-\overline{p\left(\frac{1}{\overline{z}}\right)w'\left(\frac{1}{\overline{z}}\right)}\right]}{[w(z)]^2}\\ &= \frac{z\left[z^n\overline{w\left(\frac{1}{\overline{z}}\right)}z^{n-1}\overline{p'\left(\frac{1}{\overline{z}}\right)}-z^n\overline{p\left(\frac{1}{\overline{z}}\right)}z^{n-1}\overline{w'\left(\frac{1}{\overline{z}}\right)}\right]}{[w(z)]^2}\\ &= \frac{z\left[w^*(z)(p'(z))^*-p^*(z)(w'(z))^*\right]}{[w(z)]^2}.\end{aligned}$$

Now,

$$|zp^*(z)(w'(z))^*| < |zw^*(z)(p'(z))^*| \text{ for } z \in T,$$

and both sides are analytic. As $p(z)$ has s zeros in $|z| < 1$ and $p(z)$ is self inversive, therefore, by Cohn's Theorem $[p'(z)]^*$ has s zeros inside $|z| < 1$. Also, $w(z)$ has n zeros in $|z| > 1$, therefore, $w^*(z)$ has n zeros inside $|z| < 1$. Hence $zw^*(z)(p'(z))^*$ has $s + n + 1$ zeros inside $|z| < 1$. Therefore, by Rouche's Theorem, $zw^*(z)(p'(z))^* - zp^*(z)(w')^*$ has $s+n+1$ zeros inside $|z| < 1$. Thus, $[r'(z)]^*$ has $s+n+1$ zeros inside $|z| < 1$.

□

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Idrees Qasim,
Department of Mathematics,
National Institute of Technology, Srinagar
India.
E-mail address: idreesf3@nitsri.ac.in