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A New Contraction Principle via Fuzzy $\mathcal{L}$-Simulation Functions

Abdelhamid Moussaoui[1,*], Stojan Radenović [2], Omid Taghipour Birgani[3], Said Melliani[1]

ABSTRACT: In this study, motivated by the researches of Seong-Hoon Cho [J. Abstr. Appl. Anal. V 2018, (2018)] and E. Karapinar *et al.* [Filomat 35:1 (2021), 201-224], we define the concept of fuzzy $\mathcal{L}$-contraction and we prove a new fixed point theorem for such new type of mappings in the framework of fuzzy metric spaces.

Key Words: fixed point theory, fuzzy metric spaces, simulation functions, fuzzy $\mathcal{L}$-contraction.

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1. Introduction and Preliminaries

The attractiveness of fuzzy sets has continuously grown since Zadeh's pioneering 1965 work [1]. As a result, there has been a major advancement in theory and application in the areas of logic, topology, and analysis, with various applications in the domains of computer science and engineering. Kramosil and Michaelek [2] initially introduced fuzzy metric spaces, George and Veeramani [4] further developed the idea and demonstrated that each fuzzy metric produces Hausdorff topology. A main theoretical development at the present is the approach to defining contraction mapping in fuzzy metric spaces. In actuality, Grabiec [3] first initiated the Banach and Edelstein theorems to fuzzy metric spaces in 1988. Gregori and Sapena [5] coined the concept of fuzzy contractive mappings, which is as follows: Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a fuzzy metric space. A mapping $\mathcal{G} : \Lambda \to \Lambda$ is said to be a fuzzy contractive mapping, if there exists $k \in (0, 1)$ such that

$$\frac{1}{\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)} - 1 \leq k\left(\frac{1}{\mathcal{M}(u, v, \delta)} - 1\right), \tag{1.1}$$

for all $u, v \in \Lambda$ and $\delta > 0$. The authors proved various significant fixed point results for such class of contractions. The study of Tirado [6] led to the establishment of the following theorem.

Theorem 1.1 *[6] Let $(\Lambda, \mathcal{M}, \curlywedge_L)$ be a complete fuzzy metric space and $\mathcal{G} : \Lambda \to \Lambda$ be a mapping such that*

$$1 - \mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) \leq k\,(1 - \mathcal{M}(u, v, \delta)).$$

for all $u, v \in \Lambda$, $\delta > 0$ and for some $k \in (0, 1)$. Then $\mathcal{G}$ has a unique fixed point.

Many researchers have recently attempted to generalize the Banach contraction principle by altering and changing the contraction conditions (see [10,12,13,14,16,18,19,21,22,23,24,26,27]). In particular, the concept of θ-contractions was proposed by Jleli and Samet [11], who also provided a generalization of the Banach contraction principle in generalized metric spaces. In the same direction, different researchers developed various types of contractions starting with an auxiliary function satisfying the necessary criteria and achieved intriguing fuzzy fixed point findings, the class of fuzzy ψ-contractive mappings was suggested by Mihet [8]. Wardowski introduced and researched the idea of fuzzy $\mathcal{H}$-contractive mappings [9].

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Abdelhamid Moussaoui *et al.* [15,17] initiated a simulation function approach to fuzzy metric framework and proposed the notion of $\mathcal{FZ}$-contraction, which is further developed in [23] by defining the class of extended $\mathcal{FZ}$-simulation functions.

In 2020, inspired by the research of Jleli *et al.* [11], Saleh *et al.* [25] brought in the concept of fuzzy θ_f-contractive mappings with the help of the class Ω of the functions $\theta_f : (0,1) \to (0,1)$ fulfilling the following conditions:

(Ω_1) θ_f is non-decreasing,

(Ω_2) θ_f is continuous,

(Ω_3) $\lim_{n\to+\infty} \theta_f(\omega_n) = 1$ if and only if $\lim_{n\to+\infty} \omega_n = 1$, where $\{\omega_n\}$ is a sequence in $(0,1)$.

In this study, motivated by the research of Seong [12], we define a new type of simulation function called fuzzy $\mathcal{L}$-simulation function and apply it to demonstrate a new fixed point theorem in fuzzy metric spaces. We illustrate that our approach unifies a number of previous findings, and several further findings are established as corollaries.

In order to make our study self-contained, we cover some fundamental notions in this section.

Definition 1.1 *[7] The operation $\curlywedge : [0,1] \times [0,1] \longrightarrow [0,1]$ si said to be a continuous t-norm if:*

($\mathcal{TN}_1$) *$\curlywedge$ is continuous,*

($\mathcal{TN}_2$) *$\curlywedge$ is commutative and associative,*

($\mathcal{TN}_3$) *$u \curlywedge 1 = u$ for all $u \in [0,1]$,*

($\mathcal{TN}_4$) *$u \curlywedge \nu \leq \sigma \curlywedge \pi$ whenever $u \leq \sigma$ and $\nu \leq \pi$, for all $u, \nu, \sigma, \pi \in [0,1]$.*

Example 1.1 *i)* $u \curlywedge_Z \nu = \min\{u, \nu\}$.

ii) $u \curlywedge_P \nu = u.\nu$,

iii) $u \curlywedge_L \nu = \max\{0, u + \nu - 1\}$,

Definition 1.2 *[4] The triple $(\Lambda, \mathcal{M}, \curlywedge)$ is called a fuzzy metric space if Λ is a nonempty set, $\curlywedge$ is a continuous t-norm and $\mathcal{M}$ is a fuzzy set on $\Lambda^2 \times (0, +\infty)$ satisfying:*

($\mathcal{M}1$) $\mathcal{M}(u, v, \delta) > 0$,

($\mathcal{M}2$) $\mathcal{M}(u, v, \delta) = 1$ *if and only if* $u = v$,

($\mathcal{M}3$) $\mathcal{M}(u, v, \delta) = \mathcal{M}(v, u, \delta)$,

($\mathcal{M}4$) $\mathcal{M}(u, v, \delta) \curlywedge \mathcal{M}(v, z, \gamma) \leq \mathcal{M}(u, z, \delta + \gamma)$,

($\mathcal{M}5$) $\mathcal{M}(u, v, .) : (0, +\infty) \to [0,1]$ *is continuous.*

for all $u, v, z \in \Lambda$ and $\delta, \gamma > 0$.

The number $\mathcal{M}(u, v, \delta)$ can be regarded as the degree of nearness of u and v with respect to the variable δ.

Lemma 1.1 *[3] $\mathcal{M}(u, v, .)$ is nondecreasing function for all u, v in Λ.*

Example 1.2 *[4] Let $(\Lambda, \mathcal{D})$ be a metric space, $u \curlywedge v = \curlywedge_Z$ and*

$$\mathcal{M}(u, v, \delta) = \frac{\lambda \delta^p}{\lambda \delta^p + q\mathcal{D}(x, y)} \ , \lambda, q, p \in \mathbb{R}^+$$

Then $(\Lambda, \mathcal{M}, \curlywedge)$ is a fuzzy metric space.
Setting $\lambda = q = p = 1$, we obtain

$$\mathcal{M}(u, v, \delta) = \frac{\delta}{\delta + \mathcal{D}(u, v)}.$$

Definition 1.3 *[4] Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a fuzzy metric space, let $\{u_j\} \subseteq \Lambda$ be a sequence in Λ and $u \in \Lambda$. Then we say that*

(1) *$\{u_j\}$ is convergent or converges to $u \in \Lambda$ if $\lim_{j\to+\infty} \mathcal{M}(u_j, u, \delta) = 1$ for all $\delta > 0$.*

(2) *$\{u_j\}$ is a Cauchy if for all $\varepsilon \in (0,1)$ and $\delta > 0$, there exists $j_0 \in \mathbb{N}$ such that $\mathcal{M}(u_j, u_i, \delta) > 1 - \varepsilon$ for all $j, i \geq n_0$.*

(3) *$(\Lambda, \mathcal{M}, \curlywedge)$ is complete if each Cauchy sequence is convergent in Λ.*

In order to develop a new type of fuzzy contractions known as $\mathcal{FZ}$-contractions, the following class of fuzzy simulation functions was suggested in [15], and further extended by Moussaoui *et al.* [23].

Definition 1.4 *([15])*
The function $\Gamma : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ is called an $\mathcal{FZ}$-simulation function, if the following conditions hold:

(Γ1) $\Gamma(1,1) = 1$,

(Γ2) $\Gamma(\tau, \sigma) < \frac{1}{\sigma} - \frac{1}{\tau}$ *for all* $\tau, \sigma \in (0,1)$,

(Γ3) *if $\{\tau_n\}, \{\sigma_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to+\infty} \tau_n = \lim_{n\to+\infty} \sigma_n < 1$ then $\lim_{n\to+\infty} \sup \Gamma(\tau_n, \sigma_n) < 0$.*

The class of all $\mathcal{FZ}$-simulation functions is denoted by $\mathcal{FZ}$.

Definition 1.5 *([15], [24]) Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a fuzzy metric space, $\mathcal{G} : \Lambda \longrightarrow \Lambda$ a mapping and $\Gamma \in \mathcal{FZ}$. Then $\mathcal{G}$ is called a $\mathcal{FZ}$-contraction w.r.t $\zeta \in \mathcal{FZ}$ if:*

$$\Gamma(\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta), \mathcal{M}(u, v, \delta)) \geq 0 \text{ for all } u, v \in \Lambda, \delta > 0.$$

The notion of fuzzy θ_f-contractive mapping was established in 2020 by Saleh *et al.* [25] as follows:

Definition 1.6 *[25] Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a fuzzy metric space. A mapping $\mathcal{G} : \Lambda \to \Lambda$ is called a fuzzy Θ_f-contractive mapping w.r.t $\theta_f \in \Omega$ if there exists $k \in (0,1)$ such that*

$$\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) < 1 \text{ implies } \theta_f(\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)) \geq [\theta_f(M(u, v, \delta))]^k,$$

for all $u, v \in \Lambda$ and $\delta > 0$.

The authors then proved the following theorem involving fuzzy Θ_f-contractive mapping.

Theorem 1.2 *[25] Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a complete fuzzy metric space and $\mathcal{G} : \Lambda \longrightarrow \Lambda$ be a fuzzy Θ_f-contractive mapping, then $\mathcal{G}$ has a unique fixed point.*

2. A New Type of Control Functions

In this section, we enrich the existing classes of control functions by introducing the class of fuzzy $\mathcal{L}$-simulation function.

Definition 2.1 *The function $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ is said to be a fuzzy $\mathcal{L}$-simulation function, if the following conditions hold:*

(ζ1) $\zeta(1,1) = 1$,

(ζ2) $\zeta(t, s) < \frac{t}{s}$ *for all* $t, s \in (0,1)$,

(ζ3) *if $\{t_n\}, \{s_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n < 1$ then $\lim_{n\to+\infty} \sup \zeta(t_n, s_n) < 1$.*

The collection of all fuzzy $\mathcal{L}$-simulation functions is denoted by $\mathcal{FL}$.

Example 2.1 *Let $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ be defined by*

$$\zeta(t,s) = \frac{t}{s^k} \text{ for all } s,t \in (0,1],$$

where $k \in (0,1)$. Then $\zeta \in \mathcal{FL}$.

Example 2.2 *Let $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ be defined by*

$$\zeta(t,s) = \frac{t}{\varphi(s)} \text{ for all } s,t \in (0,1],$$

where $\varphi : (0,1] \to (0,1]$ such that φ is non-decreasing, continuous and $\varphi(t) > t$, for all $t \in (0,1)$. Then $\zeta \in \mathcal{FL}$.

Example 2.3 *Let $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$ be defined by*

$$\zeta(t,s) = \frac{t\delta(t,s)}{s} \text{ for all } s,t \in (0,1],$$

where $\delta : (0,+\infty] \times (0,+\infty] \to (0,+\infty]$ such that $\delta(t,s) < 1$ for all $s,t < 1$ and $\lim_{n\to+\infty} \sup \delta(t_n,s_n) < 1$ if $\{t_n\}, \{s_n\}$ are sequences in $(0,1]$ such that $\lim_{n\to+\infty} t_n = \lim_{n\to+\infty} s_n < 1$. Then $\zeta \in \mathcal{FL}$.

Now, we define the concept of fuzzy $\mathcal{L}$-contraction mapping as follows:

Definition 2.2 *Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a fuzzy metric space and let $\mathcal{G} : \Lambda \longrightarrow \Lambda$ be a mapping. Then $\mathcal{G}$ is said to be a fuzzy $\mathcal{L}$-contraction with respect to $\zeta \in \mathcal{FL}$ and $\theta_f \in \Omega$, if for all $u,v \in \Lambda, \delta > 0$ with $\mathcal{M}(\mathcal{G}u,\mathcal{G}v,\delta) < 1$, we have*

$$\zeta(\theta_f(\mathcal{M}(\mathcal{G}u,\mathcal{G}v,\delta)), \theta_f(\mathcal{M}(u,v,\delta))) \geq 1 \text{ for all } u,v \in \Lambda, \delta > 0. \tag{2.1}$$

Remark 2.1

- *If $\zeta_1(t,s) = \frac{t}{s^k}$ for all $s,t \in (0,1]$, where $k \in (0,1)$ then Definition 2.2 yields to the concept of fuzzy θ_f-contractive mappings.*
- *If $\theta_f(\omega) = e^{1-\frac{1}{\omega}}$ for all $\omega \in (0,1)$ and $\zeta_1(t,s) = \frac{t}{s^k}$ for all $s,t \in (0,1]$, then this definition yields to the concept of fuzzy contractive mappings initiated by Gregori and Sapena [5].*
- *If $\theta_f(\omega) = \omega$ for all $\omega \in (0,1)$ and $\zeta_1(t,s) = \frac{t}{s^k}$ for all $s,t \in (0,1]$, then this definition yields to the concept of Tirado's contraction [6].*

Remark 2.2 *Every fuzzy $\mathcal{L}$-contraction mapping is continuous.*

Proof: *To prove this, let $\{u_n\} \subset \Lambda$ be any sequence and $u \in \Lambda$ such that $\lim_{n\to+\infty} \mathcal{M}(u_n,u,\delta) = 1$ and $\mathcal{M}(\mathcal{G}u_n,\mathcal{G}u,\delta) < 1$. From (2.1),*

$$\begin{aligned} 1 &\leq \zeta\Big(\theta_f(\mathcal{M}(\mathcal{G}u_n,\mathcal{G}u,\delta), \theta_f(\mathcal{M}(u_n,u,\delta))\Big) \\ &< \frac{\theta_f(\mathcal{M}(\mathcal{G}u_n,\mathcal{G}u,\delta))}{\theta_f(\mathcal{M}(u_n,u,\delta))}. \end{aligned} \tag{2.2}$$

Hence,

$$\theta_f(\mathcal{M}(u_n,u,\delta)) < \theta_f(\mathcal{M}(\mathcal{G}u_n,\mathcal{G}u,\delta)).$$

As θ_f is non decreasing, we derive

$$\mathcal{M}(u_n,u,t) < \mathcal{M}(\mathcal{G}u_n,\mathcal{G}u,\delta).$$

So that $\lim_{n\to+\infty} \mathcal{M}(\mathcal{G}u_n,\mathcal{G}u,\delta) = 1$. Thus, $\mathcal{G}$ is continuous.

□

3. Fixed Point Results

In this section, we prove some fixed point results for the newly defined class of fuzzy contraction in the framework of complete fuzzy metric spaces.

Theorem 3.1 *Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a complete fuzzy metric space and $\mathcal{G} : \Lambda \longrightarrow \Lambda$ be a fuzzy $\mathcal{L}$-contraction with respect to $\zeta \in \mathcal{FL}$. Then $\mathcal{G}$ has a unique fixed point.*

Proof: First, we demonstrate that fixed point is unique provided that it exists. We argue by contradiction, assume that $u, v \in \Lambda$ are two distinct fixed points. Thus, $\mathcal{M}(u, v, \delta) < 1$ for all $\delta > 0$. From (2.1), we get

$$\begin{aligned} 1 &\leq \zeta\Big(\theta_f(\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)), \theta_f(\mathcal{M}(u, v, \delta))\Big) \\ &= \zeta\Big(\theta_f(\mathcal{M}(u, v, \delta)), \theta_f(\mathcal{M}(u, v, \delta))\Big) \\ &< \frac{\theta_f(\mathcal{M}(u, v, \delta))}{\theta_f(\mathcal{M}(u, v, \delta))}. \end{aligned} \tag{3.1}$$

Which means

$$\theta_f(\mathcal{M}(u, v, \delta)) < \theta_f(\mathcal{M}(u, v, \delta)).$$

Which is a contradiction. Therefore $u = v$, that is, the fixed point of $\mathcal{G}$ is unique. Next, we prove the existence of the fixed point. Define $\{u_n\}$ in Λ by $\mathcal{G}u_n = u_{n+1}$ for all $n \geq 0$. If there exists $n_0 \in \mathbb{N}$ such that $u_{n_0} = u_{n_0+1}$, it follows that u_{n_0} is a fixed point of $\mathcal{G}$. Therefore, to continue our proof, we assume that $u_n \neq u_{n+1}$ for all $n \in \mathbb{N}$, then $\mathcal{M}(u_n, u_{n+1}, \delta) < 1$ for all $n \in \mathbb{N}$ and $\delta > 0$. From (2.1), we obtain

$$\begin{aligned} 1 &\leq \zeta\Big(\theta_f(\mathcal{M}(\mathcal{G}u_{n-1}, \mathcal{G}u_n, \delta)), \theta_f(\mathcal{M}(u_{n-1}, u_n, \delta))\Big) \\ &= \zeta\Big(\theta_f(\mathcal{M}(u_n, u_{n+1}, \delta)), \theta_f(\mathcal{M}(u_{n-1}, u_n, \delta))\Big) \\ &< \frac{\theta_f(\mathcal{M}(u_n, u_{n+1}, \delta))}{\theta_f(\mathcal{M}(u_{n-1}, u_n, \delta))}. \end{aligned} \tag{3.2}$$

Which implies,

$$\theta_f(\mathcal{M}(u_{n-1}, u_n, \delta)) < \theta_f(\mathcal{M}(u_n, u_{n+1}, \delta)).$$

Hence,

$$\mathcal{M}(u_{n-1}, u_n, \delta) < \mathcal{M}(u_n, u_{n+1}, \delta).$$

We deduce that $\{\mathcal{M}(u_n, u_{n+1}, \delta)\}$ is a nondecreasing sequence of positive real numbers in $[0, 1]$. Thus, there exists $a(\delta) \leq 1$ such that $\lim_{n\to+\infty} \mathcal{M}(u_n, u_{n-1}, \delta) = a(\delta) \geq 1$ for all $\delta > 0$. We prove that

$$\lim_{n\to+\infty} M(u_n, u_{n-1}, \delta) = 1.$$

On contrary assume that $a(\delta_0) < 1$ for some $\delta_0 > 0$. Now, if we consider the sequences $\{\alpha_n = \mathcal{M}(u_n, u_{n+1}, \delta_0)\}$ and $\{\beta_n = \mathcal{M}(u_{n-1}, u_n, \delta_0)\}$ and taking into account ($\zeta 3$), we derive

$$1 \leq \lim_{n\to+\infty} \sup \zeta(\alpha_n, \beta_n) < 1.$$

A contradiction, hence

$$\lim_{n\to+\infty} \mathcal{M}(u_n, u_{n-1}, \delta) = 1 \tag{3.3}$$

Next, we show that the sequence $\{u_n\}$ is Cauchy. Reasoning by contradiction, suppose that $\{u_n\}$ is not a Cauchy sequence. Then, there exists $\epsilon \in (0,1)$, $\delta_0 > 0$ and two subsequences $\{u_{n_k}\}$ and $\{u_{m_k}\}$ of $\{u_n\}$ with $m_k > n_k \geq k$ for all $k \in \mathbb{N}$ such that

$$\mathcal{M}(u_{m_k}, u_{n_k}, \delta_0) \leq 1-\epsilon. \tag{3.4}$$

Taking in account Lemma 1.1, we have

$$\mathcal{M}(u_{m_k}, u_{n_k}, \frac{\delta_0}{2}) \leq 1-\epsilon. \tag{3.5}$$

By choosing n_k as the smallest index satisfying (3.5), we have

$$\mathcal{M}(u_{m_k-1}, u_{n_k}, \frac{\delta_0}{2}) > 1-\epsilon. \tag{3.6}$$

Applying (2.1) with $u = x_{m_k-1}$ and u_{n_k-1}, we obtain

$$\begin{aligned} 1 &\leq \zeta\Big(\theta_f(\mathcal{M}(\mathcal{G}u_{m_k-1}, \mathcal{G}u_{n_k-1}, \delta_0)), \theta_f(\mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0))\Big) \\ &= \zeta\Big(\theta_f(\mathcal{M}(u_{m_k}, u_{n_k}, \delta_0)), \theta_f(\mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0))\Big) \\ &< \frac{\theta_f(\mathcal{M}(u_{m_k}, u_{n_k}, \delta_0))}{\theta_f(\mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0))}. \end{aligned} \tag{3.7}$$

Therefore,

$$\theta_f(\mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0)) < \theta_f(\mathcal{M}(u_{m_k}, u_{n_k}, \delta_0)) \tag{3.8}$$

As θ_f is nondecreasing, we derive

$$\mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0) < \mathcal{M}(u_{m_k}, u_{n_k}, \delta_0) \tag{3.9}$$

On account of (3.4), (3.6) and the triangular inequality, we obtain

$$\begin{aligned} 1-\epsilon &\geq \mathcal{M}(u_{m_k}, u_{n_k}, \delta_0) \\ &> \mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0) \\ &\geq \mathcal{M}(u_{m_k-1}, u_{n_k}, \frac{\delta_0}{2}) \curlywedge \mathcal{M}(u_{n_k}, x_{n_k-1}, \frac{\delta_0}{2}) \\ &> (1-\epsilon) \curlywedge \mathcal{M}(u_{n_k-1}, u_{n_k}, \frac{\delta_0}{2}) \end{aligned}$$

Taking limit as $k \to +\infty$ in both sides of the above inequality and using (3.3), we derive that

$$\lim_{k\to+\infty} \mathcal{M}(u_{m_k}, u_{n_k}, \delta_0) = \lim_{k\to+\infty} \mathcal{M}(u_{m_k-1}, u_{n_k-1}, \delta_0) = 1-\epsilon \tag{3.10}$$

Now, we consider the sequences $\hat{\beta}_k = \theta_f(\mathcal{M}(u_{n_k-1}, u_{m_k-1}, \delta_0))$ and $\hat{\alpha}_k = \theta_f(\mathcal{M}(u_{m_k}, u_{n_k}, \delta_0))$, then $\lim_{k\to+\infty} \hat{\beta}_k = \lim_{k\to+\infty} \hat{\alpha}_k = \theta_f(1-\epsilon) < 1$. Applying ($\zeta 3$), we get

$$1 \leq \lim_{k\to+\infty} \sup \zeta(\hat{\alpha}_k, \hat{\beta}_k) < 1$$

which is a contradiction. Hence, $\{u_n\}$ is a Cauchy sequence. Since $(\Lambda, \mathcal{M}, \curlywedge)$ a complete fuzzy metric space, there exists $u \in \Lambda$ such that $u_n \to u$. Hence

$$\lim_{n\to+\infty} \mathcal{M}(u_n, u, t) = 1, \tag{3.11}$$

As $\mathcal{T}$ is continuous, we have

$$\lim_{n\to+\infty} \mathcal{M}(u_{n+1}, \mathcal{G}u, \delta) = \mathcal{M}(\mathcal{G}u_{n+1}, \mathcal{G}, \delta) = 1$$

The uniqueness of the limit implies that $\mathcal{G}u = u$, thus u is a fixed point of $\mathcal{G}$. □

Example 3.1 *Let $\Lambda = [0,1]$ be equipped with the fuzzy metric $\mathcal{M}$ given by*

$$\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)) = \frac{\delta}{\delta + \mathcal{D}(u,v)}$$

for all $u, v \in \Lambda$, $\delta > 0$, where $\mathcal{D}$ is the usual metric. Then, $(\Lambda, \mathcal{M}, \curlywedge_p)$ is a fuzzy metric space. Consider the mapping $\mathcal{T} : \Lambda \to \Lambda$ given by $\mathcal{G}u = \frac{u}{u+1}$, for all $u \in \Lambda$, and the fuzzy $\mathcal{L}$-simulation function $\zeta : (0,1] \times (0,1] \longrightarrow \mathbb{R}$, defined by

$$\zeta(t,s) = \frac{t}{s^k}$$

for all $t, s \in (0,1]$ and $k \in (0,1)$. Define the mapping $\theta_f \in \Omega$ by

$$\theta_f(\omega) = e^{1-\frac{1}{\omega}} \text{ for all } \omega \in (0,1).$$

For all $u, v \in \Lambda$ with $\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) < 1$, we have

$$\begin{aligned}\frac{2}{9}(1 - \frac{1}{\mathcal{M}(u,v,\delta)}) &< \frac{1}{(u+1)(v+1)}(1 - \frac{1}{\mathcal{M}(u,v,\delta)}) \\ &= \frac{1}{(u+1)(v+1)}(-\frac{\mathcal{D}(u,v)}{\delta}) \\ &= -\frac{d(\mathcal{G}u, \mathcal{G}v)}{\delta} \\ &= 1 - \frac{1}{\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)}.\end{aligned}$$

Taking $k = \frac{2}{9}$, we obtain that

$$\begin{aligned}\zeta\big(\theta_f(\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)), (\theta_f(\mathcal{M}(u,v,\delta)))\big) &= \frac{\theta_f(\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta))}{\big(\theta_f(\mathcal{M}(u,v,\delta))\big)^k} \\ &= \frac{e^{1-\frac{1}{\mathcal{M}(\mathcal{G}u,\mathcal{G}v,\delta)}}}{\big(e^{1-\frac{1}{\mathcal{M}(u,v,\delta)}}\big)^k} \\ &> 1.\end{aligned}$$

Therefore, $\mathcal{G}$ is a fuzzy $\mathcal{L}$-contraction w.r.t $\zeta \in \mathcal{FL}$. Thus, by Theorem 3.1, $\mathcal{G}$ has a unique fixed point, that is $u = 0$.

Corollary 3.1 *Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a complete fuzzy metric space, $\zeta \in \mathcal{FL}$ and $\mathcal{G} : \Lambda \to \Lambda$ be a self mapping with $\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) < 1$, such that*

$$\zeta(\mathcal{M}(\mathcal{T}u, \mathcal{T}v, \delta)), \mathcal{M}(u,v,\delta)) \geq 1 \text{ for all } u, v \in \Lambda, \delta > 0.$$

Then $\mathcal{G}$ has a unique fixed point.

Proof: The conclusion can be drawn from Theorem 3.1 by defining $\theta_f(\omega) = \omega$ for all $\omega \in (0,1)$. □

Corollary 3.2 *[25] Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a complete fuzzy metric space and $\mathcal{G} : \Lambda \to \Lambda$ be a self mapping such that for all $u, v \in \Lambda$ with $\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) < 1$ we have*

$$\theta_f(\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta)) \geq [\theta_f(\mathcal{M}(u,v,\delta))]^k.$$

Then $\mathcal{G}$ has a unique fixed point.

Proof: The conclusion can be drawn from Theorem 3.1 by defining $\zeta(t,s) = \frac{t}{s^k}$ for all $s, t \in (0,1]$. □

Corollary 3.3 *[25] Let $(\Lambda, \mathcal{M}, \curlywedge)$ be a complete fuzzy metric space and $\mathcal{G} : \Lambda \to \Lambda$ be a self mapping such that for all $u, v \in \Lambda$ with $\mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) < 1$ we have*

$$[1 + \sin\Big(\frac{\pi}{2}(\mathcal{M}(u, v, \delta) - 1)\Big)]^k \leq 1 + \sin\Big(\frac{\pi}{2}((\mathcal{M}\mathcal{G}u, \mathcal{G}v, \delta) - 1)\Big).$$

Then $\mathcal{T}$ has a unique fixed point.

Proof: The proof follows from Theorem 3.1 by taking $\zeta(t, s) = \frac{t}{s^k}$ for all $s, t \in (0, 1]$ and $\theta_f(\omega) = 1 + \sin\left(\frac{\pi}{2}(\omega - 1)\right)$ for all $\omega \in (0, 1)$. □

Corollary 3.4 *[6] Let $(\Lambda, \mathcal{M}, \curlywedge_L)$ be a complete fuzzy metric space and $\mathcal{G} : \Lambda \to \Lambda$ be a mapping such that*

$$1 - \mathcal{M}(\mathcal{G}u, \mathcal{G}v, \delta) \leq k\,(1 - \mathcal{M}(u, v, \delta)).$$

for all $u, v \in \Lambda$,$\delta > 0$ and for some $k \in (0, 1)$. Then $\mathcal{G}$ has a unique fixed point.

Proof: The result follows by choosing $\zeta(t, s) = \frac{t}{s^k}$ for all $s, t \in (0, 1]$ and $\theta_f(\omega) = \omega$ for all $\omega \in (0, 1)$ in Theorem 3.1. □

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Abdelhamid Moussaoui,
[1]*Faculty of Sciences and Technics, LMACS*
Sultan Moulay Slimane University,
Beni Mellal, Morocco,
E-mail address: a.moussaoui@usms.ma; saidmelliani@gmail.com

and

Stojan Radenović,
[2]*Faculty of Mechanical Engineering,*
University of Belgrade,
Serbia
E-mail address: radens@beotel.net

and

Omid Taghipour Birgani,
[3]*School of Mathematics,*
Iran University of Science and Technology,
Tehran, Iran
E-mail address: o.taghipour69@gmail.com