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Lupaş type Bernstein operators on square with two curved sides

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ABSTRACT: The motive of this paper is to construct Lupaş type Bernstein operators $\left(B_{r,q}^{x_1}F\right)(x_1,x_2)$, $\left(B_{s,q}^{x_2}F\right)(x_1,x_2)$, their products $\left(P_{rs,q}F\right)(x_1,x_2)$ and $\left(Q_{sr,q}F\right)(x_1,x_2)$ and their Boolean sums $\left(S_{rs,q}F\right)(x_1,x_2)$ and $\left(T_{sr,q}F\right)(x_1,x_2)$ on square $\mathbb{D}_h$ with two curved side. Which interpolate a given function on the some edges and at the vertices of square. The remainders from the approximation formulas are computed using Peano's theorem.

Key Words: Lupaş q-Bernstein operators on square, Product operators, Boolean sum operators, Modulus of continuity, Peano's theorem, Error estimation.

Contents

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1. Introduction

Approximation theory, due to its large applications in Engineering sciences and related areas, has always attracted Mathematicians. Approximating functions, some data, or a member of a given set are some of the examples of the approximation calculations. It links both theoretical and applied mathematics from a need to represent functions in computer calculations to numerical analysis and development of mathematical software etc. Any development can be used in many industrial and commercial fields and will be helpful with the advances in the subject [26].

A constructive proof of the Weierstrass approximation theorem [6] by S.N. Bernstein in 1912 is based on uniform continuity and law of large numbers. These polynomials are now known as Bernstein polynomials in Approximation theory. In Computer Aided Geometric Design (CAGD), the basis of these Bernstein type polynomials play a significant role in preserving the shape of the curves and surfaces [27].

It is well known that the space of all continuous functions on compact interval is not strictly convex concerning uniform norm. Therefore best approximation may not be unique. Thus several authors constructed various operators to approximate continuous functions. Lupaş in 1987 [25], and Phillips [16] in 1997 respectively constructed the q-analogue of Bernstein polynomials via q-calculus. A survey of the obtained results and references on the subject can be found in [15].

The approximating operators and their basis on the square have applications in finite element analysis and Computer Aided Geometric Design [4]. The blending interpolation operators were considered in the papers by Barnhill et al. in [3,4,5]. One can refer [8,11,14] for interpolation on triangles and error bound. Schumaker studied fitting surfaces to scattered data in [17]. Bernstein-type operators, their product and Boolean sums to approximate any real-valued function f defined on triangle T_h [7] and square [23] were

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respectively studied. They also studied approximation properties on the domain with one and two curved sides of triangle and square.

Herein, we will recall and review some preliminary results of [19,23] for the sake of completeness. For a real-valued function f defined on region inside and on square with one curved side $\mathbb{D}_h$ and $(0,x_2)$, $(g(x_2),x_2)$ respectively $(x_1,0)$, $(x_1,f(x_1))$ be the points in which the parallel lines to the coordinate axes, passing through the point $(s,t)\in\mathbb{D}_h$, intersect the sides Γ_2, Γ_4 respectively Γ_1, Γ_3. We consider the uniform partitions of the intervals $[0,g(t)]$ and $[0,h]$, $t\in[0,h]$ with $g(h)=g(0)=h$. (See Figure 1 in [23]) Bernstein-type operators $B_r^{x_1}$ and $B_s^{x_2}$ are defined was follows:

$$(B_r^{x_1}F)(x_1,x_2)=\sum_{i=0}^{r}p_{r,i}(x_1,x_2)F\left(\frac{i}{r}g(x_2),x_2\right),$$

where

$$p_{r,i}(x_1,x_2)=\frac{\binom{r}{i}\ x_1^i\ (g(x_2)-x_1)^{r-i}}{g(x_2)^r}, \qquad ,$$

and

$$(B_s^{x_2}F)(x_1,x_2)=\sum_{j=0}^{s}q_{s,j}(x_1,x_2)F\left(x_1,\frac{j}{s}f(x_1)\right),$$

with

$$q_{s,j}(x_1,x_2)=\frac{\binom{s}{j}\ x_2^j\ (f(x_1)-x_2)^{s-j}}{(f(x_1))^s}, \qquad .$$

For results related to Lupaş, one can refer cf. [2, Chapter 10]. For results related Phillips and Lupaş type Bernstein operators on triangles, one can see recent work [18,19].
For $q>0$, the q-integer $[r]_q$ is defined by

$$[r]_q:=1+q+\cdots+q^{r-1} \quad r=1,2,\cdots, \quad [0]_q:=0.$$

Similarly for details of q-calculus and terms like q-factorial, q-Binomial or the Gaussian coefficient etc., one can refer [2,22]. Formula of q-analogue of Newton's binomial is

$$(1+x_1)(1+qx_1)\cdot\cdots\cdot(1+q^{r-1}x_1)=\sum_{i=0}^{r}\left[\begin{array}{c}r\\i\end{array}\right]_q q^{k(k-1)/2}x_1^k. \tag{1.1}$$

Following Lupaş [25]. We denote

$$b_{r,i}(q;x_1):=\left[\begin{array}{c}r\\i\end{array}\right]_q\frac{q^{i(i-1)/2}x_1^i(1-x_1)^{r-i}}{(1-x_1+qx_1)\cdots(1-x_1+q^{r-1}x_1)}. \tag{1.2}$$

It follows from (1.2) that

$$\sum_{i=0}^{r}b_{r,i}(q;x_1)=1, \qquad x_1\in[0,1]. \tag{1.3}$$

For $x_1=1$, equation (1.3) is obvious. For $x_1\neq 1$, we get

$$\begin{aligned}&\sum_{i=0}^{r}\left[\begin{array}{c}r\\i\end{array}\right]_q q^{i(i-1)/2}x_1^i(1-x_1)^{r-i}\\&=(1-x_1)^r\left(1+\frac{x_1}{1-x_1}\right)\left(1+q\frac{x_1}{1-x_1}\right)\cdots\left(1+q^{r-1}\frac{x_1}{1-x_1}\right)\\&=(1-x_1+qx_1)\cdots(1-x_1+q^{r-1}x_1).\end{aligned}$$

In this paper, we construct and study Lupaş type q-Bernstein operators on the square with one and two curve sides which interpolate the value of a given function on some edges of a square. The remainders from approximation formulas are evaluated using modulus of continuity and Peano's theorem. For some recent relevant literature related to Bernstein type operators, see: Cai et al. studied approximation properties by λ-Bernstein operators in [9,10], Stancu investigated approximation properties for Bernstein type polynomials and evaluated remainder terms in [20,21], Braha et al. studied convergence properties of λ-Bernstein operators via power series summability methods [24], Mursaleen et al studied error estimation for q-Bernstein shifted operators and generalized q-Bernstein Schurer operators in [12,13]. Other relevant papers related to Bernstein and its bivariate form with applications in CAGD, one can refer [1,24].

2. New univariate Lupaş operators on square

Let $\mathbb{D}_h$ be the region inside and on the square with two curved side with vertices $V_1=(0,0), V_2=(h,0), V_3=(h,h)$ and $V_4=(0,h)$. Consider a real-valued function F defined on $\mathbb{D}_h$ with two curved side. Through the point $(x_1,x_2)\in\mathbb{D}_h$, one considers the parallel lines to the OX_1 axis which intersect the edges Γ_2 and Γ_4 of the square at the points $(0,x_2)$ and $(g(x_2),x_2)$ and parallel line to OX_2 axis Γ_1 and Γ_3 at the points $(x_1,0)$, $(x_1,f(x_1))$ as shown in Figure 1 and 2.

Let $\square_r^{x_1}=\{[i]_q\frac{g(x_2)}{[r]_q}, i=\overline{0,r}\}$ and $\square_s^{x_2}=\{[j]_q\frac{f(x_1)}{[s]_q}, j=\overline{0,s}\}$ be uniform partitions of the intervals $[0,g(x_2)]$ and $[0,f(x_1)]$, respectively.
We define the new Lupaş type Bernstein operators $B_{r,q}^{x_1}$ and $B_{s,q}^{x_2}$ by using quantum calculus as follows:

$$(B_{r,q}^{x_1}F)=\sum_{i=0}^{r}p_{r,i}^{*}(x_1,x_2)F\left(\frac{[i]_q}{[r]_q}g(x_2),x_2\right),\qquad (x_1,x_2)\in\mathbb{D}_h \tag{2.1}$$

and

$$(B_{s,q}^{x_2}F)=\sum_{j=0}^{s}q_{s,j}^{*}(x_1,x_2)F\left(x_1,\frac{[j]_q}{[s]_q}f(x_1)\right),\qquad (x_1,x_2)\in\mathbb{D}_h \tag{2.2}$$

where,

$$p_{r,i}^{*}(x_1,x_2)=\frac{\left[\begin{array}{c} r\\ i\end{array}\right]_q q^{i(i-1)/2}\ x_1^i\ (g(x_2)-x_1)^{r-i}}{\prod_{w=0}^{r-1}(g(x_2)-x_1+q^w x_1)},$$

and

$$q_{s,j}^{*}(x_1,x_2)=\frac{\left[\begin{array}{c} s\\ j\end{array}\right]_q q^{j(j-1)/2}\ x_2^j\ (f(x_1)-x_2)^{s-j}}{\prod_{z=0}^{s-1}(f(x_1)-x_2+q^z x_2)},$$

respectively. Also, the operator(2.1) and (2.2) reduces to classical Bernstein type operators on square if $q=1$ [23].

Note that we assume $(B_{r,q}^{x_1}F)(V_i)=B_{r,q}^{x_2}F)(V_i)=F(V_i),\quad i=1,2,3,4.$

Theorem 2.1 *If F is a real-valued function defined on $\mathbb{D}_h$, then*

(i) $B_{r,q}^{x_1}F=F$ *on* $\Gamma_2\cup\Gamma_4$;

$$(ii)(B_{r,q}^{x_1}e_{ij})\ (x_1,x_2)=x_1^i x_2^j,\quad i=0,1\ j\in\mathbb{N} \tag{2.3}$$

$$(iii)(B_{r,q}^{x_1}e_{2j})\ (x_1,x_2)=x_1^2+\frac{x_1(g(x_2)-x_1)}{[r]_q}-\frac{x_1^2(g(x_2)-x_1)(1-q)}{g(x_2)-x_1+qx_1}\left(1-\frac{1}{[r]_q}\right), \tag{2.4}$$

Proof. The interpolation properties (i) and (ii) follow from the relations

$$p_{r,i}^{*}\ (0,x_2)=\begin{cases}1, & \text{if } i=0,\\ 0, & i>0,\end{cases}$$

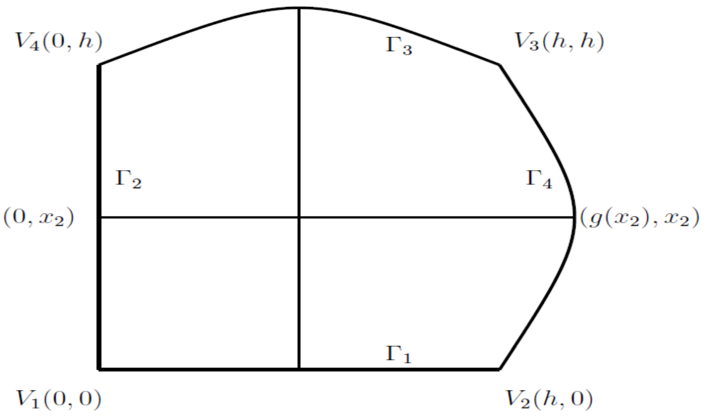

Figure 1: Square with two curved sides

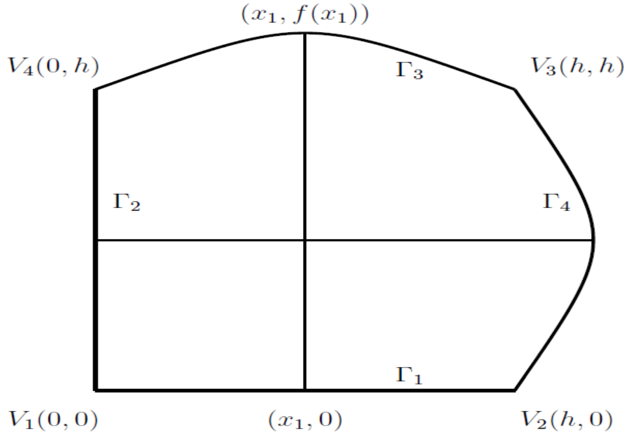

Figure 2: Square with two curved sides

and

$$p^*_{r,i}(g(x_2), x_2) = \begin{cases} 0, & \text{if } i < r, \\ 1, & i = r. \end{cases}$$

Regarding the property (i) easily follows by using the definition of above property of basis functions.

$$\begin{aligned}
\left(B^{x_1}_{r,q} e_{00}\right)(x_1, x_2) &= \sum_{i=0}^{r} \frac{\begin{bmatrix} r \\ i \end{bmatrix}_q q^{i(i-1)/2}\; x_1^i\; (g(x_2) - x_1)^{r-i}}{\prod_{w=0}^{r-1}(g(x_2) - x_1 + q^w x_1)} \\
&= \frac{\prod_{i=0}^{r-1}(g(x_2) - x_1 + q^i x_1)}{\prod_{w=0}^{r-1}(g(x_2) - x_1 + q^w x_1)} = 1; \\
\left(B^{x_1}_{r,q} e_{10}\right)(x_1, x_2) &= \sum_{i=0}^{r} \frac{\begin{bmatrix} r \\ i \end{bmatrix}_q q^{i(i-1)/2}\; x_1^i\; (g(x_2) - x_1)^{r-i}}{\prod_{w=0}^{r-1}(g(x_2) - x_1 + q^w x_1)} \frac{[i]_q}{[r]_q} g(x_2) \\
&= \sum_{i=0}^{r} \frac{\frac{[i]_q}{[r]_q}\begin{bmatrix} r \\ i \end{bmatrix}_q q^{i(i-1)/2}\; x_1^i\; (g(x_2) - x_1)^{r-i}}{\prod_{w=1}^{r-1}(g(x_2) - x_1 + q^w x_1)} \\
&= \sum_{i=0}^{r-1} \frac{\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{i(i+1)/2}\; x_1^{i+1}\; (g(x_2) - x_1)^{r-i-1}}{\prod_{w=1}^{r-1}(g(x_2) - x_1 + q^w x_1)} \\
&= x_1 \sum_{i=0}^{r-1} \frac{\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{i(i-1)/2}\; (qx_1)^i\; (g(x_2) - x_1)^{r-i-1}}{\prod_{w=0}^{r-2}(g(x_2) - x_1 + q^w (qx_1))} \\
&= x_1.
\end{aligned}$$

$$
\begin{aligned}
\left(B^{x_1}_{r,q}e_{20}\right)(x_1,x_2) &= \sum_{i=0}^{r}\frac{\begin{bmatrix} r \\ i \end{bmatrix}_q q^{i(i-1)/2}\ x_1^i\ (g(x_2)-x_1)^{r-i}}{\prod_{w=0}^{r-1}(g(x_2)-x_1+q^w x_1)}\frac{[i]_q^2}{[r]_q^2}(g(x_2))^2 \\
&= (g(x_2))^2\sum_{i=0}^{r-1}\frac{\frac{[i+1]_q}{[r]_q}\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{i(i+1)/2}\ x_1^{i+1}\ (g(x_2)-x_1)^{r-i-1}}{\prod_{w=0}^{r-1}(g(x_2)-x_1+q^w x_1)} \\
&= (g(x_2))^2 x_1\sum_{i=0}^{r-1}\frac{\frac{1+q[i]_q}{[r]_q}\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{i(i-1)/2}\ (qx_1)^i\ (g(x_2)-x_1)^{r-1-i}}{\prod_{w=0}^{r-1}(g(x_2)-x_1+q^w x_1)} \\
&= (g(x_2))\frac{x_1}{[r]_q}\sum_{i=0}^{r-1}\frac{\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{i(i-1)/2}\ (qx_1)^i\ (g(x_2)-x_1)^{r-1-i}}{\prod_{w=0}^{r-2}(g(x_2)-x_1+q^w(qx_1))} \\
&+ (g(x_2))^2 x_1\sum_{i=0}^{r-1}\frac{\frac{q[r-1]_q}{[r]_q}\frac{[i]_q}{[r-1]_q}\begin{bmatrix} r-1 \\ i \end{bmatrix}_q q^{i(i-1)/2}\ (qx_1)^i\ (g(x_2)-x_1)^{r-1-i}}{\prod_{w=0}^{r-2}(g(x_2)-x_1+q^w(qx_1))} \\
&= g(x_2)\frac{x_1}{[r]_q} \\
&+ \frac{(g(x_2))x_1}{(g(x_2)-x_1+qx_1)}\frac{q[r-1]_q}{[r]_q}\sum_{i=0}^{r-2}\frac{\begin{bmatrix} r-2 \\ i \end{bmatrix}_q q^{i(i+1)/2}\ (qx_1)^{i+1}\ (g(x_2)-x_1)^{r-2-i}}{\prod_{w=0}^{r-3}(g(x_2)-x_1+q^w(q^2x_1))} \\
&= g(x_2)\frac{x_1}{[r]_q}+\frac{(g(x_2))qx_1^2}{g(x_2)-x_1+qx_1}\left(1-\frac{1}{[r]_q}\right),
\end{aligned}
$$

or, equivalently,

$$
\begin{aligned}
\left(B^{x_1}_{r,q}e_{20}\right)(x_1,x_2) &= x_1^2\left(1-\frac{1}{[r]_q}\right)+g(x_2)\frac{x_1}{[r]_q}-\left(x_1^2-\frac{(g(x_2))qx_1^2}{g(x_2)-x_1+qx_1}\right)\left(1-\frac{1}{[r]_q}\right) \\
&= x_1^2+\frac{x_1(g(x_2)-x_1)}{[r]_q}-\frac{x_1^2(g(x_2)-x_1)(1-q)}{g(x_2)-x_1+qx_1}\left(1-\frac{1}{[r]_q}\right). \qquad \square
\end{aligned}
$$

Remark 1. The basis function $q^*_{s,j}(x_1,x_2)$ satisfies

$$
q^*_{s,j}\ (x_1,0)=\begin{cases} 1, & \text{if } j=0, \\ 0, & j>0, \end{cases}
$$

and

$$
q^*_{s,j}(x_1,f(x_1))=\begin{cases} 0, & \text{if } j<s, \\ 1, & j=s. \end{cases}
$$

Following similar steps, it can be proved that:
If F is a real-valued function defined on $\mathbb{D}_h$, then

$$(i)\ B^{x_2}_{s,q}F=F\ on\ \Gamma_1\cup\Gamma_3, \tag{2.5}$$

$$(ii)(B^{x_2}_{s,q}e_{ij})\ (x_1,x_2)=x_1^i x_2^j,\quad j=0,1;i\in\mathbb{N} \tag{2.6}$$

$$(iii)\ (B^{x_2}_{s,q}e_{02})\ (x_1,x_2)=x_2^2+\frac{x_2(f(x_1)-x_2)}{[s]_q}-\frac{x_2^2(f(x_1)-x_2)(1-q)}{f(x_1)-x_2+qx_2}\left(1-\frac{1}{[s]_q}\right), \tag{2.7}$$

Based on the following approximation formula

$$F = B_{r,q}^{x_1}F + R_{r,q}^{x_1}F,$$

error will be zero at interpolating points of $\mathbb{D}_h$, however we compute error bounds for these operators at non-interpolating points.

Theorem 2.2 *If $F(\cdot,x_2) \in C[0,g(x_2)]$, $x_2 \in [0,h]$ then*

$$\Big|(R_{r,q}^{x_1}F)(x_1,x_2)\Big| \leq \left[1+\frac{1}{\delta}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q} - \frac{x_1^2(g(x_2)-x_1)(1-q)}{g(x_2)-x_1+qx_1}\left(1-\frac{1}{[r]_q}\right)}\ \right] w\big(F(\cdot,x_2);\delta\big) \tag{2.8}$$

and

$$\Big|(R_{r,q}^{x_1}F)(x_1,x_2)\Big| \leq \left(1+\frac{g(x_2)}{2\delta\sqrt{[r]_q}}\right) w\big(F(\cdot,x_2);\delta\big), \qquad x_2 \in [0,h].$$

where modulus of continuity of the function F with respect to the variable x_1 is denoted by $w(F(\cdot,x_2);\delta)$. Further, if $\delta = \frac{1}{\sqrt{[r]_q}}$, then

$$\Big|(R_{r,q}^{x_1}F)(x_1,x_2)\Big| \leq \left(1+\frac{M}{2}\right) w\left(F(\cdot,x_2);\frac{1}{\sqrt{[r]_q}}\right), \qquad x_2 \in [0,h] \tag{2.9}$$

Proof. We have

$$\Big|(R_{r,q}^{x_1}F)(x_1,x_2)\Big| \leq \sum_{i=0}^{r} p_{r,i}^*(x_1,x_2)\left|F(x_1,x_2) - F\left(\frac{[i]_q g(x_2)}{[r]_q},x_2\right)\right|.$$

Since,

$$\left|F(x_1,x_2) - F\left(\frac{[i]_q g(x_2)}{[r]_q},x_2\right)\right| \leq \left(\frac{1}{\delta}\left|x_1 - \frac{[i]_q g(x_2)}{[r]_q}\right| + 1\right) w(F(\cdot,x_2);\delta),$$

one obtains,

$$\begin{aligned}
\Big|(R_{r,q}^{x_1}F)(x_1,x_2)\Big| &\leq \sum_{i=0}^{r} p_{r,i}^*(x_1,x_2)\left(\frac{1}{\delta}\left|x_1 - \frac{[i]_q g(x_2)}{[r]_q}\right| + 1\right) w(F(\cdot,x_2);\delta) \\
&\leq \left[1+\frac{1}{\delta}\left(\sum_{i=0}^{r} p_{r,i}^*(x_1,x_2)\left(x_1 - \frac{[i]_q g(x_2)}{[r]_q}\right)^2\right)^{1/2}\right] w(F(\cdot,x_2);\delta) \\
&= \left[1+\frac{1}{\delta}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q} - \frac{x_1^2(g(x_2)-x_1)(1-q)}{g(x_2)-x_1+qx_1}\left(1-\frac{1}{[r]_q}\right)}\ \right] w(F(\cdot,x_2);\delta).
\end{aligned}$$

If $0 < q \leq 1$, then we can see that

$$\frac{x_1(g(x_2)-x_1)}{[r]_q} \geq \frac{x_1^2(g(x_2)-x_1)(1-q)}{g(x_2)-x_1+qx_1}\left(1-\frac{1}{[r]_q}\right) \qquad \textit{for all } (x_1,x_2) \in \mathbb{D}_h,$$

and the term on the right side of inequality is always non-negative for all $x_1, x_2 \in \mathbb{D}_h$.
We have,

$$\left|(R_{r,q}^{x_1}F)(x_1,x_2)\right| \le \left[1+\frac{1}{\delta}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q}}\,\right] w(F(\centerdot,x_2);\delta).$$

Since,

$$\max_{\mathbb{D}_h}[x_1(g(x_2)-x_1)] = \frac{(g(x_2))^2}{4},$$

it follows that

$$\left|(R_{r,q}^{x_1}F)(x_1,x_2)\right| \le \left(1+\frac{g(x_2)}{2\delta\sqrt{[r]_q}}\right) w(F(\centerdot,x_2);\delta).$$

For $\delta = \frac{1}{\sqrt{[r]_q}}$ and let $M = \underset{0\le x_2\le h}{max}\, g(x_2)$, we obtain

$$\left|(R_{r,q}^{x_1}F)(x_1,x_2)\right| \le \left(1+\frac{M}{2}\right)\, w\left(F(\centerdot,x_2);\frac{1}{\sqrt{[r]_q}}\right).$$

Theorem 2.3 *If $F(\centerdot,x_2) \in C^2[0,g(x_2)]$, then*

$$(R_{r,q}^{x_1}F)(x_1,x_2) = -\left(\frac{x_1(g(x_2)-x_1)}{2[r]_q} - \frac{x_1^2(g(x_2)-x_1)(1-q)}{2(g(x_2)-x_1+qx_1)}\left(1-\frac{1}{[r]_q}\right)\right)F^{(2,0)}(\xi,x_2), \quad \xi\in[0,g(x_2)], \tag{2.10}$$

for all $x_2 \in [0,h]$ and $0 < q \le 1$, we have

$$\left|(R_{r,q}^{x_1}F)(x_1,x_2)\right| \le \frac{M^2}{8[r]_q}M_{20}F, \qquad (x_1,x_2)\in\mathbb{D}_h,$$

where,

$$M_{ij}F = \max_{\mathbb{D}_h}\left|F^{(i,j)}(x_1,x_2)\right|.$$

Proof. As $dex(B_{r,q}^{x_1}) = 1$, by Peano's theorem, one obtains '

$$\left(R_{r,q}^{x_1}F\right)(x_1,x_2) = \int_0^{g(x_2)} K_{20}(x_1,x_2;t)F^{(2,0)}(t,x_2)dt,$$

where the kernel

$$K_{20}(x_1,x_2;t) := R_{r,q}^{x_1}\left[(x_1-t)_+\right] = (x_1-t)_+ - \sum_{i=0}^{r} p_{r,i}^*(x_1,x_2)\left([i]_q\frac{g(x_2)}{[r]_q}-t\right)_+$$

does not change the sign $(K_{20}(x_1,x_2;t)\le 0, \quad x_1\in[0,g(x_2)]$. By the Mean Value Theorem, it follows that

$$\left(R_{r,q}^{x_1}F\right)(x_1,x_2) = F^{(2,0)}(\xi,x_2)\int_0^{g(x_2)} K_{20}(x_1,x_2;t)dt, \qquad \xi\in[0,g(x_2)].$$

After some computation, we get

$$(R_{r,q}^{x_1}F)(x_1,x_2) = -\left(\frac{x_1(g(x_2)-x_1)}{2[r]_q} - \frac{x_1^2(g(x_2)-x_1)(1-q)}{2(g(x_2)-x_1+qx_1)}\left(1-\frac{1}{[r]_q}\right)\right)F^{(2,0)}(\xi,x_2),$$

where $\xi\in[0,g(x_2)]$.
If $0<q\le 1$, then

$$\frac{x_1(g(x_2)-x_1)}{2[r]_q} \ge \frac{x_1^2((g(x_2)-x_1))(1-q)}{2(g(x_2)-x_1+qx_1)}\left(1-\frac{1}{[r]_q}\right), \qquad for\ all\ (x_1,x_2)\in\mathbb{D}_h,$$

After certain steps, we obtain

$$\left|(R^{x_1}_{r,q}F)(x_1,x_2)\right| \leq \frac{M^2}{8[r]_q}M_{20}F, \qquad (x_1,x_2)\in \mathbb{D}_h. \qquad \square$$

Remark 2.1 From (2.13) it follows that
- If $F(.,x_2)$ is a concave function, then $\big(R^{x_1}_{r,q}F\big)(x_1,x_2)\geq 0$, i.e. $\big(B^{x_1}_{r,q}F\big)(x_1,x_2)\leq F(x_1,x_2)$.
- If $F(.,x_2)$ is a convex function, then $\big(R^{x_1}_{r,q}F\big)(x_1,x_2)\leq 0$, i.e. $\big(B^{x_1}_{r,q}F\big)(x_1,x_2)\geq F(x_1,x_2)$, for $x_1\in[0,g(x_2)]$ and $x_2\in[0,f(x_1)]$. $\square$

Remark 2.2 For the remainder $R^{x_2}_{s,q}F$ of the approximation formula

$$F = B^{x_2}_{s,q}F + R^{x_2}_{s,q}F.$$

We also have:
A. if $F(x_1,.)\in C[0,f(x_1)]$, then

$$\left|(R^{x_2}_{s,q}F)(x_1,x_2)\right| \leq \left[1+\frac{1}{\delta}\sqrt{\frac{x_2(f(x_1)-x_2)}{[s]_q} - \frac{x_2^2(f(x_1)-x_2)(1-q)}{(F(x_1)-x_2+qx_2)}\left(1-\frac{1}{[s]_q}\right)}\ \right] w\big(F(x_1,.);\delta\big), \tag{2.11}$$

for all $x_1\in[0,h]$ If $0<q\leq 1$, and $N=\underset{0\leq x_1\leq h}{max}(f(x_1))$

$$\left|(R^{x_2}_{s,q}F)(x_1,x_2)\right| \leq \left(1+\frac{N}{2\delta\sqrt{[s]_q}}\right) w\big(F(x_1,.);\delta\big), \qquad x_1\in[0,h],$$

where modulus of continuity of the function F with respect to the variable x_2 is denoted by $w(f(x_1,.);\delta)$. Further, if $\delta=\frac{1}{\sqrt{[s]_q}}$, then

$$\left|(R^{x_2}_{s,q}F)(x_1,x_2)\right| \leq \left(1+\frac{N}{2}\right) w\left(F(x_1,.);\frac{1}{\sqrt{[s]_q}}\right), \qquad x_1\in[0,h]. \tag{2.12}$$

B. If $F(x_1,.)\in C^2[0,f(x_1)]$, then

$$(R^{x_2}_{s,q}F)(x_1,x_2) = -\left(\frac{x_2(f(x_1)-x_2)}{2[s]_q} - \frac{x_2^2(f(x_1)-x_2)(1-q)}{2(f(x_1)-x_2+qx_2)}\left(1-\frac{1}{[s]_q}\right)\right)F^{(0,2)}(x_1,\eta),\ \ \eta\in[0,f(x_1)], \tag{2.13}$$

for all $x_1\in[0,h]$ and if $0<q\leq 1$, we have

$$\left|(R^{x_2}_{s,q}F)(x_1,x_2)\right| \leq \frac{N^2}{8[s]_q}M_{02}F, \qquad (x_1,x_2)\in\mathbb{D}_h,$$

where,

$$M_{ij}f = \max_{\mathbb{D}_h}\left|F^{(i,j)}(x_1,x_2)\right|. \qquad \square$$

3. Product operators

Let $P_{rs,q}=B^{x_1}_{r,q}B^{x_2}_{s,q}$ and $Q_{rs,q}=B^{x_2}_{s,q}B^{x_1}_{r,q}$ be the products of operators $B^{x_1}_{r,q}$ and $B^{x_2}_{s,q}$. We have

$$(P_{rs,q}F)(x_1,x_2) = \sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{(g(x_2))}{[r]_q},x_2\right)\ F\left([i]_q\frac{g(x_2)}{[r]_q},\frac{[j]_q}{[s]_q}f(\frac{[i]_q}{[r]_q}g(x_2))\right).$$

Remark 3.1 The nodes used in the operator $P_{rs,q}$ are the q-analogue of the nodes, which are given in [23, Figure 12], for $i = \overline{0,r}$; $j = \overline{0,s}$, and $x_2 \in [0, f(x_1)]$. □

Theorem 3.1 *For all $x_1 \in [0, g(x_2)]$ and $x_2 \in [0, f(x_1)]$, The product operator $P_{rs,q}$ satisfies the following relations:*
(i) $(P_{rs,q}F)(x_1, 0) = (B^{x_1}_{r,q}F)(x_1, 0)$,
(ii) $(P_{rs,q}F)(0, x_2) = (B^{x_2}_{s,q}F)(0, x_2)$,
(iii) $(P_{rs,q}F)(x_1, f(x_1)) = (B^{x_1}_{r,q}F)(x_1, f(x_1))$,
(iv) $(P_{rs,q}F)(g(x_2), x_2) = (B^{x_2}_{s,q}F)(g(x_2), x_2)$.
Above proofs follow from some simple computation.

Remark 3.2 From the above theorem it is clear that the operators $P_{rs,q}$ interpolate the given function on the vertices of the square $\mathbb{D}_h$, i.e, $(P_{rs,q}F(V_i)) = V_i$ for $i = 1, 2, 3, 4$.

The product operator $Q_{rs,q}$ given by

$$(Q_{sr,q}F)(x_1, x_2) = \sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}\left(x_1, [j]_q \frac{f(x_1)}{[s]_q}\right) q^*_{s,j}(x_1, x_2)\ F\left(\frac{[i]_q}{[r]_q} g(\frac{[j]_q}{[s]_q} f(x_1)), [j]_q \frac{f(x_1)}{[s]_q}\right),$$

has the nodes, which are q-analogue of nodes given in [23, Figure 13], for $i = \overline{0,r}$, $j = \overline{0,s}$, $x_1 \in [0; g(x_2)]$, and $x_2 \in [0, f(x_1)]$.
For all $x_1 \in [0, g(x_2)]$ and $x_2 \in [0, f(x_1)]$, the operator $Q_{rs,q}$ satisfies the following relations:
(i) $(Q_{sr,q}F)(x_1, 0) = (B^{x_1}_{r,q}F)(x_1, 0)$,
(ii) $(Q_{sr,q}F)(0, x_2) = (B^{x_2}_{s,q}F)(0, x_2)$,
(iii) $(Q_{sr,q}F)(g(x_2), x_2) = (B^{x_2}_{s,q}F)(g(x_2), x_2)$,
(iv) $(Q_{sr,q}F)(x_1, f(x_1)) = (B^{x_1}_{r,q}F)(x_1, f(x_1))$,

Remark 3.3 From the above properties of operator $Q_{sr,q}$ it is clear that the operators $Q_{sr,q}$ interpolate the given function on the vertices of the square $\mathbb{D}_h$, i.e, $Q_{sr,q}F(V_i) = V_i$ for $i = 1, 2, 3, 4$.

Let us consider the approximation formula

$$F = P_{rs,q}F + R^P_{rs,q}F.$$

Theorem 3.2 *If $F \in C(\mathbb{D}_h)$ and $0 < q$ then*

$$\begin{aligned}\left|(R^P_{rs,q}F)(x_1, x_2)\right| \le \Bigg(1 &+ \frac{1}{\delta_1}\sqrt{\frac{x_1(g(x_2) - x_1)}{[r]_q} - \frac{x_1^2(g(x_2) - x_1)(1 - q)}{(g(x_2) - x_1 + qx_1)}\left(1 - \frac{1}{[r]_q}\right)} \\ &+ \frac{1}{\delta_2}\sqrt{\frac{x_2(f(x_1) - x_2)}{[s]_q} - \frac{x_2^2(f(x_1) - x_2)(1 - q)}{(f(x_1) - x_2 + qx_2)}\left(1 - \frac{1}{[s]_q}\right)}\ \Bigg),\end{aligned} \tag{3.1}$$

for all $(x_1, x_2) \in \mathbb{D}_h \setminus \{V_1, V_2, V_3, V_4\}$ and at the vertices the remainder $R^P_{rs,q}F$ is zero by Remark 3.2. If $0 < q \le 1$, then

$$\left|(R^P_{rs,q}F)(x_1, x_2)\right| \le (1 + \frac{M}{2} + \frac{N}{2})w\left(F; \frac{1}{\sqrt{[r]_q}}, \frac{1}{\sqrt{[s]_q}}\right) \qquad \textit{for all } (x_1, x_2) \in \mathbb{D}_h. \tag{3.2}$$

Proof. We have

$$\Big|\big(R^P_{rs,q}F\big)(x_1,x_2)\Big| \le \Bigg[\frac{1}{\delta_1}\sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{g(x_2)}{[r]_q},x_2\right)\left|x_1-[i]_q\frac{g(x_2)}{[r]_q}\right|$$
$$+\frac{1}{\delta_2}\sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{g(x_2)}{[r]_q},x_2\right)\left|x_2-\frac{[j]_q}{[s]_q}f(\frac{[i]_q}{[r]_q}g(x_2))\right|$$
$$+\sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{g(x_2)}{[r]_q},x_2\right)\Bigg]w(F;\delta_1,\delta_2).$$

After some transformations, one obtains

$$\sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{g(x_2)}{[r]_q},x_2\right)\left|x_1-\frac{[j]_q}{[s]_q}F(\frac{[i]_q}{[r]_q}g(x_2))\right|$$
$$\le\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q}-\frac{x_1^2(g(x_2)-x_1)(1-q)}{(g(x_2)-x_1+qx_1)}\left(1-\frac{1}{[r]_q}\right)},$$

$$\sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{g(x_2)}{[r]_q},x_2\right)\left|x_2-\frac{[j]_q}{[s]_q}f(\frac{[i]_q}{[r]_q}g(x_2))\right|$$
$$\le\sqrt{\frac{x_2(f(x_1)-x_2)}{[s]_q}-\frac{x_2^2(f(x_1)-x_2)(1-q)}{(f(x_1)-x_2+qx_2)}\left(1-\frac{1}{[s]_q}\right)},$$

while,

$$\sum_{i=0}^{r}\sum_{j=0}^{s} p^*_{r,i}(x_1,x_2)q^*_{s,j}\left([i]_q\frac{g(x_2)}{[r]_q},x_2\right)=1.$$

It follows,

$$\Big|\big(R^P_{rs,q}F\big)(x_1,x_2)\Big| \le \Bigg(\frac{1}{\delta_1}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q}-\frac{x_1^2(g(x_2)-x_1)(1-q)}{(g(x_2)-x_1+qx_1)}\left(1-\frac{1}{[r]_q}\right)}$$
$$+\frac{1}{\delta_2}\sqrt{\frac{x_2(f(x_1)-x_2)}{[s]_q}-\frac{x_2^2(f(x_1)-x_2)(1-q)}{(f(x_1)-x_2+qx_2)}\left(1-\frac{1}{[s]_q}\right)}\ +\ 1\Bigg)w(F;\delta_1,\delta_2). \tag{3.3}$$

Taking into acccount that if $0<q\le 1$ then

$$\Big|\big(R^P_{rs,q}F\big)(x_1,x_2)\Big| \le \left(\frac{1}{\delta_1}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q}}+\frac{1}{\delta_2}\sqrt{\frac{x_2(f(x_1)-x_2)}{[s]_q}}\ +\ 1\right)w(F;\delta_1,\delta_2).$$

Since,

$$\frac{x_1(g(x_2)-x_1)}{[r]_q}\le\frac{g(x_2)^2}{4[r]_q},\qquad \frac{x_2(f(x_1)-x_2)}{[s]_q}\le\frac{f(x_1)^2}{4[s]_q},\qquad for\ all\ (x_1,x_2)\in\mathbb{D}_h,$$

we have

$$\Big|\big(R^P_{rs,q}F\big)(x_1,x_2)\Big| \le \left(\frac{g(x_2)}{2\delta_1\sqrt{[r]_q}}+\frac{f(x_1)}{2\delta_2\sqrt{[s]_q}}\ +\ 1\right)w(F;\delta_1,\delta_2)$$

$$\left|(R^P_{rs,q}F)(x_1,x_2)\right| \leq \left(\frac{M}{2\delta_1\sqrt{[r]_q}} + \frac{N}{2\delta_2\sqrt{[s]_q}} + 1\right)$$

if $\delta_1 = \frac{1}{\sqrt{[r]_q}}$ and $\delta_2 = \frac{1}{\sqrt{[s]_q}}$ we have

$$\left|(R^P_{rs,q}F)(x_1,x_2)\right| \leq \left(\frac{M}{2} + \frac{N}{2} + 1\right)w(F; \frac{1}{\sqrt{[r]_q}}, \frac{1}{\sqrt{[s]_q}})$$

4. Boolean sum operators

Let

$$S_{rs,q} := B^{x_1}_{r,q} \oplus B^{x_2}_{s,q} = B^{x_1}_{r,q} + B^{x_2}_{s,q} - B^{x_1}_{r,q}B^{x_2}_{s,q},$$

$$T_{sr,q} := B^{x_2}_{s,q} \oplus B^{x_1}_{r,q} = B^{x_2}_{s,q} + B^{x_1}_{r,q} - B^{x_2}_{s,q}B^{x_1}_{r,q},$$

be the Boolean sums of the Lupaş type Bernstein operators $B^{x_1}_{r,q}$ and $B^{x_2}_{s,q}$.

Theorem 4.1 *For the real valued function F defined on $\mathbb{D}_h$, we have*

$$S_{rs,q}F\Big|_{\partial\mathbb{D}_h} = F\Big|_{\partial\mathbb{D}_h}.$$

Proof. We have

$$S_{rs,q}F = (B^{x_1}_{r,q} + B^{x_2}_{s,q} - B^{x_1}_{r,q}B^{x_2}_{s,q})F.$$

The interpolation properties of $B^{x_1}_{r,q}$ and $B^{x_2}_{s,q}$ together with the properties $(i)-(iii)$ of $P_{rs,q}$, imply that

$$(S_{rs,q}F)(x_1,0) = (B^{x_1}_{r,q}f)(x_1,0) + F(x_1,0) - (B^{x_1}_{r,q}F)(x_1,0) = F(x_1,0),$$

$$(S_{rs,q}F)(0,x_2) = F(0,x_2) + (B^{x_2}_{s,q}F)(0,x_2) - (B^{x_2}_{s,q}F)(0,x_2) = F(0,x_2),$$

$$(S_{rs,q}F)(x_1,f(x_1)) = (B^{x_1}_{r,q}F)(x_1,f(x_1)) + F(x_1,f(x_1)) - (B^{x_1}_{r,q}F)(x_1,f(x_1)) = F(x_1,f(x_1)),$$

$$(S_{rs,q}F)(g(x_2),x_2) = F(g(x_2),x_2) + (B^{x_2}_{s,q}F)(g(x_2),x_2) - (B^{x_2}_{s,q}F)(g(x_2),x_2) = F(g(x_2),x_2),$$

for all $(x_1,x_2) \in \mathbb{D}_h$ □

Let $R^S_{rs,q}F$ be the remainder of the Boolean sum approximation formula

$$F = S_{rs,q}F + R^S_{rs,q}F.$$

Theorem 4.2 *If $F \in C(\mathbb{D}_h)$, then* $\left|\left(R^S_{rs,q}F\right)(x_1,x_2)\right|$

$$\begin{aligned}
&\leq \left[1 + \frac{1}{\delta_1}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q} - \frac{x_1^2(g(x_2)-x_1)(1-q)}{(g(x_2)-x_1+qx_1)}\left(1-\frac{1}{[r]_q}\right)}\ \right] w\big(F(\cdot,x_2);\delta_1\big)\\
&+ \left[1 + \frac{1}{\delta_2}\sqrt{\frac{x_2(f(x_1)-x_2)}{[s]_q} - \frac{x_2^2(f(x_1)-x_2)(1-q)}{(f(x_1)-x_2+qx_2)}\left(1-\frac{1}{[s]_q}\right)}\ \right] w\big(F(x_1,\cdot);\delta_2\big)\\
&+ \left[1 + \frac{1}{\delta_1}\sqrt{\frac{x_1(g(x_2)-x_1)}{[r]_q} - \frac{x_1^2(g(x_2)-x_1)(1-q)}{(h-x_1-x_2+qx_1)}\left(1-\frac{1}{[r]_q}\right)}\right.\\
&\left.+ \frac{1}{\delta_2}\sqrt{\frac{x_2(f(x_1)-x_2)}{[s]_q} - \frac{x_2^2(f(x_1)-x_2)(1-q)}{(f(x_1)-x_2+qx_2)}\left(1-\frac{1}{[s]_q}\right)}\ \right] w(F;\delta_1,\delta_2),
\end{aligned} \tag{4.1}$$

for all $(x_1, x_2) \in \mathbb{D}_h \setminus \{\partial\mathbb{D}_h\}$ *and at the boundary points the remainder* $R^P_{ms,q}F$ *is zero by Theorem 4.1. Moreover, if* $0 < q \leq 1$, $\delta_1 = \frac{1}{\sqrt{[r]_q}}$ *and* $\delta_2 = \frac{1}{\sqrt{[s]_q}}$ *then*

$$\left|\left(R^S_{rs,q}F\right)(x_1, x_2)\right| \leq \left(1 + \frac{M}{2}\right) w\left(F(.,x_2); \frac{1}{\sqrt{[r]_q}}\right) + \left(1 + \frac{N}{2}\right) w\left(F(x_1,.); \frac{1}{\sqrt{[s]_q}}\right)$$
$$+ \left(1 + \frac{M}{2} + \frac{N}{2}\right) w\left(F; \frac{1}{\sqrt{[r]_q}}, \frac{1}{\sqrt{[s]_q}}\right). \tag{4.2}$$

for all $(x_1, x_2) \in \mathbb{D}_h$

Proof. From the equality

$$F - S_{rs,q}F = F - B^{x_1}_{r,q}f + F - B^{x_2}_{s,q} - (F - P_{rs,q}F),$$

we get

$$\left|\left(R^S_{rs,q}F\right)(x_1, x_2)\right| \leq \left|\left(R^{x_1}_{r,q}F\right)(x_1, x_2)\right| + \left|\left(R^{x_2}_{s,q}F\right)(x_1, x_2)\right| + \left|\left(R^P_{rs,q}F\right)(x_1, x_2)\right|.$$

Now, from previous discussion proof follows immediately. □

Remark 4.1 One can obtain similar results for the remainders of the product approximation formula

$$F = Q_{sr,q}F + R^Q_{sr,q}F = B^{x_2}_{s,q}B^{x_1}_{r,q}F + R^Q_{sr,q}F$$

and for the Boolean sum formula

$$F = T_{sr,q}F + R^T_{sr,q}F = (B^{x_2}_{s,q} \oplus B^{x_1}_{r,q})F + R^T_{sr,q}F. \qquad \square$$

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