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Some results on cyclic Meir-Keeler Kannan-Chatterjea-Reich type contraction mappings on complete metric space

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ABSTRACT: In this paper, using the notions of cyclic contractions and Meir-Keeler mappings, we define a generalised version of cyclic Meir-Keeler Kannan-Chatterjea-Reich type contraction mappings and cyclic Meir-Keeler Kannan-Chatterjea-Reich type contractive pairs. We establish some results on fixed point and best proximity point for these generalized contraction mappings in the framework of metric space. Our results generalize many existing results on fixed points and best proximity points.

Key Words: Fixed point; Cyclic Contraction; Best proximity points, Complete Metric Space; Cyclic Meir-Keeler Kannan-Chatterjea-Reich (MKKCR) type contraction.

Contents

1. Introduction

Banach Contraction Principle have different type of popular extensions, for example, Kannan [8], Chatterjea [2] and Reich [18]. These extensions have been redecorated and established in various frameworks. One of the extension is in the direction of cyclical representations of mappings. The idea of cyclic representation was put forth by William Kirk et.al. [10], in 2003. They defined cyclic mapping as follows.

If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$. Then a mapping $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclical mapping if $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$ is a subset of $\mathcal{A}$. Kirk et al. [10] presented the results on fixed point for cyclical contraction mappings. Further, in 2006, Eldred and Veeramani [6] presented some results on existence of best proximity points for cyclical contractions. A point $z \in \mathcal{A} \cup \mathcal{B}$ is a best proximity point of the mapping $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ if $\mathfrak{d}(z, \mathcal{H}z) = \mathfrak{d}(\mathcal{A}, \mathcal{B}) = \inf\{\mathfrak{d}(x, y) : x \in \mathcal{A},\, y \in \mathcal{B}\}$.

Rhoades [19] provided a comparison of various definations of contractive mappings. Aydi and Felhi [1] discussed about best proximity points for cyclic Kannan-Chatterjea Ćirić type contractions on metric-like spaces. In [4], Debnath studied on Banach-Kannan-Chatterjea-Reich-type contractive inequalities for multivalued mappings. Debnath et al. [5] worked on common fixed points of Kannan-Chatterjea-Reich type pairs of self-maps in a complete metric space. Karapınar et al. [9] obtained the best proximity point theorems for two weak cyclic contractions on metric-like spaces. De la Sen and Agarwal [20] investigated some fixed point-type results for a class of extended cyclic self-mappings.

Contraction Principle can be seen holding more generally. A mapping $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping (see [11]) if for any $\alpha > 0$, $\exists\ \delta > 0$ such that $\alpha \le t < \alpha + \delta$ implies $\mu(t) < \alpha, \forall\ t \in \mathbb{R}^+$. Further, for any Meir-Keeler mapping μ, $\mu(t) < t$, for all t.

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Fakhar et al. [7] obtained some results on best proximity points of cyclic Meir–Keeler contraction mappings. In 2019, Chen and Kuo [3] established the notions of cyclic Meir-Keeler-Kannan-Chatterjea (MKKC) contractions and cyclic Meir-Keeler-Kannan-Chatterjea (MKKC) contractive pairs and established some best proximity results for these contractions.

Definition 1.1 *[3] If $\mathcal{A}$ and $\mathcal{B}$ are two non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mapping $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be cyclic MKKC-contraction, if*

(1) $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$ is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}$ and $b \in \mathcal{B}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{H}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{H}b) + \mathfrak{d}(d(a, \mathcal{H}b) + \mathfrak{d}(b, \mathcal{H}a)}{4} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Definition 1.2 *[3] If $\mathcal{A}$ and $\mathcal{B}$ are two non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ are said to form a cyclic MKKC-contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

(1) the mappings $\mathcal{H}$ and $\mathcal{G}$ are cyclical mappings.

(2) for $a \in \mathcal{A}$ and $b \in \mathcal{B}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{G}b) + \mathfrak{d}(a, \mathcal{G}b) + \mathfrak{d}(b, \mathcal{H}a)}{4} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Many applications of fixed point theory can be seen in nonlinear analysis, see [12], [13], [14], [15], [16], [17] for some recent works related to existence of solutions of different type of equations. Motivated by the varied applications of fixed point results and the works of Chen and Kuo [3], we define a generalised version of cyclical MKKCR-type contraction mappings and cyclical MKKCR-type contractive pairs using the notions of cyclic contractions and Meir-Keeler mappings. We establish some fixed point and best proximity point results for these generalized contractions in the framework of metric space. Our results generalize the results of Chen and Kuo [3] and many other existing results on fixed points and best proximity points. Here, consider $\mathbb{N}_n = \{1, 2, 3, ..., n\}$.

2. Best Proximity Point Results for cyclic MKKCR-type contraction mappings

In this section, Some results on best proximity points have been established for following generalized cyclical contraction mappings of MKKCR-type.

Definition 2.1 *If $\mathcal{A}$ and $\mathcal{B}$ are two non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mapping $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be cyclical MKKCR-type contraction, if*

(1) $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$ is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{H}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{H}b) + \mathfrak{d}(a, \mathcal{H}b) + \mathfrak{d}(b, \mathcal{H}a) + 4l\mathfrak{d}(a, b)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Cyclic MKKC contraction defined in [3] is a particular case of the above definition.

Lemma 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty closed subsets of a metric space $(\mathcal{X}, \mathfrak{d})$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping, and $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclical MKKCR-type contraction mapping. Then, for any $x_0 \in \mathcal{A} \cup \mathcal{B}$, $\lim_{m \to \infty} \mathfrak{d}(x_m, x_{m+1}) = \mathfrak{d}(\mathcal{A}, \mathcal{B})$.*

Proof: Let $x_0 \in \mathcal{A} \cup \mathcal{B}$ such that $x_{m+1} = \mathcal{H}x_m$, where $m \in \mathbb{N} \cup \{0\}$. Then,

$$
\begin{aligned}
&\mathfrak{d}(x_{m+2}, x_{m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&= \mathfrak{d}(\mathcal{H}x_{m+1}, x_m) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&\leq \mu \left(\frac{\mathfrak{d}(x_{m+1}, \mathcal{H}x_{m+1}) + \mathfrak{d}(x_m, \mathcal{H}x_m) + \mathfrak{d}(x_{m+1}, \mathcal{H}x_m) + \mathfrak{d}(x_m, \mathcal{H}x_{m+1}) + 4l\mathfrak{d}(x_{m+1}, x_m)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \right) \\
&\leq \mu \left(\frac{\mathfrak{d}(x_{m+1}, x_{m+2}) + \mathfrak{d}(x_m, x_{m+1}) + \mathfrak{d}(x_{m+1}, x_{m+1}) + 4l\mathfrak{d}(x_{m+1}, x_m)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \right) \\
&\leq \mu \left(\frac{2\mathfrak{d}(x_{m+1}, x_{m+2}) + 4(l+2)\mathfrak{d}(x_{m+1}, x_m)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \right) \\
&\leq \mathfrak{d}(x_{m+1}, x_m) - \mathfrak{d}(\mathcal{A}, \mathcal{B}).
\end{aligned}
$$

So, $\{\mathfrak{d}(x_{m+1}, x_m) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\}$ is a decreasing sequence which is bounded below and hence, converges to some $\alpha \geq 0$. It is obvious that $\alpha = \inf\{\mathfrak{d}(x_{m+1}, x_m) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\}$, let us assume that $\alpha > 0$.
Then, μ being Meir-Keeler mapping, there exists a δ and a natural number r_0 corresponding to $\alpha > 0$ such that

$$
\begin{aligned}
&\alpha \leq \mathfrak{d}(x_r, x_{r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \alpha + \delta, \ \forall r \geq r_0, \\
\Longrightarrow &\mu(\mathfrak{d}(x_r, x_{r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})) < \alpha.
\end{aligned}
$$

Now, $\mathcal{H}$ being cyclical MKKCR-type contraction mapping and μ being an increasing Meir-Keeler mapping,

$$
\begin{aligned}
&\mathfrak{d}(x_{r+2}, x_{r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&= \mathfrak{d}(\mathcal{H}x_{r+1}, \mathcal{H}x_r) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&\leq \mu \left(\frac{\mathfrak{d}(x_{r+1}, \mathcal{H}x_{r+1}) + \mathfrak{d}(x_r, \mathcal{H}x_r) + \mathfrak{d}(x_{r+1}, \mathcal{H}x_r) + \mathfrak{d}(x_r, \mathcal{H}x_{r+1}) + 4l\mathfrak{d}(x_{r+1}, x_r)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \right) \\
&\leq \mu \left(\frac{\mathfrak{d}(x_{r+1}, x_{r+2}) + \mathfrak{d}(x_r, x_{r+1}) + \mathfrak{d}(x_{r+1}, x_{r+1}) + \mathfrak{d}(x_r, x_{r+2}) + 4l\mathfrak{d}(x_{r+1}, x_r)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \right) \\
&\leq \mu \left(\frac{2\mathfrak{d}(x_{r+1}, x_{r+2}) + (4l+2)\mathfrak{d}(x_r, x_{r+1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \right) \\
&\leq \mathfrak{d}(x_r, x_{r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&\leq \alpha,
\end{aligned}
$$

which is a contradiction. Hence, $\mathfrak{d}(x_m, x_{m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \to 0$, as $m \to \infty$.
That is, $\lim_{m \to \infty} \mathfrak{d}(x_m, x_{m+1}) = \mathfrak{d}(\mathcal{A}, \mathcal{B})$. □

Theorem 2.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping, and $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclical MKKCR-type contraction mapping. Then, for any $x_0 \in \mathcal{A} \cup \mathcal{B}$ such that $x_{m+1} = \mathcal{H}x_m$, where $m \in \mathbb{N} \cup \{0\}$, the following holds:*

(1) If $x_0 \in \mathcal{A}$ and $\{x_{2m}\}$ has a subsequence $\{x_{2m_k}\}$ that converges to $\nu \in \mathcal{A}$, then ν is a best proximity point of $\mathcal{H}$.

(2) If $x_0 \in \mathcal{B}$ and $\{x_{2m-1}\}$ has a subsequence $\{x_{2m_k-1}\}$ that converges to $\nu \in \mathcal{B}$, then ν is a best proximity point of $\mathcal{H}$.

Proof: Let $x_0 \in \mathcal{A}$ such that $x_{m+1} = \mathcal{H}x_m$, where $m \in \mathbb{N} \cup \{0\}$.
Also, let $\{x_{2m}\}$ has a subsequence $\{x_{2m_k}\}$ that converges to $\nu \in \mathcal{A}$.

Then, μ and $\mathcal{H}$, being increasing Meir-Keeler mapping and cyclical MKKCR-type contraction mapping, respectively, give

$$\begin{aligned}
&\mathfrak{d}(\nu, \mathcal{H}\nu) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\\
&\leq \mathfrak{d}(\nu, x_{2m_k}) + \mathfrak{d}(x_{2m_k}, \mathcal{H}\nu) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\\
&\leq \mathfrak{d}(\nu, x_{2m_k}) + \mu\left(\frac{\mathfrak{d}(x_{2m_k-1}, \mathcal{H}x_{2m_k-1}) + \mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m_k-1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, \mathcal{H}x_{2m_k-1}) + 4l\mathfrak{d}(x_{2m_k-1}, \nu)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right)\\
&\leq \mathfrak{d}(\nu, x_{2m_k}) + \mu\left(\frac{\mathfrak{d}(x_{2m_k-1}, x_{2m_k}) + \mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m_k-1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, x_{2m_k}) + 4l\mathfrak{d}(x_{2m_k-1}, \nu)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right)\\
&\leq \mathfrak{d}(\nu, x_{2m_k}) + \mu\left(\frac{(4l+2)\mathfrak{d}(x_{2m_k-1}, x_{2m_k}) + \mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m_k}, \mathcal{H}\nu) + (4l+1)\mathfrak{d}(\nu, x_{2m_k})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right)\\
&< \mathfrak{d}(\nu, x_{2m_k}) + \frac{(4l+2)\mathfrak{d}(x_{2m_k-1}, x_{2m_k}) + \mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m_k}, \mathcal{H}\nu) + (4l+1)\mathfrak{d}(\nu, x_{2m_k})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

As $k \to \infty$,

$$\begin{aligned}
\mathfrak{d}(\nu, \mathcal{H}\nu) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) &< \frac{(4l+2)\mathfrak{d}(\mathcal{A}, \mathcal{B}) + 2\mathfrak{d}(\nu, \mathcal{H}\nu)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}),\\
\Longrightarrow \mathfrak{d}(\nu, \mathcal{H}\nu) &< \frac{(4l+2)\mathfrak{d}(\mathcal{A}, \mathcal{B}) + 2\mathfrak{d}(\nu, \mathcal{H}\nu)}{4(l+1)}.
\end{aligned}$$

Thus, $\mathfrak{d}(\nu, \mathcal{H}\nu) = \mathfrak{d}(\mathcal{A}, \mathcal{B})$. That is, ν is a best proximity point of $\mathcal{H}$.
The proof of (2) is analogous to that of (1). □

For $l = 0$, the above Theorem reduces to the Theorem 2.4 of [3].

Definition 2.2 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mapping $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be cyclical MKKR-type contraction mapping, if*

(1) $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$ is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{H}a) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}b) + \mathfrak{d}(b, \mathcal{H}b) + 2l\mathfrak{d}(a, b)}{2(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Cyclic Meir-Keeler-Kannan contraction defined in [3] is a particular case of the above definition.

Definition 2.3 *If $\mathcal{A}$ and $\mathcal{B}$ are a non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mapping $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be cyclical MKKCR-type contraction mapping, if*

(1) $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$ is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{H}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}b) + \mathfrak{d}(b, \mathcal{H}a) + 2l\mathfrak{d}(a, b)}{2(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Cyclic Meir-Keeler-Chatterjea contraction defined in [3] is a particular case of the above definition. Correponding to the above two definitions, we have the following two results.

Theorem 2.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping, and $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclical MKKCR-type contraction mapping. Then, for any $x_0 \in \mathcal{A} \cup \mathcal{B}$ such that $x_{m+1} = \mathcal{H}x_m$, where $m \in \mathbb{N} \cup \{0\}$, the following holds:*

(1) If $x_0 \in \mathcal{A}$ and $\{x_{2m}\}$ has a subsequence $\{x_{2m_k}\}$ that converges to $\nu \in \mathcal{A}$, then ν is a best proximity point of $\mathcal{H}$.

(2) If $x_0 \in B$ and $\{x_{2m-1}\}$ has a subsequence $\{x_{2m_k-1}\}$ that converges to $\nu \in \mathcal{B}$, then ν is a best proximity point of $\mathcal{H}$.

For $l = 0$, the above Theorem reduces to the Theorem 2.4 of [3].

Theorem 2.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$. Let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping, and $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ be a cyclical MKKCR-type contraction mapping. Then, for any $x_0 \in \mathcal{A} \cup \mathcal{B}$ such that $x_{m+1} = \mathcal{H}x_m$, where $m \in \mathbb{N} \cup \{0\}$, the following holds:*

(1) If $x_0 \in \mathcal{A}$ and $\{x_{2m}\}$ has a subsequence $\{x_{2m_k}\}$ that converges to $\nu \in \mathcal{A}$, then ν is a best proximity point of $\mathcal{H}$.

(2) If $x_0 \in \mathcal{B}$ and $\{x_{2m-1}\}$ has a subsequence $\{x_{2m_k-1}\}$ that converges to $\nu \in \mathcal{B}$, then ν is a best proximity point of $\mathcal{H}$.

Corollaries 2.7 and 2.8 of [3] are particular case $l = 0$ of the above Theorems 2.2 and 2.3, respectively.

3. Best Proximity points for cyclic MKKCR-type contractive pairs

In this section, some results on best proximity point have been established for the following generalized cyclical contractive pairs of MKKCR-type.

Definition 3.1 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ are said to form a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

*(1) the mappings $\mathcal{H}$ and $\mathcal{G}$ are cyclical mappings.
i.e., $\mathcal{H}(\mathcal{A})$is a subset of $\mathcal{B}$ and $\mathcal{G}(\mathcal{B})$is a subset of $\mathcal{A}$.*

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \le \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{G}b) + \mathfrak{d}(a, \mathcal{G}b) + \mathfrak{d}(b, \mathcal{H}a) + 4l\mathfrak{d}(a, b)}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Cyclic MKKC contractive pair defined in [3] is a particular case ($l = 0$) of the above definition.

Lemma 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$ and let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping. Also, let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ forms a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. Then, for any $x_0 \in \mathcal{A}$, $\lim_{m\to\infty} \mathfrak{d}(x_m, x_{m+1}) = \mathfrak{d}(\mathcal{A}, \mathcal{B})$.*

Proof: For $x_0 \in \mathcal{A}$ such that $x_{2m+1} = \mathcal{H}x_{2m}$ and $x_{2m+2} = \mathcal{G}x_{2m+1}$.

$$\begin{aligned}
&\mathfrak{d}(x_{2m+1}, x_{2m+2}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\\
&= \mathfrak{d}(\mathcal{H}x_{2m}, \mathcal{G}x_{2m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\\
&\le \mu\left(\frac{\mathfrak{d}(x_{2m}, \mathcal{H}x_{2m}) + \mathfrak{d}(x_{2m+1}, \mathcal{G}x_{2m+1}) + \mathfrak{d}(x_{2m}, \mathcal{G}x_{2m+1}) + \mathfrak{d}(x_{2m+1}, \mathcal{H}x_{2m}) + 4l\mathfrak{d}(x_{2m}, x_{2m+1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right)\\
&\le \mu\left(\frac{\mathfrak{d}(x_{2m}, x_{2m+1}) + \mathfrak{d}(x_{2m+1}, x_{2m+2}) + \mathfrak{d}(x_{2m}, x_{2m+2}) + \mathfrak{d}(x_{2m+1}, x_{2m+1}) + 4l\mathfrak{d}(x_{2m}, x_{2m+1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right)\\
&\le \mu\left(\frac{(4l+2)\mathfrak{d}(x_{2m}, x_{2m+1}) + 2\mathfrak{d}(x_{2m+1}, x_{2m+2})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right)\\
&\le \frac{(4l+2)\mathfrak{d}(x_{2m}, x_{2m+1}) + 2\mathfrak{d}(x_{2m+1}, x_{2m+2})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

This gives, $\mathfrak{d}(x_{2m+1}, x_{2m+2}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) < \mathfrak{d}(x_{2m}, x_{2m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})$.
Similarly, we get
$$\mathfrak{d}(x_{2m}, x_{2m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) < \mathfrak{d}(x_{2m-1}, x_{2m}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}).$$

Thus, $\{\mathfrak{d}(x_m, x_{m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\}$ is decreasing and bounded below. So, there exists $\alpha \geq 0$ such that $\mathfrak{d}(x_m, x_{m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \to \alpha$, as $n \to \infty$.
It is obvious that $\alpha = \inf\{\mathfrak{d}(x_{m+1}, x_m) - \mathfrak{d}(\mathcal{A}, \mathcal{B})\}$.
Let us assume that $\alpha > 0$.
Then, μ being Meir-Keeler mapping, there exists a δ and a natural number r_0, corresponding to $\alpha > 0$, such that

$$\begin{aligned}
&\alpha \leq \mathfrak{d}(x_r, x_{r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \alpha + \delta, \ \forall r \geq r_0, \\
&\Longrightarrow \mu(\mathfrak{d}(x_r, x_{r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})) < \alpha.
\end{aligned}$$

Now, $\mathcal{H}$ and $\mathcal{G}$ being a pair of cyclical MKKCR-type contractive pair and μ being an increasing Meir-Keeler mapping,

$$\begin{aligned}
&\mathfrak{d}(x_{2r+1}, x_{2r+2}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&= \mathfrak{d}(\mathcal{H}x_{2r}, \mathcal{G}x_{2r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{2r}, \mathcal{H}x_{2r}) + \mathfrak{d}(x_{2r+1}, \mathcal{G}x_{2r+1}) + \mathfrak{d}(x_{2r+1}, \mathcal{H}x_{2r}) + \mathfrak{d}(x_{2r}, \mathcal{G}x_{2r+1}) + 4l\mathfrak{d}(x_{2r}, x_{2r+1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{2r}, x_{2r+1}) + \mathfrak{d}(x_{2r+1}, x_{2r+2}) + \mathfrak{d}(x_{2r}, x_{2r+2}) + 4l\mathfrak{d}(x_{2r}, x_{2r+1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&\leq \mu\left(\frac{2\mathfrak{d}(x_{2r+1}, x_{2r+2}) + (4l+2)\mathfrak{d}(x_{2r}, x_{2r+1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&\leq \mu(\mathfrak{d}(x_{2r}, x_{2r+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B})) \\
&\leq \alpha,
\end{aligned}$$

which is a contradiction.
Thus, $\alpha = 0$.
Hence, $\mathfrak{d}(x_m, x_{m+1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \to 0$, as $m \to \infty$.
That is, $\lim_{m\to\infty} \mathfrak{d}(x_m, x_{m+1}) = \mathfrak{d}(\mathcal{A}, \mathcal{B})$. □

Lemma 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$ and let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping. Also, let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ forms a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. Then, for any $x_0 \in \mathcal{A}$, the sequence $\{x_m\}$ is bounded.*

Proof: Lemma 3.1 accounts for boundedness of the sequence $\{\mathfrak{d}(x_m, x_{m+1})\}$. Let $x_0 \in A$ such that $x_{2m+1} = \mathcal{H}x_{2m}$ and $x_{2m+2} = \mathcal{G}x_{2m+1}$.
Then we have

$$\begin{aligned}
&\mathfrak{d}(x_{2m}, \mathcal{H}x_0) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&= \mathfrak{d}(\mathcal{G}x_{2m-1}, \mathcal{H}x_0) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&= \mathfrak{d}(\mathcal{H}x_0, \mathcal{G}x_{2m-1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_0, \mathcal{H}x_0) + \mathfrak{d}(x_{2m-1}, \mathcal{G}x_{2m-1}) + \mathfrak{d}(x_{2m-1}, \mathcal{H}x_0) + \mathfrak{d}(x_0, \mathcal{G}x_{2m-1}) + 4l\mathfrak{d}(x_0, x_{2m-1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&\leq \mu\left(\frac{(4l+2)[\mathfrak{d}(x_0, \mathcal{H}x_0) + \mathfrak{d}(x_{2m-1}, x_{2m}) + \mathfrak{d}(x_{2m}, \mathcal{H}x_0)]}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&< \frac{(2l+1)[\mathfrak{d}(x_0, \mathcal{H}x_0) + \mathfrak{d}(x_{2m-1}, x_{2m}) + \mathfrak{d}(x_{2m}, \mathcal{H}x_0)]}{2(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

Thus,

$$\mathfrak{d}(x_{2m}, \mathcal{H}x_0) < \frac{(2l+1)[\mathfrak{d}(x_0, \mathcal{H}x_0) + \mathfrak{d}(x_{2m-1}, x_{2m}) + \mathfrak{d}(x_{2m}, \mathcal{H}x_0)]}{2(l+1)}.$$

So, $\{x_{2m}\}$ is bounded. Similarly, it can be shown that $\{x_{2m+1}\}$ is bounded. □

Theorem 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$ and let $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing Meir-Keeler mapping. Also, let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ forms a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. If $\nu \in \mathcal{A}$ is the point of convergence of a subsequence of the sequence $\{x_{2m}\}$, then ν is a best proximity point of $\mathcal{H}$.*

Proof: Let $x_0 \in A$ such that $x_{2m+1} = \mathcal{H}x_{2m}$ and $x_{2m+2} = \mathcal{G}x_{2m+1}$. Also, let the sequence $\{x_{2m}\}$ has a subsequence $\{x_{2m_k}\}$ that converges to $\nu \in A$

$$\begin{aligned}
&\mathfrak{d}(x_{2m_k}, \mathcal{H}\nu) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&= \mathfrak{d}(\mathcal{H}\nu, \mathcal{G}x_{2m_k-1}) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m_k-1}, \mathcal{G}x_{2m_k-1}) + \mathfrak{d}(x_{2m_k-1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, \mathcal{G}x_{2m_k-1}) + 4l\mathfrak{d}(\nu, x_{2m_k-1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m_k-1}, x_{2m_k}) + \mathfrak{d}(x_{2m_k-1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, x_{2m_k}) + 4l\mathfrak{d}(\nu, x_{2m_k-1})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(x_{2m_k-1}, x_{2m_k}) + \mathfrak{d}(x_{2m_k}, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(\nu, x_{2m_k})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right) \\
&< \frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(x_{2m_k-1}, x_{2m_k}) + \mathfrak{d}(x_{2m_k}, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(\nu, x_{2m_k})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

As $k \to \infty$,

$$\begin{aligned}
&\mathfrak{d}(\nu, \mathcal{H}\nu) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) < \frac{2\mathfrak{d}(\nu, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(\mathcal{A}, \mathcal{B})}{4(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}), \\
&\implies \mathfrak{d}(\nu, \mathcal{H}\nu) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) < \frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + (2l+1)\mathfrak{d}(\mathcal{A}, \mathcal{B})}{2(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B}).
\end{aligned}$$

Thus, $\mathfrak{d}(\nu, \mathcal{H}\nu) = \mathfrak{d}(\mathcal{A}, \mathcal{B})$. Hence, ν is a best proximity of $\mathcal{H}$. □

For $l = 0$, Theorem 3.4 reduces to the Theorem 3.6 of [3].

Definition 3.2 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ are said to form a cyclical Meir-Keeler-Kannan-Reich (MKKR)-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

(1) $\mathcal{H}(\mathcal{A})$is a subset of $\mathcal{B}$ and $\mathcal{G}(\mathcal{B})$is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{G}b) + 2l\mathfrak{d}(a, b)}{2(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Cyclic Meir-Keeler-Kannan (MKK)-contractive pair defined in [3] is a particular case of the above definition.

Definition 3.3 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ are said to form a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

(1) $\mathcal{H}(\mathcal{A})$is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) - \mathfrak{d}(\mathcal{A}, \mathcal{B}) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{G}b) + d(b, \mathcal{H}a) + 2l\mathfrak{d}(a, b)}{2(l+1)} - \mathfrak{d}(\mathcal{A}, \mathcal{B})\right).$$

Cyclic Meir-Keeler-Chatterjea (MKC)-contractive pair defined in [3] is a particular case of the above definition. Corresponding to the above definitions, we have the following results.

Theorem 3.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ form a cyclical MKKR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. If $\nu \in \mathcal{A}$ is the point of convergence of a subsequence of the sequence $\{x_{2m}\}$, then ν is a best proximity point of $\mathcal{H}$.*

Theorem 3.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ form a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. If $\nu \in \mathcal{A}$ is the point of convergence of a subsequence of the sequence $\{x_{2m}\}$, then ν is a best proximity point of $\mathcal{H}$.*

Corollaries 3.9 and 3.10 of [3] are particular case $l = 0$ of the above Theorems 3.2 and 3.3, respectively.

4. Fixed Point results for cyclic MKKCR-type contraction mappings

In this section, we assume that $\mathcal{A} \cap \mathcal{B} \neq \emptyset$ and consequently, established fixed point results for cyclical MKKCR-type contraction mappings. Here, Definition 2.1 may be rewritten as

Definition 4.1 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is said to be cyclical MKKCR-type contraction mapping, if*

(1) $\mathcal{H}(\mathcal{A})$is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$is a subset of $\mathcal{A}$.

(2) for any $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,

$$\mathfrak{d}(\mathcal{H}a, \mathcal{H}b) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{H}b) + \mathfrak{d}(a, \mathcal{H}b) + \mathfrak{d}(b, \mathcal{H}a) + 4l\mathfrak{d}(a, b)}{4(l+1)}\right).$$

Lemma 4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. If $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclical MKKCR-type contraction mapping. Then, for any $x_0 \in \mathcal{A}$, $\{x_m\}$ is Cauchy.*

Proof: Let $x_0 \in \mathcal{A}$ such that $\mathcal{H}x_{m+1} = x_m$, where $m \in \mathbb{N} \cup \{0\}$.

$$\begin{aligned}
&\mathfrak{d}(x_{m+2}, x_{m+1}) \\
&= \mathfrak{d}(\mathcal{H}x_{m+1}, \mathcal{H}x_m) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{m+1}, \mathcal{H}x_{m+1}) + \mathfrak{d}(x_m, \mathcal{H}x_m) + \mathfrak{d}(x_{m+1}, \mathcal{H}x_m) + \mathfrak{d}(x_m, \mathcal{H}x_{m+1}) + 4l\mathfrak{d}(x_{m+1}, x_m)}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{m+1}, x_{m+2}) + \mathfrak{d}(x_m, x_{m+1}) + \mathfrak{d}(x_{m+1}, x_{m+1}) + 4l\mathfrak{d}(x_{m+1}, x_m)}{4(l+1)}\right) \\
&\leq \mu\left(\frac{2\mathfrak{d}(x_{m+1}, x_{m+2}) + 4(l+2)\mathfrak{d}(x_{m+1}, x_m)}{4(l+1)}\right).
\end{aligned}$$

Thus, $\mathfrak{d}(x_{m+2}, x_{m+1}) \leq \mathfrak{d}(x_{m+1}, x_m)$.
That is, $\{\mathfrak{d}(x_{m+1}, x_m)\}$ is a decreasing sequence which is bounded below. Hence, $\{\mathfrak{d}(x_{m+1}, x_m)\}$ converges to some $\alpha \geq 0$. It is obvious that
$\alpha = \inf\{\mathfrak{d}(x_{m+1}, x_m) : m \in \mathbb{N} \cup \{0\}\}$.
Now, let us assume that $\alpha > 0$.
Then, μ being Meir-Keeler mapping, there exists a δ and a natural number r_0, corresponding to $\alpha > 0$, such that

$$\begin{aligned}
&\alpha \leq \mathfrak{d}(x_r, x_{r+1}) \leq \alpha + \delta, \ \forall r \geq r_0, \\
&\implies \mu(\mathfrak{d}(x_r, x_{r+1})) < \alpha.
\end{aligned}$$

Now, $\mathcal{H}$ being cyclical MKKCR-type contraction mapping and μ being an increasing Meir-Keeler mapping,

$$\begin{aligned}
&\mathfrak{d}(x_{r+2}, x_{r+1})\\
&= \mathfrak{d}(\mathcal{H}x_{r+1}, \mathcal{H}x_r)\\
&\leq \mu\left(\frac{\mathfrak{d}(x_{r+1}, \mathcal{H}x_{r+1}) + \mathfrak{d}(x_r, \mathcal{H}x_r) + \mathfrak{d}(x_{r+1}, \mathcal{H}x_r) + \mathfrak{d}(x_r, \mathcal{H}x_{r+1}) + 4l\mathfrak{d}(x_{r+1}, x_r)}{4(l+1)}\right)\\
&\leq \mu\left(\frac{\mathfrak{d}(x_{r+1}, x_{r+2}) + \mathfrak{d}(x_r, x_{r+1}) + \mathfrak{d}(x_{r+1}, x_{r+1}) + \mathfrak{d}(x_r, x_{r+2}) + 4l\mathfrak{d}(x_{r+1}, x_r)}{4(l+1)}\right)\\
&\leq \mu\left(\frac{2\mathfrak{d}(x_{r+1}, x_{r+2}) + (4l+2)\mathfrak{d}(x_r, x_{r+1})}{4(l+1)}\right)\\
&\leq \mathfrak{d}(x_r, x_{r+1})\\
&< \alpha,
\end{aligned}$$

which is a contradiction.
Hence, $\{\mathfrak{d}(x_m, x_{m+1})\} \to 0$, as $m \to \infty$.
Now, let $p, q \in \mathbb{N}$ such that $q > p$.
Then, $\mathfrak{d}(x_p, x_q) \leq \mathfrak{d}(x_p, x_{p+1}) + \mathfrak{d}(x_{p+1}, x_{p+2}) + ... + \mathfrak{d}(x_{q-1}, x_q) \to 0$, as $p, q \to \infty$.
Thus, $\{x_m\}$ is a Cauchy sequence. □

Theorem 4.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. If $\mathcal{H} : \mathcal{A} \cup \mathcal{B} \to \mathcal{A} \cup \mathcal{B}$ is a cyclical MKKCR-type contraction mapping. Then, $\mathcal{H}$ has a unique fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Proof: Let $x_0 \in \mathcal{A}$ such that $\mathcal{H}x_{m+1} = x_m$, where $m \in \mathbb{N} \cup \{0\}$. Then, by Lemma 4.1, sequence $\{x_m\}$ is Cauchy. Further, $(\mathcal{X}, \mathfrak{d})$ being complete, Cauchy sequence $\{x_m\}$ converges to some $\nu \in \mathcal{A} \cap \mathcal{B}$.
Now,

$$\begin{aligned}
&\mathfrak{d}(x_m, \mathcal{H}\nu) = \mathfrak{d}(\mathcal{H}x_{m-1}, \mathcal{H}\nu)\\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{m-1}, \mathcal{H}x_{m-1}) + \mathfrak{d}(x_{m-1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, \mathcal{H}x_{m-1}) + 4l\mathfrak{d}(\nu, x_{m-1})}{4(l+1)}\right)\\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{m-1}, x_m) + \mathfrak{d}(x_{m-1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, x_m) + 4l\mathfrak{d}(\nu, x_{m-1})}{4(l+1)}\right)\\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(x_{m-1}, x_m) + \mathfrak{d}(x_m, \mathcal{H}\nu) + (4l+1)\mathfrak{d}(\nu, x_m)}{4(l+1)}\right)\\
&< \frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + (4l+2)\mathfrak{d}(x_{m-1}, x_m) + \mathfrak{d}(x_m, \mathcal{H}\nu) + (4l+1)\mathfrak{d}(\nu, x_m)}{4(l+1)}.
\end{aligned}$$

As $m \to \infty$,

$$\mathfrak{d}(\nu, \mathcal{H}\nu) < \frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(\nu, \mathcal{H}\nu)}{4(l+1)}.$$

Thus, $\mathfrak{d}(\nu, \mathcal{H}\nu) = 0$. That is, $\nu = \mathcal{H}\nu$. Hence, ν is a fixed point of $\mathcal{H}$.
Let γ be another fixed point of $\mathcal{H}$.

Then,

$$\begin{aligned}
\mathfrak{d}(\nu,\gamma) &= \mathfrak{d}(\mathcal{H}\nu,\mathcal{H}\gamma)\\
&\le \mu\left(\frac{\mathfrak{d}(\nu,\mathcal{H}\nu)+\mathfrak{d}(\gamma,\mathcal{H}\gamma)+\mathfrak{d}(\nu,\mathcal{H}\gamma)+\mathfrak{d}(\gamma,\mathcal{H}\nu)+4l\mathfrak{d}(\nu,\gamma)}{4(l+1)}\right)\\
&\le \mu\left(\frac{\mathfrak{d}(\nu,\nu)+\mathfrak{d}(\gamma,\gamma)+\mathfrak{d}(\nu,\gamma)+\mathfrak{d}(\gamma,\nu)+4l\mathfrak{d}(\nu,\gamma)}{4(l+1)}\right)\\
&\le \mu\left(\frac{(4l+2)\mathfrak{d}(\nu,\gamma)}{4(l+1)}\right)\\
&< \frac{(4l+2)\mathfrak{d}(\nu,\gamma)}{4(l+1)}.
\end{aligned}$$

Thus, $\mathfrak{d}(\nu,\gamma)=0$. That is, $\nu=\gamma$. Hence, $\mathcal{H}$ has unique fixed point. □

Now, we rewrite the Definitions 2.2 and 2.3 as follows.

Definition 4.2 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X},\mathfrak{d})$, and $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be a Meir-Keeler mapping. Then, the mapping $\mathcal{H}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ is said to be cyclical MKKR-type contraction mapping, if*

(1) $\mathcal{H}(\mathcal{A})$is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$is a subset of $\mathcal{A}$.

(2) for $a\in\mathcal{A}, b\in\mathcal{B}$ and $l\in\mathbb{N}_n\cup\{0\}$,
$$\mathfrak{d}(\mathcal{H}a,\mathcal{H}b)\le\mu\left(\frac{\mathfrak{d}(a,\mathcal{H}a)+\mathfrak{d}(b,\mathcal{H}b)+2l\mathfrak{d}(a,b)}{2(l+1)}\right).$$

Definition 4.3 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X},\mathfrak{d})$, and $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be a Meir-Keeler mapping. Then, the mapping $\mathcal{H}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ is said to be cyclical MKCR-type contraction mapping, if*

(1) $\mathcal{H}(\mathcal{A})$is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$is a subset of $\mathcal{A}$.

(2) for $a\in\mathcal{A}, b\in\mathcal{B}$ and $l\in\mathbb{N}_n\cup\{0\}$,
$$\mathfrak{d}(\mathcal{H}a,\mathcal{H}b)\le\mu\left(\frac{\mathfrak{d}(a,\mathcal{H}b)+\mathfrak{d}(b,\mathcal{H}a)+2l\mathfrak{d}(a,b)}{2(l+1)}\right).$$

Thus, corresponding to the above definitions, we have the following results.

Theorem 4.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X},\mathfrak{d})$. Let $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing Meir-Keeler mapping, and $\mathcal{H}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ be a cyclical MKKR-type contraction mapping. Then, $\mathcal{H}$ has a unique fixed point in $\mathcal{A}\cap\mathcal{B}$.*

Theorem 4.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X},\mathfrak{d})$. Let $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be an increasing Meir-Keeler mapping, and $\mathcal{H}:\mathcal{A}\cup\mathcal{B}\to\mathcal{A}\cup\mathcal{B}$ be a cyclical MKCR-type contraction mapping. Then, $\mathcal{H}$ has a unique fixed point in $\mathcal{A}\cap\mathcal{B}$.*

5. Fixed Point Results for cyclic MKKCR-type contractive pairs

In this section, assuming $\mathcal{A}\cap\mathcal{B}\neq\emptyset$, some results on fixed point have been established for cyclical MKKCR-type contractive pairs between $\mathcal{A}$ and $\mathcal{B}$. Here, Definition 3.1 may be rewritten as

Definition 5.1 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X},\mathfrak{d})$, and $\mu:\mathbb{R}^+\to\mathbb{R}^+$ be a Meir-Keeler mapping. Then, the mappings $\mathcal{H}:\mathcal{A}\to\mathcal{B}$ and $\mathcal{H}:\mathcal{B}\to\mathcal{A}$ are said to form a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

(1) the mappings $\mathcal{H}$ and $\mathcal{G}$ are cyclical mappings.
i.e., $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{G}(\mathcal{B})$ is a subset of$\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,

$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{G}b) + \mathfrak{d}(a, \mathcal{G}b) + \mathfrak{d}(b, \mathcal{H}a) + 4l\mathfrak{d}(a, b)}{4(l+1)}\right).$$

Lemma 5.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ forms a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. Then, for any $x_0 \in \mathcal{A}$, $\{x_m\}$ is Cauchy.*

Proof: Let $x_0 \in \mathcal{A}$ such that $\mathcal{H}x_{2m} = x_{2m+1}$ and $\mathcal{G}x_{2m+1} = x_{2m+2}$, where $m \in \mathbb{N} \cup \{0\}$.
Then,

$$\begin{aligned}
&\mathfrak{d}(x_{2m+1}, x_{2m+2}) \\
&= \mathfrak{d}(\mathcal{H}x_{2m}, \mathcal{G}x_{2m+1}) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{2m}, \mathcal{H}x_{2m}) + \mathfrak{d}(x_{2m+1}, \mathcal{G}x_{2m+1}) + \mathfrak{d}(x_{2m}, \mathcal{G}x_{2m+1}) + \mathfrak{d}(x_{2m+1}, \mathcal{H}x_{2m}) + 4l\mathfrak{d}(x_{2m}, x_{2m+1})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{2m}, x_{2m+1}) + \mathfrak{d}(x_{2m+1}, x_{2m+2}) + \mathfrak{d}(x_{2m}, x_{2m+2}) + \mathfrak{d}(x_{2m+1}, x_{2m+1}) + 4l\mathfrak{d}(x_{2m}, x_{2m+1})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{(4l+2)\mathfrak{d}(x_{2m}, x_{2m+1}) + 2\mathfrak{d}(x_{2m+1}, x_{2m+2})}{4(l+1)}\right) \\
&< \frac{(4l+2)\mathfrak{d}(x_{2m}, x_{2m+1}) + 2\mathfrak{d}(x_{2m+1}, x_{2m+2})}{4(l+1)} \\
&< \mathfrak{d}(x_{2m}, x_{2m+1}).
\end{aligned}$$

Thus, $\mathfrak{d}(x_{2m+1}, x_{2m+2}) < \mathfrak{d}(x_{2m}, x_{2m+1})$.
Similarly, it can be proved that $\mathfrak{d}(x_{2m}, x_{2m+1}) < \mathfrak{d}(x_{2m-1}, x_{2m})$.
Thus, $\{\mathfrak{d}(x_m, x_{m+1})\}$ is a decreasing sequence which is bounded below. Hence, $\{\mathfrak{d}(x_{m+1}, x_m)\}$ converges to some $\alpha \geq 0$. It is obvious that

$$\alpha = \inf\{\mathfrak{d}(x_{m+1}, x_m) : m \in \mathbb{N} \cup \{0\}\}.$$

Now, let us assume that $\alpha > 0$. Then, μ being Meir-Keeler mapping, there exists a δ and a natural number r_0, corresponding to $\alpha > 0$, such that

$$\begin{aligned}
&\alpha \leq \mathfrak{d}(x_r, x_{r+1}) \leq \alpha + \delta, \ \forall r \geq r_0 \\
&\implies \mu(\mathfrak{d}(x_r, x_{r+1})) < \alpha.
\end{aligned}$$

Now, $\mathcal{H}$ and $\mathcal{G}$ being cyclical MKKCR-type contractive pair between $\mathcal{A}$and $\mathcal{B}$ and μ being an increasing Meir-Keeler mapping,

$$\begin{aligned}
&\mathfrak{d}(x_{2r+1}, x_{2r+2}) = d(\mathcal{H}x_{2r}, \mathcal{G}x_{2r+1}) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{2r}, \mathcal{H}x_{2r}) + \mathfrak{d}(x_{2r+1}, \mathcal{G}x_{2r+1}) + \mathfrak{d}(x_{2r}, \mathcal{G}x_{2r+1}) + \mathfrak{d}(x_{2r+1}, \mathcal{H}x_{2r}) + 4l\mathfrak{d}(x_{2r}, x_{2r+1})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(x_{2r}, x_{2r+1}) + \mathfrak{d}(x_{2r+1}, x_{2r+2}) + \mathfrak{d}(x_{2r}, x_{2r+2}) + \mathfrak{d}(x_{2r+1}, x_{2r+1}) + 4l\mathfrak{d}(x_{2r}, x_{2r+1})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{2\mathfrak{d}(x_{2r+1}, x_{2r+2}) + (4l+2)\mathfrak{d}(x_{2r}, x_{2r+1})}{4(l+1)}\right) \\
&\leq \mu\left(\mathfrak{d}(x_{2r}, x_{2r+1})\right) \\
&< \alpha,
\end{aligned}$$

which is a contradiction.
So, $\alpha = 0$. And hence, $\{\mathfrak{d}(x_m, x_{m+1})\} \to 0$, as $m \to \infty$.
Now, let $p, q \in \mathbb{N}$ such that $q > p$.
Then, $\mathfrak{d}(x_p, x_q) \leq \mathfrak{d}(x_p, x_{p+1}) + \mathfrak{d}(x_{p+1}, x_{p+2}) + ... + \mathfrak{d}(x_{q-1}, x_q) \to 0$, as $p, q \to \infty$.
Thus, $\{x_m\}$ is a Cauchy sequence. □

Theorem 5.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ form a cyclical MKKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. Then, $\mathcal{H}$ and $\mathcal{G}$ have a common fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Proof: Let $x_0 \in \mathcal{A}$ such that $\mathcal{H}x_{2m} = x_{2m+1}$ and $\mathcal{G}x_{2m+1} = x_{2m+2}$, where $m \in \mathbb{N} \cup \{0\}$. Then, by Lemma 5.1, sequence $\{x_m\}$ is Cauchy. Further, $(\mathcal{X}, \mathfrak{d})$ being complete, Cauchy sequence $\{x_m\}$ converges to some $\nu \in \mathcal{A} \cap \mathcal{B}$. Now,

$$\begin{aligned}
&\mathfrak{d}(x_{2m+2}, \mathcal{H}\nu) = \mathfrak{d}(\mathcal{G}x_{2m+1}, \mathcal{H}\nu) \\
&\leq \mu(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m+1}, \mathcal{G}x_{2m+1}) + \mathfrak{d}(x_{2m+1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, \mathcal{G}x_{2m+1}) + 4l\mathfrak{d}(\nu, x_{2m+1})}{4(l+1)}) \\
&\leq \mu(\frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m+1}, x_{2m+2}) + \mathfrak{d}(x_{2m+1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, x_{2m+2}) + 4l\mathfrak{d}(\nu, x_{2m+1})}{4(l+1)}) \\
&< \frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(x_{2m+1}, x_{2m+2}) + \mathfrak{d}(x_{2m+1}, \mathcal{H}\nu) + \mathfrak{d}(\nu, x_{2m+2}) + 4l\mathfrak{d}(\nu, x_{2m+1})}{4(l+1)}.
\end{aligned}$$

As $m \to \infty$,

$$\mathfrak{d}(\nu, \mathcal{H}\nu) < \frac{\mathfrak{d}(\nu, \mathcal{H}\nu) + \mathfrak{d}(\nu, \mathcal{H}\nu)}{4(l+1)}.$$

Thus, $\mathfrak{d}(\nu, \mathcal{H}\nu) = 0$. That is, $\nu = \mathcal{H}\nu$. Hence, ν is a fixed point of $\mathcal{H}$.

Further,

$$\begin{aligned}
&\mathfrak{d}(x_{2m+1}, \mathcal{G}\nu) = \mathfrak{d}(\mathcal{H}x_{2m}, \mathcal{G}\nu) \\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{G}\nu) + \mathfrak{d}(x_{2m}, \mathcal{H}x_{2m}) + \mathfrak{d}(x_{2m}, \mathcal{G}\nu) + \mathfrak{d}(\nu, \mathcal{H}x_{2m}) + 4l\mathfrak{d}(\nu, x_{2m})}{4(l+1)}\right) \\
&\leq \mu\left(\frac{\mathfrak{d}(\nu, \mathcal{G}\nu) + \mathfrak{d}(x_{2m}, x_{2m+1}) + \mathfrak{d}(x_{2m}, \mathcal{G}\nu) + \mathfrak{d}(\nu, x_{2m+1}) + 4l\mathfrak{d}(\nu, x_{2m})}{4(l+1)}\right) \\
&< \frac{\mathfrak{d}(\nu, \mathcal{G}\nu) + \mathfrak{d}(x_{2m}, x_{2m+1}) + \mathfrak{d}(x_{2m}, \mathcal{G}\nu) + \mathfrak{d}(\nu, x_{2m+1}) + 4l\mathfrak{d}(\nu, x_{2m})}{4(l+1)}.
\end{aligned}$$

As $m \to \infty$,

$$\mathfrak{d}(\nu, \mathcal{G}\nu) < \frac{\mathfrak{d}(\nu, \mathcal{G}\nu) + \mathfrak{d}(\nu, \mathcal{G}\nu)}{4(l+1)}.$$

Thus, $\mathfrak{d}(\nu, \mathcal{G}\nu) = 0$. That is, $\nu = \mathcal{G}\nu$. Hence, ν is a common fixed point of $\mathcal{H}$ and $\mathcal{G}$. □

On similar lines, Definitions 3.2 and 3.3 may be rewritten as

Definition 5.2 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{H} : \mathcal{B} \to \mathcal{A}$ are said to form a cyclical MKKR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

(1) $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{G}(\mathcal{B})$ is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{H}a) + \mathfrak{d}(b, \mathcal{G}b) + 2l\mathfrak{d}(a, b)}{2(l+1)}\right).$$

Definition 5.3 *If $\mathcal{A}$ and $\mathcal{B}$ are non-empty subsets of a metric space $(\mathcal{X}, \mathfrak{d})$, and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ is a Meir-Keeler mapping. Then, the mappings $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{H} : \mathcal{B} \to \mathcal{A}$ are said to form a cyclical MKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$, if*

(1) $\mathcal{H}(\mathcal{A})$ is a subset of $\mathcal{B}$ and $\mathcal{H}(\mathcal{B})$ is a subset of $\mathcal{A}$.

(2) for $a \in \mathcal{A}, b \in \mathcal{B}$ and $l \in \mathbb{N}_n \cup \{0\}$,
$$\mathfrak{d}(\mathcal{H}a, \mathcal{G}b) \leq \mu\left(\frac{\mathfrak{d}(a, \mathcal{G}b) + \mathfrak{d}(b, \mathcal{H}a) + 2l\mathfrak{d}(a, b)}{2(l+1)}\right).$$

Corresponding to the above definitions, we have the following results.

Theorem 5.2 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ form a cyclical MKKR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. Then, $\mathcal{H}$ and $\mathcal{G}$ have a common fixed point in $\mathcal{A} \cap \mathcal{B}$.*

Theorem 5.3 *Let $\mathcal{A}$ and $\mathcal{B}$ be two non-empty and closed subsets of a complete metric space $(\mathcal{X}, \mathfrak{d})$ and $\mu : \mathbb{R}^+ \to \mathbb{R}^+$ be a Meir-Keeler mapping. Let $\mathcal{H} : \mathcal{A} \to \mathcal{B}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{A}$ form a cyclical MKCR-type contractive pair between $\mathcal{A}$ and $\mathcal{B}$. Then, $\mathcal{H}$ and $\mathcal{G}$ have a common fixed point in $\mathcal{A} \cap \mathcal{B}$.*

6. Conclusion

A generalised version of cyclical MKKCR-type contraction mappings and cyclical MKKCR-type contractive pairs is defined in this paper using the notions of cyclic contractions and Meir-Keeler mapping. Some results for these contraction mappings were established from which many existing results on fixed points and best proximity points follows as particular case. One may also generalize other contractive mappings in similar manner.

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