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A Study on Generalized Absolute Matrix Summability

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ABSTRACT: In the present paper, a theorem dealing with the $\left|\bar{N}, p_n\right|_k$ summability factors of an infinite series has been generalized to the absolute matrix summability under weaker conditions by using a quasi σ-power increasing sequence.

Key Words: Absolute matrix summability, almost increasing sequence, Hölder's inequality, infinite series, Minkowski's inequality, quasi power increasing sequence, summability factor.

Contents

1. Introduction

A positive sequence (c_n) is said to be almost increasing if there exists a positive increasing sequence (d_n) and two positive constants A and B such that $Ad_n \leq c_n \leq Bd_n$ (see [1]). A sequence (λ_n) is said to be of bounded variation, denote by $(\lambda_n) \in \mathcal{BV}$, if $\sum_{n=1}^{\infty} |\Delta\lambda_n| = \sum_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$. Let $\sum a_n$ be a given infinite series with the partial sums (s_n). Let (p_n) be a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad as \quad n \to \infty, \quad (P_{-i} = p_{-i} = 0, \quad i \geq 1).$$

Let $A = (a_{nv})$ be a normal matrix, i.e., a lower triangular matrix of nonzero diagonal entries. Then A defines the sequence-to-sequence transformation, mapping the sequence $s = (s_n)$ to $As = (A_n(s))$, where

$$A_n(s) = \sum_{v=0}^{n} a_{nv} s_v, \quad n = 0, 1, ...$$

The series $\sum a_n$ is said to be summable $|A, p_n|_k$, $k \geq 1$, if (see [14])

$$\sum_{n=1}^{\infty} \left(\frac{P_n}{p_n}\right)^{k-1} |A_n(s) - A_{n-1}(s)|^k < \infty.$$

For $a_{nv} = \frac{p_v}{P_n}$, $|A, p_n|_k$ summability reduces to $\left|\bar{N}, p_n\right|_k$ summability (see [2]).
Given a normal matrix $A = (a_{nv})$, two lower semimatrices $\bar{A} = (\bar{a}_{nv})$ and $\hat{A} = (\hat{a}_{nv})$ are defined as follows:

$$\bar{a}_{nv} = \sum_{i=v}^{n} a_{ni}, \quad n, v = 0, 1, ... \tag{1.1}$$

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \quad \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \quad n = 1, 2, ... \tag{1.2}$$

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and

$$A_n(s)=\sum_{v=0}^{n} a_{nv}s_v=\sum_{v=0}^{n}\bar{a}_{nv}a_v \tag{1.3}$$

$$\bar{\Delta}A_n(s)=\sum_{v=0}^{n}\hat{a}_{nv}a_v. \tag{1.4}$$

2. Known Result

In [4], Bor has proved the following theorem by using an almost increasing sequence.

Theorem 2.1 *Let (X_n) be an almost increasing sequence, and let there be sequences (β_n) and (λ_n) such that*

$$|\Delta\lambda_n|\leq\beta_n, \tag{2.1}$$

$$\beta_n\to 0 \quad as \quad n\to\infty, \tag{2.2}$$

$$\sum_{n=1}^{\infty} n|\Delta\beta_n|X_n<\infty, \tag{2.3}$$

$$|\lambda_n|X_n=O(1) \quad as \quad n\to\infty. \tag{2.4}$$

If

$$\sum_{v=1}^{n}\frac{|t_v|^k}{v}=O(X_n) \quad as \quad n\to\infty, \tag{2.5}$$

where (t_n) is the n-th $(C,1)$ mean of the sequence (na_n), and (p_n) is a sequence such that

$$P_n=O(np_n), \tag{2.6}$$

$$P_n\Delta p_n=O(p_np_{n+1}), \tag{2.7}$$

then the series $\sum_{n=1}^{\infty}a_n\frac{P_n\lambda_n}{np_n}$ is summable $|\bar{N},p_n|_k$, $k\geq 1$.

3. Main Result

Recently, many studies have been done concerning absolute matrix summability methods (see [5,7,8,9,10,11,12,13]). The purpose of this paper is to generalize Theorem 2.1 to $|A,p_n|_k$ summability under weaker conditions. Therefore we need the concept of quasi σ-power increasing sequence. A positive sequence (γ_n) is said to be quasi σ-power increasing sequence if there exists a constant $K=K(\sigma,\gamma)\geq 1$ such that $Kn^\sigma\gamma_n\geq m^\sigma\gamma_m$ holds for all $n\geq m\geq 1$ (see [6]). It should be noted that every almost increasing sequence is quasi σ-power increasing sequence for any nonnegative σ, but the converse need not be true as can be seen by taking the example, say $\gamma_n=n^{-\sigma}$ for $\sigma>0$.
Now, we prove the following theorem.

Theorem 3.1 *Let $A=(a_{nv})$ be a positive normal matrix such that*

$$\bar{a}_{n0}=1, \quad n=0,1,... \tag{3.1}$$

$$a_{n-1,v}\geq a_{nv}, \quad for \quad n\geq v+1, \tag{3.2}$$

$$a_{nn} = O\left(\frac{p_n}{P_n}\right), \tag{3.3}$$

$$|\hat{a}_{n,v+1}| = O\left(v\left|\Delta_v\left(\hat{a}_{nv}\right)\right|\right), \tag{3.4}$$

and (X_n) *be a quasi* σ*-power increasing sequence for some* $0<\sigma<1$*. If all conditions of Theorem 2.1 and*

$$(\lambda_n) \in \mathcal{BV} \tag{3.5}$$

are satisfied, then the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$ *is summable* $|A, p_n|_k$*,* $k \geq 1$*.*

If we take $a_{nv} = \frac{p_v}{P_n}$ and (X_n) as an almost increasing sequence, then Theorem 3.1 reduces to Theorem 2.1. In this case, the condition (3.5) is not needed.

We need following lemmas for the proof of Theorem 3.1.

Lemma 3.1 ([6]) *Let* (X_n) *be a quasi* σ*-power increasing sequence for some* $0<\sigma<1$*. If the conditions (2.2) and (2.3) are satisfied, then*

$$nX_n\beta_n = O(1) \quad as \quad n \to \infty, \tag{3.6}$$

$$\sum_{n=1}^{\infty} X_n \beta_n < \infty. \tag{3.7}$$

Lemma 3.2 ([3]) *If the conditions (2.6) and (2.7) are satisfied, then we have*

$$\Delta\left(\frac{P_n}{n^2 p_n}\right) = O\left(\frac{1}{n^2}\right).$$

4. Proof of Theorem 3.1

Let (I_n) denotes A-transform of the series $\sum_{n=1}^{\infty} a_n \frac{P_n \lambda_n}{n p_n}$. By (1.3) and (1.4), we have

$$\begin{aligned} \bar{\Delta} I_n &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{a_v P_v \lambda_v}{v p_v} \\ &= \sum_{v=1}^{n} \hat{a}_{nv} \frac{P_v \lambda_v v a_v}{v^2 p_v}. \end{aligned}$$

Using Abel's transformation, we get

$$\begin{aligned} \bar{\Delta} I_n &= \sum_{v=1}^{n-1} \Delta_v\left(\hat{a}_{nv}\frac{P_v\lambda_v}{v^2 p_v}\right)\sum_{r=1}^{v} r a_r + \frac{a_{nn}P_n\lambda_n}{n^2 p_n}\sum_{r=1}^{n} r a_r \\ &= \sum_{v=1}^{n-1} \frac{\hat{a}_{n,v+1}P_v\Delta\lambda_v}{v^2 p_v}(v+1)t_v + \sum_{v=1}^{n-1}\frac{\Delta_v(\hat{a}_{nv})P_v\lambda_v}{v^2 p_v}(v+1)t_v \\ &\quad + \sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\left(\frac{P_v}{v^2 p_v}\right)\lambda_{v+1}(v+1)t_v + \frac{a_{nn}P_n\lambda_n}{n^2 p_n}(n+1)t_n \\ &= I_{n,1}+I_{n,2}+I_{n,3}+I_{n,4}. \end{aligned}$$

To complete the proof of Theorem 3.1, it is sufficient to show that

$$\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,r}|^k < \infty, \quad for \quad r = 1,2,3,4.$$

Case-I: For r = 1, we need to show that $\sum_{n=1}^{\infty}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,1}|^k<\infty$. By applying Hölder's inequality and the condition (2.6), we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,1}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{\hat{a}_{n,v+1}P_v\Delta\lambda_v}{v^2p_v}(v+1)t_v\right|^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v||t_v|^k\right)\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}||\Delta\lambda_v|\right)^{k-1}.\end{aligned}$$

Here, using (1.2), (1.1) and (3.2), we have

$$\begin{aligned}\hat{a}_{n,v+1} &= \bar{a}_{n,v+1}-\bar{a}_{n-1,v+1}\\ &= \sum_{i=v+1}^{n}a_{ni}-\sum_{i=v+1}^{n-1}a_{n-1,i}\\ &= a_{nn}+\sum_{i=v+1}^{n-1}(a_{ni}-a_{n-1,i})\le a_{nn}.\end{aligned}$$

Also, using the fact that $(\lambda_n)\in\mathcal{BV}$ and the condition (2.1), we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,1}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}|\hat{a}_{n,v+1}|\beta_v|t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|.\end{aligned}$$

By (1.2), (1.1), (3.1) and (3.2), we obtain $|\hat{a}_{n,v+1}|=\sum_{i=0}^{v}(a_{n-1,i}-a_{ni})$. Thence, using (1.1) and (3.1), we have

$$\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|=\sum_{n=v+1}^{m+1}\sum_{i=0}^{v}(a_{n-1,i}-a_{ni})\le 1, \tag{4.1}$$

by Abel's transformation, we have

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,1}|^k &= O(1)\sum_{v=1}^{m}\beta_v|t_v|^k=O(1)\sum_{v=1}^{m}(v\beta_v)\frac{|t_v|^k}{v}\\ &= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{r=1}^{v}\frac{|t_r|^k}{r}+O(1)m\beta_m\sum_{v=1}^{m}\frac{|t_v|^k}{v}\\ &= O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)X_v+O(1)m\beta_mX_m\\ &= O(1)\sum_{v=1}^{m-1}(v+1)|\Delta\beta_v|X_v+O(1)\sum_{v=1}^{m-1}\beta_vX_v\\ &+ O(1)m\beta_mX_m=O(1),\quad m\to\infty\end{aligned}$$

by (2.5), (2.3), (3.7) and (3.6).

Case-II: For r = 2, again by using Hölder's inequality, we get

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,2}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\frac{\Delta_v(\hat{a}_{nv})P_v\lambda_v}{v^2p_v}(v+1)t_v\right|^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\frac{P_v}{vp_v}|\Delta_v\left(\hat{a}_{nv}\right)||\lambda_v||t_v|\right)^k\\ &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v\left(\hat{a}_{nv}\right)||\lambda_v|^k|t_v|^k\right)\\ &\times\left(\sum_{v=1}^{n-1}|\Delta_v\left(\hat{a}_{nv}\right)|\right)^{k-1}.\end{aligned}$$

By (1.2) and (1.1), we obtain

$$\begin{aligned}\Delta_v(\hat{a}_{nv}) &= \hat{a}_{nv}-\hat{a}_{n,v+1}\\ &= \bar{a}_{nv}-\bar{a}_{n-1,v}-\bar{a}_{n,v+1}+\bar{a}_{n-1,v+1}\\ &= a_{nv}-a_{n-1,v}.\end{aligned}\tag{4.2}$$

and so (1.1), (3.1) and (3.2) imply that

$$\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|=\sum_{v=1}^{n-1}(a_{n-1,v}-a_{nv})\leq a_{nn}.\tag{4.3}$$

Then, from (3.3)

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,2}|^k &= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}\left(\frac{P_v}{vp_v}\right)^k|\Delta_v\left(\hat{a}_{nv}\right)||\lambda_v|^k|t_v|^k\right)\\ &= O(1)\sum_{v=1}^{m}\left(\frac{P_v}{vp_v}\right)^k|\lambda_v|^k|t_v|^k\sum_{n=v+1}^{m+1}|\Delta_v\left(\hat{a}_{nv}\right)|.\end{aligned}$$

Now, using (4.2) and (3.2), we have

$$\sum_{n=v+1}^{m+1}|\Delta_v(\hat{a}_{nv})|=\sum_{n=v+1}^{m+1}(a_{n-1,v}-a_{nv})\leq a_{vv}$$

and consequently,

$$\begin{aligned}\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,2}|^k &= O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_v|^{k-1}|\lambda_v||t_v|^k=O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_v||t_v|^k\\ &= O(1)\sum_{v=1}^{m-1}\Delta|\lambda_v|\sum_{r=1}^{v}\frac{|t_r|^k}{r}+O(1)|\lambda_m|\sum_{v=1}^{m}\frac{|t_v|^k}{v}\\ &= O(1)\sum_{v=1}^{m-1}\beta_vX_v+O(1)|\lambda_m|X_m\\ &= O(1),\quad m\to\infty\end{aligned}$$

by (3.3), (2.6), (2.1), (2.5), (3.7) and (2.4).

Case-III: For r = 3, using the fact that $\Delta\left(\frac{P_n}{n^2p_n}\right)=O\left(\frac{1}{n^2}\right)$ by Lemma 3.2, also using (3.4), (4.3), (3.3), (4.1), we have

$$\begin{aligned}
\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,3}|^k &= \sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\sum_{v=1}^{n-1}\hat{a}_{n,v+1}\Delta\left(\frac{P_v}{v^2p_v}\right)\lambda_{v+1}(v+1)t_v\right|^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_{v+1}||t_v|\right)^k\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}\left(\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k|t_v|^k\right)\\
&\times\left(\sum_{v=1}^{n-1}|\Delta_v(\hat{a}_{nv})|\right)^{k-1}\\
&= O(1)\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^{k-1}\left(\sum_{v=1}^{n-1}\frac{1}{v}|\hat{a}_{n,v+1}||\lambda_{v+1}|^k|t_v|^k\right)\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_{v+1}|^k|t_v|^k\sum_{n=v+1}^{m+1}|\hat{a}_{n,v+1}|\\
&= O(1)\sum_{v=1}^{m}\frac{1}{v}|\lambda_{v+1}||t_v|^k.
\end{aligned}$$

Here, similar to Case-II, we obtain

$$\sum_{n=2}^{m+1}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,3}|^k=O(1)\quad as\quad m\to\infty.$$

Case-IV: For r = 4, using the hypotheses of Theorem 3.1 and Lemma 3.1, we get

$$\begin{aligned}
\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}|I_{n,4}|^k &= \sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}\left|\frac{a_{nn}P_n\lambda_n}{n^2p_n}(n+1)t_n\right|^k\\
&= O(1)\sum_{n=1}^{m}\left(\frac{P_n}{p_n}\right)^{k-1}a_{nn}^k\left(\frac{P_n}{np_n}\right)^k|\lambda_n|^k|t_n|^k\\
&= O(1)\sum_{n=1}^{m}\frac{1}{n}|\lambda_n||t_n|^k=O(1)\quad as\quad m\to\infty,
\end{aligned}$$

by following analogously to Case-II. This completes the proof of Theorem 3.1.

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