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Some Weighted Midpoint Type Inequalities For Differentiable log-Convex Functions

Badreddine Meftah, Djaber Chemseddine Benchettah*, Abdelghani Lakhdari and Meriem Merad

ABSTRACT: On the basis of a given integral identity, this paper purports to establish some novel weighted midpoint type inequalities for functions whose first derivatives are log-convex. Traditional integral inequalities, such as Holder's inequality, are among the integral inequalities that are utilised in proofs, in addition to fundamental definitions and conventional methods of mathematical analysis.

Key Words: weighted functions, log-convex functions, Hölder inequality, midpoint inequality.

Contents

1. Introduction

Finding new inequalities and making traditional methods better by using new ideas and methods is what inequality studies are all about. The theory of inequality is a part of mathematics that has been around for a long time and has stayed important. Inequalities are still a branch that is studied a lot, is important to research, and is very fascinating. The concept of convexity is closely related to the development of inequality theory, which is a significant tool for studying the properties of the solutions of some differential equations and the error estimates of quadrature formulas.

Let I be an interval of real numbers.

Definition 1.1 ([18]) A function $f : I \to \mathbb{R}$ is said to be convex, if for all $x, y \in I$ and all $t \in [0,1]$, we have

$$f(tx + (1-t)y) \le tf(x) + (1-t)f(y).$$

Definition 1.2 ([18]) A positive function $f : I \to \mathbb{R}$ is said to be logarithmically convex, if

$$f(tx + (1-t)y) \le [f(x)]^t [f(y)]^{1-t}$$

holds for all $x, y \in I$ and all $t \in [0,1]$.

One of the famous inequalities for the class of convex functions is the so-called Hermite-Hadamard inequality (see [5,6]), which can be stated as follows: For every convex function f on the interval $[a, b]$ with $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

About some recent papers dealing generalization, variants and improvements of inequalities of type (1.1) we refer readers to [1,2,4,8,10,11,12,13,14,15,16,20,21] and references therein.

Fejér [3], gave a generalization of inequality (1.1) by proving that if $w : [a, b] \to \mathbb{R}$ is nonnegative, integrable, and symmetric function with respect to $\frac{a+b}{2}$, then for every convex function f on the interval $[a, b]$ with $a < b$, the following inequality holds

$$f\left(\frac{a+b}{2}\right)\int_a^b w(x)\,dx \le \frac{1}{b-a}\int_a^b f(x)\,w(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b w(x)\,dx. \tag{1.2}$$

* Corresponding author

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In [19], Sarikaya et al. investigated the following midpoint type inequalities for log-preinvex derivatives and derived the following inequalities for log-convex functions

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left(\frac{|f'(b)|^{\frac{1}{2}} - |f'(a)|^{\frac{1}{2}}}{\ln|f'(b)| - \ln|f'(a)|}\right)^2$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left\{\frac{|f'(a)|^{\frac{1}{2}}}{2^{\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left(\frac{|f'(b)|^{\frac{q}{2}} - |f'(a)|^{\frac{q}{2}}}{q(\ln|f'(b)| - \ln|f'(a)|)}\right)^{\frac{1}{q}}\right\}.$$

Note that in [9], Latif and Dragomir gave generalization of the results given in [19] for n-times differentiable log-preinvex functions.

More recently, Jain et al. [7], established the following midpoint type inequalities for logarithmically convex derivatives.

Theorem 1.1 *Let $f : I^\circ = [a,b] \to \mathbb{R}$ be a differentiable mapping on I° with $a < b$. If $|f'|^q$, $q > 1$ is* log-*convex on $[a,b]$, such that $|f'(a)| \neq 1$ and $|f'(b)| \neq 1$, then for $p, q, \alpha, \beta > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left\{\left(\frac{|f'(a)|^{\frac{\alpha q}{2}} - 1}{\alpha^2 q \ln|f'(a)|} + \frac{|f'(b)|^{\frac{\beta q}{2}}\left(|f'(b)|^{\frac{\beta q}{2}} - 1\right)}{\beta^2 q \ln|f'(b)|}\right)^{\frac{1}{q}} + \left(\frac{|f'(a)|^{\frac{\beta q}{2}}\left(|f'(a)|^{\frac{\beta q}{2}} - 1\right)}{\alpha^2 q \ln|f'(a)|} + \frac{|f'(b)|^{\frac{\alpha q}{2}} - 1}{\beta^2 q \ln|f'(b)|}\right)^{\frac{1}{q}}\right\}.$$

Theorem 1.2 *Let $f : I^\circ = [a,b] \to \mathbb{R}$ be a differentiable mapping on I° with $a < b$. If $|f'|^q$, $q > 1$ is* log-*convex on $[a,b]$, such that $|f'(a)| \neq 1$ and $|f'(b)| \neq 1$, then we have*

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left\{\left(\frac{|f'(a)|^{\frac{\alpha}{2}} - 1}{\alpha^{\frac{3}{2}}\ln|f'(a)|}\right)^2 + \left(\frac{|f'(b)|^{\frac{\beta}{2}} - 1}{\beta^{\frac{3}{2}}\ln|f'(b)|}\right)^2\right\}.$$

In this paper, By using the identity given in [17], we derive some new weighted midpoint type inequalities for functions whose first derivatives are log-convex.

2. Main results

Lemma 2.1 ([17]) *Let $f : I = [a,b] \to \mathbb{R}$ be a differentiable function on I°, with $a < b$, and let $w : [a,b] \to \mathbb{R}$ be symmetric with respect to $\frac{a+b}{2}$. If $f, w \in L[a,b]$, then*

$$\int_a^b w(u)f(u)\,du - \left(\int_a^b w(u)\,du\right)f\left(\frac{a+b}{2}\right)$$
$$= \frac{(b-a)^2}{4}\left\{\int_0^1 p_1(t)f'\left(tb + (1-t)\frac{a+b}{2}\right)dt - \int_0^1 p_2(t)f'\left(ta + (1-t)\frac{a+b}{2}\right)dt\right\},$$

where

$$p_1(t) = \int_t^1 w\left(sb + (1-s)\frac{a+b}{2}\right)ds \tag{2.1}$$

and

$$p_2(t) = \int_t^1 w\left(sa + (1-s)\frac{a+b}{2}\right) ds. \tag{2.2}$$

Theorem 2.1 *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L([a,b])$ with $0 \le a < b$, and let $w : [a,b] \to \mathbb{R}$ be continuous and symmetric function with respect to $\frac{a+b}{2}$. If $|f'|$ is* log*-convex, then we have*

1. *For* $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$

$$\left|\int_a^b w(u) f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right| \le \frac{(b-a)^2}{4}\left|f'\left(\frac{a+b}{2}\right)\right| \|w\|_{[a,b]\infty}.$$

2. *For* $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\left|\int_a^b w(u) f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^2}{4}\left\{\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \xi\left(|f'(b)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right\} \|w\|_{[a,b]\infty}.$$

3. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$

$$\left|\int_a^b w(u) f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^2}{4}\left\{\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \xi\left(|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right\} \|w\|_{[a,b]\infty}.$$

4. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\left|\int_a^b w(u) f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^2}{4}\left\{\xi\left(|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right) + \xi\left(|f'(b)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right\} \|w\|_{[a,b]\infty},$$

where

$$\xi(x,y) = \frac{x-y}{(\ln x - \ln y)^2} - \frac{y}{\ln x - \ln y}; x, y > 0 \text{ and } x \neq y. \tag{2.3}$$

Proof: From Lemma 2.1 and properties of modulus, we have

$$\left|\int_a^b w(u) f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\le \frac{(b-a)^2}{4}\left\{\int_0^1 |p_1(t)| \left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right| dt + \int_0^1 |p_2(t)| \left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right| dt\right\}. \tag{2.4}$$

Since $|f'|$ is log-convex, we have

$$\left|f'(tb+(1-t)\frac{a+b}{2}\right| \leq |f'(b)|^{t}\left|f'\left(\frac{a+b}{2}\right)\right|^{1-t} \tag{2.5}$$

and

$$\left|f'(ta+(1-t)\frac{a+b}{2}\right| \leq |f'(a)|^{t}\left|f'\left(\frac{a+b}{2}\right)\right|^{1-t}. \tag{2.6}$$

Using (2.1), (2.2), (2.5) and (2.6) in (2.4), we get

$$\begin{aligned}
&\left|\int_a^b w(u)\, f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{(b-a)^2}{4}\left(\int_0^1 \left|\int_t^1 w\left(sb+(1-s)\frac{a+b}{2}\right) ds\right| |f'(b)|^{t}\left|f'\left(\frac{a+b}{2}\right)\right|^{1-t} dt\right. \\
&\left.+\int_0^1 \left|\int_t^1 w\left(sa+(1-s)\frac{a+b}{2}\right) ds\right| |f'(a)|^{t}\left|f'\left(\frac{a+b}{2}\right)\right|^{1-t} dt\right) \\
\leq& \frac{(b-a)^2}{4}\|w\|_{[a,b]\infty}\left|f'\left(\frac{a+b}{2}\right)\right|\left(\int_0^1 (1-t)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a+b}{2}\right)\right|}\right)^{t} dt + \int_0^1 (1-t)\left(\frac{|f'(a)|}{\left|f'\left(\frac{a+b}{2}\right)\right|}\right)^{t} dt\right).
\end{aligned} \tag{2.7}$$

If $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$, then from (2.7), we have

$$\begin{aligned}
&\left|\int_a^b w(u)\, f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{(b-a)^2}{2}\|w\|_{[a,b]\infty}\left|f'\left(\frac{a+b}{2}\right)\right|\int_0^1 (1-t)dt \\
=& \frac{(b-a)^2}{4}\|w\|_{[a,b]\infty}\left|f'\left(\frac{a+b}{2}\right)\right|.
\end{aligned} \tag{2.8}$$

If $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$, then from (2.7), we have

$$\begin{aligned}
&\left|\int_a^b w(u)\, f(u)\, du - \left(\int_a^b w(u)\, du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{(b-a)^2}{4}\|w\|_{[a,b]\infty}\left|f'\left(\frac{a+b}{2}\right)\right|\left(\int_0^1 (1-t)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a+b}{2}\right)\right|}\right)^{t} dt + \int_0^1 (1-t)dt\right) \\
=& \frac{(b-a)^2}{4}\|w\|_{[a,b]\infty}\left(\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \frac{|f'(b)| - \left|f'\left(\frac{a+b}{2}\right)\right|}{\left(\ln|f'(b)| - \ln\left|f'\left(\frac{a+b}{2}\right)\right|\right)^2} - \frac{\left|f'\left(\frac{a+b}{2}\right)\right|}{\ln|f'(b)| - \ln\left|f'\left(\frac{a+b}{2}\right)\right|}\right) \\
=& \frac{(b-a)^2}{4}\|w\|_{[a,b]\infty}\left(\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \xi\left(|f'(b)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right),
\end{aligned} \tag{2.9}$$

where $\xi(.,.)$ is defined in (2.3).
If $|f'(b)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(a)|$, then from (2.7), we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{(b-a)^2}{4} \|w\|_{[a,b]\infty} \left|f'\left(\frac{a+b}{2}\right)\right| \left(\int_0^1 (1-t)dt + \int_0^1 (1-t)\left(\frac{|f'(a)|}{\left|f'\left(\frac{a+b}{2}\right)\right|}\right)^t dt\right) \\
=& \frac{(b-a)^2}{4} \|w\|_{[a,b]\infty} \left(\frac{1}{2}\left|f'\left(\frac{a+b}{2}\right)\right| + \xi\left(|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right). \qquad (2.10)
\end{aligned}$$

If $|f'(b)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(a)|$, then from (2.7), we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{(b-a)^2}{4} \|w\|_{[a,b]\infty} \left|f'\left(\frac{a+b}{2}\right)\right| \left(\int_0^1 (1-t)\left(\frac{|f'(b)|}{\left|f'\left(\frac{a+b}{2}\right)\right|}\right)^t dt + \int_0^1 (1-t)\left(\frac{|f'(a)|}{\left|f'\left(\frac{a+b}{2}\right)\right|}\right)^t dt\right) \\
=& \frac{(b-a)^2}{4} \|w\|_{[a,b]\infty} \left(\xi\left(|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right) + \xi\left(|f'(b)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right), \qquad (2.11)
\end{aligned}$$

where we have used the fact that for $\chi > 0$ and $\chi \neq 1$ we have

$$\int_0^1 (1-t)\chi^t dt = \frac{\chi - 1}{(\ln\chi)^2} - \frac{1}{\ln\chi}. \qquad (2.12)$$

The desired result follows from (2.8)-(2.11). The proof is completed. □

Corollary 2.1A *In Theorem 2.1, if we take $w(u) = \frac{1}{b-a}$, we obtain*

1. *For $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$*

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

2. *For $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$*

$$\begin{aligned}
&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{b-a}{8}\left(\left|f'\left(\frac{a+b}{2}\right)\right| + 2\xi\left(|f'(b)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right).
\end{aligned}$$

3. *For $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$*

$$\begin{aligned}
&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq& \frac{b-a}{8}\left(\left|f'\left(\frac{a+b}{2}\right)\right| + 2\xi\left(|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right).
\end{aligned}$$

4. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{b-a}{4}\left(\xi\left(|f'(a)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right) + \xi\left(|f'(b)|, \left|f'\left(\frac{a+b}{2}\right)\right|\right)\right),$$

where $\xi(.,.)$ *is defined by* (2.3).

Theorem 2.2 *Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable function on* (a,b) *such that* $f' \in L([a,b])$ *with* $0 \leq a < b$, *and let* $w : [a,b] \to \mathbb{R}$ *be continuous and symmetric function with respect to* $\frac{a+b}{2}$. *If* $|f'|^q$ *is* log-*convex, where* $q > 1$ *with* $p^{-1} + q^{-1} = 1$, *then we have:*

1. *For* $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$

$$\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{2(p+1)^{\frac{1}{p}}}\left|f'\left(\frac{a+b}{2}\right)\right| \|w\|_{[a,b],\infty}.$$

2. *For* $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \mu\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\} \|w\|_{[a,b],\infty}.$$

3. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$

$$\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \mu\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\} \|w\|_{[a,b],\infty}.$$

4. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right|$$
$$\leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\left\{\mu\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) + \mu\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\} \|w\|_{[a,b],\infty},$$

where

$$\mu(x,y) = \left(\frac{x-y}{\ln x - \ln y}\right)^{\frac{1}{q}}, x, y > 0, x \neq y \text{ and } q > 1. \tag{2.13}$$

Proof: From Lemma 2.1, properties of modulus, Hölder inequality and log-convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{4}\left\{\int_0^1 |p_1(t)|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|dt + \int_0^1 |p_2(t)|\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|dt\right\} \\
\leq &\frac{(b-a)^2}{4}\left\{\left(\int_0^1 |p_1(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 |p_2(t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left(\int_0^1 (1-t)^p\,dt\right)^{\frac{1}{p}}\left\{\left(\int_0^1 \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\|w\|_{[a,b],\infty}\left\{\left(\int_0^1 \left(|f'(b)|^q\right)^t\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1-t} dt\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\int_0^1 \left(|f'(a)|^q\right)^t\left(\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{1-t} dt\right)^{\frac{1}{q}}\right\} \\
= &\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\|w\|_{[a,b],\infty}\left|f'\left(\frac{a+b}{2}\right)\right|\left\{\left(\int_0^1\left(\frac{|f'(b)|^q}{\left|f'\left(\frac{a+b}{2}\right)\right|^q}\right)^t dt\right)^{\frac{1}{q}} + \left(\int_0^1\left(\frac{|f'(a)|^q}{\left|f'\left(\frac{a+b}{2}\right)\right|^q}\right)^t dt\right)^{\frac{1}{q}}\right\}.
\end{aligned} \tag{2.14}$$

If $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$, then from (2.14), we have

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2\|w\|_{[a,b],\infty}}{2(p+1)^{\frac{1}{p}}}\left|f'\left(\frac{a+b}{2}\right)\right|. \tag{2.15}$$

If $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$, then from (2.14), we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}}\|w\|_{[a,b],\infty}\left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \mu\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\},
\end{aligned} \tag{2.16}$$

where $\mu(.,.)$ is defined by (2.13).

If $|f'(b)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(a)|$, then from (2.14), we have

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}} \|w\|_{[a,b],\infty} \left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \mu\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\}. \tag{2.17}$$

If $|f'(b)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(a)|$, then from (2.14), we have

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{4(p+1)^{\frac{1}{p}}} \|w\|_{[a,b],\infty} \left\{\mu\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) + \mu\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\}. \tag{2.18}$$

The desired result follows from (2.15)-(2.18). The proof is completed. □

Corollary 2.2B *In Theorem 2.2, if we take $w(u) = \frac{1}{b-a}$, we obtain*

1. *For $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$*

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}} \left|f'\left(\frac{a+b}{2}\right)\right|.$$

2. *For $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$*

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \mu\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\}.$$

3. *For $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$*

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \mu\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\}.$$

4. *For $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$*

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4(p+1)^{\frac{1}{p}}} \left\{\mu\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right) + \mu\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right\},$$

where μ is defined by (2.13).

Theorem 2.3 *Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f'\in L([a,b])$ with $0\leq a<b$, and let $w:[a,b]\to\mathbb{R}$ be continuous and symmetric function with respect to $\frac{a+b}{2}$. If $|f'|^{q}$ is s-convex in the second sense for some fixed $s\in(0,1]$, and $q\geq 1$, then we have*

1. *For* $|f'(a)|=\left|f'\left(\frac{a+b}{2}\right)\right|=|f'(b)|$

$$\left|\int_a^b w(u)f(u)\,du-\left(\int_a^b w(u)\,du\right)f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{4}\left|f'\left(\frac{a+b}{2}\right)\right|\|w\|_{[a,b],\infty}.$$

2. *For* $|f'(a)|=\left|f'\left(\frac{a+b}{2}\right)\right|\neq|f'(b)|$

$$\begin{aligned}&\left|\int_a^b w(u)f(u)\,du-\left(\int_a^b w(u)\,du\right)f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^2}{8}\left\{\left|f'\left(\frac{a+b}{2}\right)\right|+\left(2\xi\left(|f'(b)|^{q},\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)\right)^{\frac{1}{q}}\right\}\|w\|_{[a,b],\infty}.\end{aligned}$$

3. *For* $|f'(a)|\neq\left|f'\left(\frac{a+b}{2}\right)\right|=|f'(b)|$

$$\begin{aligned}&\left|\int_a^b w(u)f(u)\,du-\left(\int_a^b w(u)\,du\right)f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^2}{8}\left\{\left|f'\left(\frac{a+b}{2}\right)\right|+\left(2\xi\left(|f'(a)|^{q},\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)\right)^{\frac{1}{q}}\right\}\|w\|_{[a,b],\infty}.\end{aligned}$$

4. *For* $|f'(a)|\neq\left|f'\left(\frac{a+b}{2}\right)\right|\neq|f'(b)|$

$$\begin{aligned}&\left|\int_a^b w(u)f(u)\,du-\left(\int_a^b w(u)\,du\right)f\left(\frac{a+b}{2}\right)\right|\\&\leq\frac{(b-a)^2}{8}\left\{\left[2\xi\left(|f'(a)|^{q},\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)\right]^{\frac{1}{q}}+\left[2\xi\left(|f'(b)|^{q},\left|f'\left(\frac{a+b}{2}\right)\right|^{q}\right)\right]^{\frac{1}{q}}\right\}\|w\|_{[a,b],\infty},\end{aligned}$$

where ξ is defined by (2.3).

Proof: From Lemma 2.1, properties of modulus, power mean inequality, and log-convexity of $|f'|^{q}$, we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{4}\left\{\int_0^1 |p_1(t)| \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right| dt + \int_0^1 |p_2(t)| \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right| dt\right\} \\
\leq &\frac{(b-a)^2}{4}\left\{\left(\int_0^1 |p_1(t)| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 |p_1(t)| \left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 |p_2(t)| dt\right)^{1-\frac{1}{q}} \left(\int_0^1 |p_2(t)| \left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left(\int_0^1 (1-t) dt\right)^{1-\frac{1}{q}} \left\{\left(\int_0^1 (1-t)\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 (1-t)\left|f'\left(ta+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}}\right\} \\
\leq &\frac{(b-a)^2}{4\times 2^{1-\frac{1}{q}}}\|w\|_{[a,b],\infty}\left|f'\left(\frac{a+b}{2}\right)\right| \\
&\times\left\{\left(\int_0^1 (1-t)\left(\frac{|f'(b)|^q}{\left|f'\left(\frac{a+b}{2}\right)\right|^q}\right)^t dt\right)^{\frac{1}{q}} + \left(\int_0^1 (1-t)\left(\frac{|f'(a)|^q}{\left|f'\left(\frac{a+b}{2}\right)\right|^q}\right)^t dt\right)^{\frac{1}{q}}\right\}. \qquad (2.19)
\end{aligned}$$

If $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$, then from (2.19), we have

$$\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{(b-a)^2}{4}\|w\|_{[a,b],\infty}\left|f'\left(\frac{a+b}{2}\right)\right|. \qquad (2.20)$$

If $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$, then from (2.19), we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{4\times 2^{1-\frac{1}{q}}}\|w\|_{[a,b],\infty}\left\{\left(\frac{1}{2}\right)^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\xi\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right\}, \qquad (2.21)
\end{aligned}$$

where we have used (2.12), and ξ is defined by (2.3).
If $|f'(b)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(a)|$, then from (2.19), we have

$$\begin{aligned}
&\left|\int_a^b w(u) f(u) du - \left(\int_a^b w(u) du\right) f\left(\frac{a+b}{2}\right)\right| \\
\leq &\frac{(b-a)^2}{4\times 2^{1-\frac{1}{q}}}\|w\|_{[a,b],\infty}\left\{\left(\frac{1}{2}\right)^{\frac{1}{q}}\left|f'\left(\frac{a+b}{2}\right)\right| + \left(\xi\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right)^{\frac{1}{q}}\right\}, \qquad (2.22)
\end{aligned}$$

where we have used (2.12), and ξ is defined by (2.3).
If $|f'(b)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(a)|$, then from (2.19), we have

$$\begin{aligned}&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{(b-a)^2}{4\times 2^{1-\frac{1}{q}}} \|w\|_{[a,b],\infty} \left\{\left[\xi\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}} + \left[\xi\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}}\right\},\end{aligned} \tag{2.23}$$

where we have used (2.12) and (2.3).
The desired result follows from (2.20)-(2.23). The proof is completed. □

Corollary 2.3C *In Theorem 2.3, if we take $w(u) = \frac{1}{b-a}$, we obtain:*

1. *For* $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$

$$\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{4}\left|f'\left(\frac{a+b}{2}\right)\right|.$$

2. *For* $|f'(a)| = \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\begin{aligned}&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{8}\left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \left[2\xi\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

3. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| = |f'(b)|$

$$\begin{aligned}&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{8}\left\{\left|f'\left(\frac{a+b}{2}\right)\right| + \left[2\xi\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

4. *For* $|f'(a)| \neq \left|f'\left(\frac{a+b}{2}\right)\right| \neq |f'(b)|$

$$\begin{aligned}&\left|\int_a^b w(u) f(u)\,du - \left(\int_a^b w(u)\,du\right) f\left(\frac{a+b}{2}\right)\right| \\ &\leq \frac{b-a}{8}\left\{\left[2\xi\left(|f'(a)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}} + \left[2\xi\left(|f'(b)|^q, \left|f'\left(\frac{a+b}{2}\right)\right|^q\right)\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

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Badreddine Meftah,

Department of Mathematics,

University 8 may 1945 Guelma, Algeria.

E-mail address: badrimeftah@yahoo.fr

and

Djaber Chemseddine Benchettah,
Department CP,
Higher School of Management Sciences, Annaba, Algeria.
E-mail address: benchettah.djaber@essg-annaba.dz

and

Abdelghani Lakhdari,
Department CPST,
National Higher School of Technology and Engineering, Annaba 23005, Algeria.
E-mail address: a.lakhdari@ensti-annaba.dz

and

Meriem Merad,
Department of Mathematics,
University 8 may 1945 Guelma, Algeria.
E-mail address: mrad.meriem@gmail.com