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New generalized results of $(\mathfrak{A}, \imath)$-expansive operators on Hilbert spaces with practical comparison

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ABSTRACT: In this research, we obtain some new result related to a category of linear bounded operators, which is known as $(\mathfrak{A}, \imath)$-expansive operators acting on infinite Hilbert space $\imath$. Further, we establish sufficient conditions which $(\mathfrak{A}, \imath)$-expansive operators are not supercyclic. We supply some spectral properties of $(\mathfrak{A}, \imath)$-expansive operators, too. Also, some practical examples are discussed which confirm our generalized results.

Key Words: $(\mathfrak{A}, \imath)$-expansive operators, supercyclicity, spectrum.

Contents

1. Introduction

Let $\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ be the algebra of all bounded linear operators on separable complex Hilbert space $\boldsymbol{S}_{\mathcal{H}}$. At first the concept of $(\mathfrak{A}, \imath)$-isometric operators was innovated by Ahmed *et al.* [27]. They generalized the concept of $\imath$-isometry on $\boldsymbol{S}_{\mathcal{H}}$ when an extra semi-linear item is considered [27].

We say an operator $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is $\imath$-isometric if,

$$\sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\mathfrak{G}^{*^{\mathrm{k}}}\mathfrak{G}^{\mathrm{k}} = 0, \qquad \imath \in \mathbb{Z}^{+},$$

where $\binom{\imath}{\mathrm{k}}$ be the binomial coefficient [2]. It's clearly an isometric operator (i.e., a 1-isometric operator) is $\imath$-isometric for each positive numbers $\imath$, which is inferred that the category of all $\imath$-isometric operators is containing the lesson of isometric operators.

Agler *et al.* in [2,3,4], Richter [24], Shimorin [25], Patel [22] and Duggal in [12] and [13] studiedy of 1-isometric and 2-isometric operators on $\boldsymbol{S}_{\mathcal{H}}$. Botelho [10] and Ahmed [27] exhibited a generalization of $\imath$-isometric operators on general Banach spaces $(\boldsymbol{S}_{\mathcal{B}})$. In 2015, Gu defined,

$$\beta_{(\imath,p)}(\mathfrak{G}, \tau) := \sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|^{p}, \qquad \forall \tau \in (\boldsymbol{S}_{\mathcal{B}}),\ \imath \in \mathbb{N}, \tag{1.1}$$

where $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{B}})$, proved $\beta_{(\imath,p)}(\mathfrak{G}, \tau) \leq 0, \forall \tau \in \boldsymbol{S}_{\mathcal{B}}$ which it implies $\beta_{(\imath,p)}(\mathfrak{G}, \tau) \geq 0, \forall \tau \in \boldsymbol{S}_{\mathcal{B}}$, and extended several results for $(\imath, p)$-isometries $(\beta_{(\imath,p)}(\mathfrak{G}, \tau) = 0)$ to operators only satisfying $\beta_{(\imath,p)}(\mathfrak{G}, \tau) \leq$

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0 [17]. Also, ABerger–Shaw type result was proved for a class of $\imath$-expansive operators $(\imath-\mathbb{E}\mathbb{O})s$ on a $\boldsymbol{S}_{\mathcal{H}}$ [17]. In 2021, Al-Ahmadi introduced three new classes of mappings satisfying the following conditions

$$\max_{\substack{0\le k\le \imath\\ k\text{ even}}}\left\|\mathfrak{G}^{k}\tau-\mathfrak{G}^{k}\acute{\tau}\right\| = \max_{\substack{0\le k\le \imath\\ k\text{ odd}}}\left\|\mathfrak{G}^{k}\tau-\mathfrak{G}^{k}\acute{\tau}\right\|,$$
$$\max_{\substack{0\le k\le \imath\\ k\text{ even}}}\left\|\mathfrak{G}^{k}\tau-\mathfrak{G}^{k}\acute{\tau}\right\| \le \max_{\substack{0\le k\le \imath\\ k\text{ odd}}}\left\|\mathfrak{G}^{k}\tau-\mathfrak{G}^{k}\acute{\tau}\right\|,$$
$$\max_{\substack{0\le k\le \imath\\ k\text{ even}}}\left\|\mathfrak{G}^{k}\tau-\mathfrak{G}^{k}\acute{\tau}\right\| \ge \max_{\substack{0\le k\le \imath\\ k\text{ odd}}}\left\|\mathfrak{G}^{k}\tau-\mathfrak{G}^{k}\acute{\tau}\right\|,$$

$\forall\tau,\acute{\tau}$ in normed space $\boldsymbol{S}_{\mathcal{N}}$, where $\imath\in\mathbb{N}$, $\mathfrak{G}$ is a self-mapping on $\boldsymbol{S}_{\mathcal{N}}$ and proved some properties of the mappings [5].

Motivated by the mentioned works, in this paper, for an operator $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$, we denote

$$\beta_{\imath}(\mathfrak{G}) := \sum_{k=0}^{\imath}(-1)^{\imath-k}\binom{\imath}{k}\mathfrak{G}^{*^{k}}\mathfrak{G}^{k}, \qquad \imath\in\mathbb{N}. \tag{1.2}$$

We defined the concept of $(\mathfrak{A},\imath)$-expansive operators $((\mathfrak{A},\imath)-\mathbb{E}\mathbb{O}s)$ and presented a generalization of $\imath$-isometries to the operators on $\boldsymbol{S}_{\mathcal{H}}$.

The contents of this paper are listed as follows: In Section 2, we set up terminology and notation, also we define the concept of $(\mathfrak{A},\imath)-\mathbb{E}\mathbb{O}s$. In Section 3, at first, we shall specialize to the case $\imath=2$ and explore some properties of $(\mathfrak{A},2)-\mathbb{E}\mathbb{O}s$ and then several properties of $(\mathfrak{A},\imath)-\mathbb{E}\mathbb{O}s$ are proved. In Section 4, we focus on the supercyclicity, spectrum and approximated point spectrum of $(\mathfrak{A},\imath)-\mathbb{E}\mathbb{O}s$.

2. Prerequisite concepts of $\imath$-expansive & $\imath$-hyperexpasive operators

Now we recall the definition of some concepts such as $\imath$-expansive, $\imath$-hyperexpasive and completely hyperexpansive operators on $\boldsymbol{S}_{\mathcal{H}}$ which have been attracted to various authors. For example, Agler showed the relation between subnormality and the positivity of $(-1)^{\imath}\beta_{\imath}(\mathfrak{G})$ [4]. The following definition is our main idea obtained from [13].

Definition 2.1 ([14]) *An operator $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is called*

i) *$\imath$-isometry whenever $\beta_{\imath}(\mathfrak{G})=0$,*

ii) *$\imath$-expansive whenever $(-1)^{\imath}\beta_{\imath}(\mathfrak{G})\le 0$,*

iii) *$\imath$-hyperexpansive whenever $(-1)^{k}\beta_{k}(\mathfrak{G})\le 0$, $k=1,2,\ldots,\imath$,*

iv) *completely hyperexpansive whenever $(-1)^{\imath}\beta_{\imath}(\mathfrak{G})\le 0$ for all $\imath\ge 1$.*

For more instance consider [6,8,7,14,19,26]. For a fixed positive operator $\mathfrak{A}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$, we denominate

$$\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}} := \sum_{k=0}^{\imath}(-1)^{\imath-k}\binom{\imath}{k}\mathfrak{G}^{*^{k}}\mathfrak{A}\mathfrak{G}^{k},$$

or equivalently

$$\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}} := \langle\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}}\tau,\tau\rangle = \sum_{k=0}^{\imath}(-1)^{\imath-k}\binom{\imath}{k}\|\mathfrak{G}^{k}\tau\|_{\mathfrak{A}}^{2},$$

for an operator $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$, a non-negative integer $\imath$ and $\tau\in\boldsymbol{S}_{\mathcal{H}}$. An operator $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is said an $(\mathfrak{A},\imath)$-isometry if $\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}}=0$. The category of all $(\mathfrak{A},\imath)$-isomtries has been defined by Ahmed *et al.* [29] which studied by other authors (see [12,15,18,20,23,28]). Recall that

$$(-1)^{\imath}\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}} = \sum_{k=0}^{\imath}(-1)^{k}\binom{\imath}{k}\|\mathfrak{G}^{k}\tau\|_{\mathfrak{A}}^{2}, \qquad \mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}}).$$

Overall of this paper, in particular in the next definition, $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$ truly means $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$, for $\tau \in \boldsymbol{S}_{\mathcal{H}}$. As an extension of the classes of expansive and hyperexpansive operators on $\boldsymbol{S}_{\mathcal{H}}$, the following definition identifies the categories of operators which will study in this paper. Consider $\mathfrak{G}$, $\mathfrak{A} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}}), \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})^+$ respectively and $\imath \geq 1$. Then,

i) $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive if $(-1)^\imath \beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$;

ii) $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-hyperexpansive if $(-1)^{\mathrm{k}} \beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0, \ \forall\, 0 \leq \mathrm{k} \leq \imath$;

iii) $\mathfrak{G}$ is completely hyperexpansive if $(-1)^{\mathrm{k}} \beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$, for each $\mathrm{k} \geq 1$;

iv) $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-alternatively expansive if $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0$;

v) $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-alternatigly hyperexpansive if $\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0$, for $0 \leq \mathrm{k} \leq \imath$,

iv) $\mathfrak{G}$ is alternatively hyperexpansive if $\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0$, for $\mathrm{k} \geq 1$.

We say $(\mathfrak{A}, \imath)$-expansive operators are simply $\imath - \mathbb{EO}s$ and $(\mathfrak{A}, 1) - \mathbb{EO}s$ are $\mathfrak{A}$-expansive. When $(-1)^\imath \beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0$, we state that $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-contractive. If $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-contractive $\forall\, \imath \in \mathbb{N}$, then $\mathfrak{G}$ is called, completely hypercontractive. Agler in [1], proved the norm of each subnormal operator $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is not greater than one if and only if $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0$ for all positive integers $\imath$ and then they extended these inequalities to the $\imath$-isometric operators. In particular, they generalized the structure of 2-isometric operators [2,3]. Since every completely hyperexpansive operator is 2-isometric operator, hence some of the mathematicians have started working on completely hyperexpansive operators [6,26]. Hence, the study of $\imath$-expansive operators is very important. We refer the reader to [14] for more information about $\imath$-expansivity. Recently, the concept of $(\mathfrak{A}, \imath)$-isometric operators on semi-hilbertian space is introduced by Ahmed *et al.* and several results of these operators is given.

Definition 2.2 *An operator $\mathfrak{G}$ is a strict $(\mathfrak{A}, \imath)$-expansive whenever $\mathfrak{G}$ is an $(\mathfrak{A}, \imath)$-expansive and is not $(\mathfrak{A}, \imath - 1)$-expansive. If $\imath = 1$, it is an $\mathfrak{A}$-expansive, that is, $\mathfrak{G}$ is an $\mathfrak{A}$-expansive if $\mathfrak{G}^*\mathfrak{A}\mathfrak{G} \geq \mathfrak{A}$.*

Example 2.1 *If $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-isometry then $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive. If $\mathfrak{A} := I$, then $\mathfrak{G}$ is $\imath$-expansive if and only if $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive. If $\mathfrak{A} := 0$, then any operator on $\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is an $\mathfrak{A}$-expansive.*

Note: If $\mathfrak{G}$ is an $(\mathfrak{A}, \imath)$-expansive, $N(\mathfrak{G}) \subseteq N(\mathfrak{A})$. In particular, if $\mathfrak{A}$ is one-to-one, then $\mathfrak{G}$ is also one-to-one.

Example 2.2 *Let $\mathfrak{G} = \alpha I$, where $\alpha \in \mathbb{C}$ and I is the identity operator. It's clearly*

$$\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} = \sum_{\mathrm{k}=0}^{\imath} (-1)^{\imath-\mathrm{k}} \binom{\imath}{\mathrm{k}} \left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2 = \|\tau\|_{\mathfrak{A}}^2 \sum_{\mathrm{k}=0}^{\imath} (-1)^{\imath-\mathrm{k}} \binom{\imath}{\mathrm{k}} |\alpha|^{2\mathrm{k}} = \|\tau\|_{\mathfrak{A}}^2 (|\alpha|^2 - 1)^\imath.$$

Hence, $\forall\, \alpha \in \mathbb{C}$ with $|\alpha| > 1$ and all odd integer number $\imath$ the map $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive.

Example 2.3 *Let $\boldsymbol{S}_{\mathcal{H}} = \mathcal{C}^2$ be equipped with the norm $\|(\alpha_1, \alpha_2)\|^2 = \|\alpha_1\|^2 + \|\alpha_2\|^2$, for $\alpha_1, \alpha_2 \in \mathcal{C}$, and consider the operator*

$$\mathfrak{A} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})^+, \quad \alpha > 0, \qquad \mathfrak{G} = \begin{bmatrix} 2 & 1 \\ 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}}).$$

A simple calculation shows

$$\begin{aligned} \beta_\imath(\mathfrak{G}, (\alpha_1, \alpha_2))_{\mathfrak{A}} &= \sum_{\mathrm{k}=0}^{\imath} (-1)^{\imath-\mathrm{k}} \binom{\imath}{\mathrm{k}} \left\|\mathfrak{G}^{\mathrm{k}}(\alpha_1, \alpha_2)\right\|_{\mathfrak{A}}^2 = \sum_{\mathrm{k}=0}^{\imath} (-1)^{\imath-\mathrm{k}} \binom{\imath}{\mathrm{k}} \tfrac{\alpha}{2^{\mathrm{k}}} \|\alpha_2\|^2 \\ &= \alpha \|\alpha_2\|^2 \sum_{\mathrm{k}=0}^{\imath} (-1)^{\imath-\mathrm{k}} \binom{\imath}{\mathrm{k}} \tfrac{1}{2^{\mathrm{k}}} = \tfrac{\alpha\|\alpha_2\|^2}{2^\imath} (-1)^\imath. \end{aligned}$$

Hence, $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-contractive and if take $\mathfrak{G} = \begin{bmatrix} 2 & 1 \\ 0 & \sqrt{2} \end{bmatrix}$, then $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive for all positive odd integer number $\imath$.

3. Main results

In this section we collect some further results about our class of operatoprs,

Lemma 3.1 *Let $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ be an $(\mathfrak{A}, 2)$-expansive, then for $\imath \geq 1$ & $\tau \in \boldsymbol{S}_{\mathcal{H}}$ the following properties hold:*

1. $\|\mathfrak{G}\tau\|_{\mathfrak{A}} \geq \frac{\imath-1}{\imath}\|\tau\|_{\mathfrak{A}}$;
2. $\|\mathfrak{G}\tau\|_{\mathfrak{A}} \geq \|\tau\|_{\mathfrak{A}}$.
3. *If $\mathfrak{A}$ is left invertible, then $\mathfrak{G}$ is one-to-one.*
4. $\|\mathfrak{G}^{\imath}\tau\|^2_{\mathfrak{A}} \leq n(\tau\|^2_{\mathfrak{A}}) + \|\tau\|^2_{\mathfrak{A}}$.
5. $\|\mathfrak{G}^{2\imath}\tau\|^2_{\mathfrak{A}} \leq \imath \left\|\mathfrak{G}^{\imath+1}\tau\right\|^2_{\mathfrak{A}} - \imath(\imath-1)\|\mathfrak{G}\tau\|^2_{\mathfrak{A}} + (\imath-1)^2\|\tau\|^2_{\mathfrak{A}}$.
6. $\|\mathfrak{G}\tau\|_{\mathfrak{A}} \leq \sqrt{2}\|\tau\|_{\mathfrak{A}}$ *for each $\tau \in \mathrm{R}(\mathfrak{G})$ (the range of $\mathfrak{G}$).*
7. *If $\ker(\mathfrak{G}^*) = \{0\}$, and $\mathfrak{A}$ is left invertible then $\mathfrak{G}$ is $\mathfrak{A}$-isometry.*

Proof: Since $\mathfrak{G}$ is $(\mathfrak{A}, 2)$-expansive thus $\left\|\mathfrak{G}^2\tau\right\|^2_{\mathfrak{A}} - \|\mathfrak{G}\tau\|^2_{\mathfrak{A}} \leq \|\mathfrak{G}\tau\|^2_{\mathfrak{A}} - \|\tau\|^2_{\mathfrak{A}}$. Replacing τ by $\mathfrak{G}^{\mathrm{k}}\tau$ leads to

$$\left\|\mathfrak{G}^{\mathrm{k}+2}\tau\right\|^2_{\mathfrak{A}} - \left\|\mathfrak{G}^{\mathrm{k}+1}\tau\right\|^2_{\mathfrak{A}} \leq \left\|\mathfrak{G}^{\mathrm{k}+1}\tau\right\|^2_{\mathfrak{A}} - \left\|\mathfrak{G}^{\mathrm{k}}\right\|^2_{\mathfrak{A}}, \qquad \mathrm{k} \geq 0.$$

Hence

$$\begin{aligned}\left\|\mathfrak{G}^{\imath}\tau\right\|^2_{\mathfrak{A}} &= \sum_{\mathrm{k}=1}^{\imath}\left(\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|^2_{\mathfrak{A}} - \left\|\mathfrak{G}^{\mathrm{k}-1}\tau\right\|^2_{\mathfrak{A}}\right) + \|\tau\|^2_{\mathfrak{A}} \\ &\leq \sum_{\mathrm{k}=1}^{\imath}\left(\|\mathfrak{G}\tau\|^2_{\mathfrak{A}} - \|\tau\|^2_{\mathfrak{A}}\right) + \|\tau\|^2_{\mathfrak{A}} = \imath\|\mathfrak{G}\tau\|^2_{\mathfrak{A}} - (\imath-1)\|\tau\|^2_{\mathfrak{A}}.\end{aligned}$$

which implies 1, 2 and 4. The relations 3 and 5 will obtain easily from 2 and 4 respectively. The relation 6 is easily obtained from

$$\left\|\mathfrak{G}^2\tau\right\|^2_{\mathfrak{A}} \leq 2\|\mathfrak{G}\tau\|^2_{\mathfrak{A}} - \|\tau\|^2_{\mathfrak{A}} \leq 2\|\mathfrak{G}\tau\|^2_{\mathfrak{A}}, \qquad \tau \in \boldsymbol{S}_{\mathcal{H}}.$$

Thus $\|\mathfrak{G}\acute{\tau}\|_{\mathfrak{A}} \leq \sqrt{2}\|\acute{\tau}\|_{\mathfrak{A}}$, for each $\acute{\tau} = \mathfrak{G}\tau \in \mathrm{R}(\mathfrak{G})$. Now, we prove (7). It is clearly, ran($\mathfrak{G}$) is dense in $\mathcal{H}$ because of $\ker(\mathfrak{G}^*) = (0)$. This matched with the property 2 of Lemma 3.1 suggests that $\mathfrak{G}$ is invertible. Then, since

$$(-1)^2\beta_2(\mathfrak{G}^{-1}) = (\mathfrak{G}^{-2})^*\mathfrak{G}^{-2} - 2(\mathfrak{G}^{-1})^*\mathfrak{G}^{-1} + I = (\mathfrak{G}^{-2})^*[I - 2\mathfrak{G}^*\mathfrak{G} + \mathfrak{G}^{2^*}\mathfrak{G}^2]\mathfrak{G}^{-2} \leq 0.$$

We have $\mathfrak{G}^{-1}$ is $(\mathfrak{A}, 2)$-expansive, and hence $\|\mathfrak{G}^{-1}\tau\|_{\mathfrak{A}} \geq \|\tau\|_{\mathfrak{A}}$, $\forall\tau \in \boldsymbol{S}_{\mathcal{H}}$. Combined with the property that $\|\mathfrak{G}\tau\|_{\mathfrak{A}} \geq \|\tau\|_{\mathfrak{A}}$, we conclude that $\mathfrak{G}$ is $\mathfrak{A}$-isometry. □

Corollary 3.1 *Every $(\mathfrak{A}, 2)$-expansive operator on a finite dimensional $\boldsymbol{S}_{\mathcal{H}}$ is unitary.*

Proposition 3.1 *Let $\mathfrak{G}_1, \mathfrak{G}_2 \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ be commuting operators such that $\mathrm{R}(\mathfrak{G}_2) \subset \ker(\mathfrak{A})$, then the following are true:*

i) *$\mathfrak{G}_1$ is $(\mathfrak{A}, \imath)$-expansive if and only if, $\mathfrak{G}_1 + \mathfrak{G}_2$ is $(\mathfrak{A}, \imath)$-expansive;*

ii) *$\mathfrak{G}_1$ is $(\mathfrak{A}, \imath)$-expansive if and only if, $\lambda\mathfrak{G}_1$ is $(\mathfrak{A}, \imath)$-expansive $\forall\,\lambda$ with $|\lambda| = 1$.*

Proof: (i) Let $\tau \in \boldsymbol{S}_{\mathcal{H}}$,

$$\begin{aligned}\beta_\imath(\mathfrak{G}_1+\mathfrak{G}_2,\tau)_{\mathfrak{A}} &= \sum_{\mathrm{k}=1}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|(\mathfrak{G}_1+\mathfrak{G}_2)^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2\\ &= \sum_{\mathrm{k}=1}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\sum_{j=0}^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\mathfrak{G}_1^j\mathfrak{G}_2^{\mathrm{k}-j}\tau\right\|_{\mathfrak{A}}^2\\ &= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}_1^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2=\beta_\imath(\mathfrak{G}_1,\tau)_{\mathfrak{A}}.\end{aligned}$$

Hence $\mathfrak{G}_1+\mathfrak{G}_2$ is $(\mathfrak{A},\imath)$-expansive if and only if $\mathfrak{G}_1$ is $(\mathfrak{A},\imath)$-expansive. (ii) Let $\tau \in \boldsymbol{S}_{\mathcal{H}}$ and $\lambda\in\mathbb{C}$ with $|\lambda|=1$, clearly we can show that $\beta_\imath(\lambda\mathfrak{G}_1,\tau)_{\mathfrak{A}}=\beta_\imath(\mathfrak{G}_1,\tau)_{\mathfrak{A}}$, thus $\mathfrak{G}_1$ and $\lambda\mathfrak{G}_1$ are same operators. □

Proposition 3.2 *Let $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$, $\tau\in\boldsymbol{S}_{\mathcal{H}}$ and $\imath\in\mathbb{N}$. Then*

$$\beta_\imath(\mathfrak{G},\tau)_{\mathfrak{A}}=\beta_{\imath-1}(\mathfrak{G},\mathfrak{G}\tau)_{\mathfrak{A}}-\beta_{\imath-1}(\mathfrak{G},\tau)_{\mathfrak{A}}, \tag{3.1}$$

and

$$\beta_\imath(\mathfrak{G},\tau)_{\mathfrak{A}}=\left\|\mathfrak{G}^\imath\tau\right\|_{\mathfrak{A}}^2-\sum_{\mathrm{k}=1}^{\imath-1}\binom{\imath}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}}. \tag{3.2}$$

Proof: By apply the formula $\binom{\imath}{\mathrm{k}}=\binom{\imath-1}{\mathrm{k}}+\binom{\imath-1}{\mathrm{k}-1}$ for the binomial coefficient, we have

$$\begin{aligned}\beta_\imath(\mathfrak{G},\tau) &= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2=(-1)^\imath\|\tau\|_{\mathfrak{A}}^2+\left\|\mathfrak{G}^\imath\tau\right\|_{\mathfrak{A}}^2+\sum_{\mathrm{k}=1}^{\imath-1}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2\\ &= (-1)^\imath\|\tau\|_{\mathfrak{A}}^2+\left\|\mathfrak{G}^\imath\tau\right\|_{\mathfrak{A}}^2+\sum_{\mathrm{k}=1}^{\imath}(-1)^{\imath-\mathrm{k}}\left[\binom{\imath-1}{\mathrm{k}}+\binom{\imath-1}{\mathrm{k}-1}\right]\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2\\ &= \sum_{\mathrm{k}=0}^{\imath-1}(-1)^{\imath-\mathrm{k}}\binom{\imath-1}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2+\sum_{\mathrm{k}=1}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath-1}{\mathrm{k}-1}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2\\ &= -\sum_{\mathrm{k}=0}^{\imath-1}(-1)^{(\imath-1)-\mathrm{k}}\binom{\imath-1}{\mathrm{k}}\|\mathfrak{G}^{\mathrm{k}}\tau\|_{\mathfrak{A}}^2+\sum_{\mathrm{k}=0}^{\imath-1}(-1)^{(\imath-1)-\mathrm{k}}\binom{\imath-1}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}+1}\tau\right\|_{\mathfrak{A}}^2\\ &= \beta_{\imath-1}(\mathfrak{G},\mathfrak{G}\tau)_{\mathfrak{A}}-\beta_{\imath-1}(\mathfrak{G},\tau)_{\mathfrak{A}}.\end{aligned}$$

Thus, the first equality is proved. The proof of second identity is demonstrated by induction on $\imath$. For $\imath=1$,

$$\beta_1(\mathfrak{G},\tau)_{\mathfrak{A}}=\sum_{\mathrm{k}=0}^{1}(-1)^{1-\mathrm{k}}\binom{1}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2=-\|\tau\|_{\mathfrak{A}}^2+\|\mathfrak{G}\tau\|_{\mathfrak{A}}^2=\|\mathfrak{G}\tau\|_{\mathfrak{A}}^2-\beta_0(\mathfrak{G},\tau)_{\mathfrak{A}}.$$

Now assume that

$$\beta_\imath(\mathfrak{G})_{\mathfrak{A}}:=\mathfrak{G}^{*^\imath}\mathfrak{A}\mathfrak{G}^\imath-\sum_{\mathrm{k}=0}^{\imath-1}\binom{\imath}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}},$$

and multiplies this relation on the left by $\mathfrak{G}^*$ and on the right by $\mathfrak{G}$,

$$\begin{aligned}
\mathfrak{G}^*\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}}\mathfrak{G} &= \mathfrak{G}^{*^{m+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \sum_{\mathrm{k}=0}^{\imath-1}\binom{\imath}{\mathrm{k}}\mathfrak{G}^*\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}}\mathfrak{G} \\
&= \mathfrak{G}^{*^{\imath+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \sum_{\mathrm{k}=0}^{\imath-1}\binom{\imath}{\mathrm{k}}[\beta_{\mathrm{k}+1}(\mathfrak{G})_{\mathfrak{A}} + \beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}}] \\
&= \mathfrak{G}^{*^{\imath+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \sum_{\mathrm{k}=1}^{\imath}\binom{\imath}{\mathrm{k}-1}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}} - \sum_{\mathrm{k}=0}^{\imath-1}\binom{\imath}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}} \\
&= \mathfrak{G}^{*^{\imath+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \binom{\imath}{\imath-1}\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}} - \beta_0(\mathfrak{G})_{\mathfrak{A}} - \sum_{\mathrm{k}=1}^{\imath-01}\binom{\imath+1}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}} \\
&= \mathfrak{G}^{*^{\imath+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \binom{\imath}{\imath-1}\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}} - \sum_{\mathrm{k}=0}^{\imath-1}\binom{\imath+1}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}}.
\end{aligned}$$

Therefore by the relation (3.1) identity again and above formula we have

$$\begin{aligned}
\beta_{\imath+1}(\mathfrak{G})_{\mathfrak{A}} &= \mathfrak{G}^*\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}}\mathfrak{G} - \beta_{\imath}(\mathfrak{G})_{\mathfrak{A}} \\
&= \mathfrak{G}^{*^{\imath+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \binom{\imath}{\imath-1}\beta_{\imath}(\mathfrak{G})_{\mathfrak{A}} - \sum_{\mathrm{k}=0}^{\imath-1}\binom{\imath+1}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}} - \beta_{\imath}(\mathfrak{G})_{\mathfrak{A}} \\
&= \mathfrak{G}^{*^{\imath+1}}\mathfrak{A}\mathfrak{G}^{\imath+1} - \sum_{\mathrm{k}=0}^{\imath}\binom{\imath+1}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}}.
\end{aligned}$$

This complete the proof. □

Corollary 3.2 *If $\mathfrak{G}$ is $(\mathfrak{A},\imath)$-expansive and $(\mathfrak{A},\imath-1)$-expansive on $\mathrm{R}(\mathfrak{G})$, then $\mathfrak{G}$ is $(\mathfrak{A},\imath-1)$-expansive.*

Theorem 3.1 *Let $\mathfrak{G}$ be a $(\mathfrak{A},2)$-expansive and assume that $\mathfrak{G}$ is $(\mathfrak{A},\imath)$-expansive for some $\imath \geq 2$. Then $\mathfrak{G}$ is $(\mathfrak{A},\imath)$-hyperexpansive.*

Proof: The conditions $\|\mathfrak{G}\tau\|_{\mathfrak{A}} \geq \|\tau\|_{\mathfrak{A}}$ and

$$\left\|\mathfrak{G}^2\tau\right\|_{\mathfrak{A}}^2 - 2\left\|\mathfrak{G}\tau\right\|_{\mathfrak{A}}^2 + \left\|\tau\right\|_{\mathfrak{A}}^2 \leq 0,$$

guarantee that the sequence

$$\left(\|\mathfrak{G}^{n+1}\tau\|_{\mathfrak{A}}^2 - \|\mathfrak{G}^n\tau\|_{\mathfrak{A}}^2\right)_{n\geq 0},$$

is monotonically non-increasing and bounded so that is convergent. Let $\|\mathfrak{G}^{n+1}\tau\|_{\mathfrak{A}}^2 - \|\mathfrak{G}^n\tau\|_{\mathfrak{A}}^2 \to a$, for some $a \geq 0$ as $n \to \infty$. By assumption, select $\imath > 2$ such that $(-1)^{\imath}\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}} \leq 0$. Since

$$\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}} = \beta_{\imath-1}(\mathfrak{G},\mathfrak{G}\tau)_{\mathfrak{A}} - \beta_{\imath-1}(\mathfrak{G},\tau)_{\mathfrak{A}},$$

we have $(-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G},\tau)_{\mathfrak{A}} \leq (-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G},\mathfrak{G}\tau)_{\mathfrak{A}}$. An induction argument shows that

$$(-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G},\tau)_{\mathfrak{A}} \leq (-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G},\mathfrak{G}^n\tau)_{\mathfrak{A}},$$

for each positive integer number n. On the other hand

$$\begin{aligned}
(-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G},\mathfrak{G}^n\tau)_{\mathfrak{A}} &= (-1)^{\imath-1}\beta_{\imath-2}(\mathfrak{G},\mathfrak{G}^{n+1}\tau)_{\mathfrak{A}} - (-1)^{\imath-1}\beta_{\imath-2}(\mathfrak{G},\mathfrak{G}^n\tau)_{\mathfrak{A}} \\
&= -\sum_{\mathrm{k}=0}^{\imath-2}(-1)^{\mathrm{k}}\binom{\imath-2}{\mathrm{k}}\left\|\mathfrak{G}^{n+\mathrm{k}+1}\tau\right\|_{\mathfrak{A}}^2 + \sum_{\mathrm{k}=0}^{\imath-2}(-1)^{\mathrm{k}}\binom{\imath-2}{\mathrm{k}}\left\|\mathfrak{G}^{n+\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2 \\
&= \sum_{\mathrm{k}=0}^{\imath-2}(-1)^{\mathrm{k}}\binom{\imath-2}{\mathrm{k}}\left[\left\|\mathfrak{G}^{n+\mathrm{k}}\tau\right\|_{\mathfrak{A}}^2 - \left\|\mathfrak{G}^{n+\mathrm{k}+1}\tau\right\|_{\mathfrak{A}}^2\right].
\end{aligned}$$

By approaching n to infinity in the preceding equality, we leads to

$$(-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G}, \mathfrak{G}^n\tau)_{\mathfrak{A}} \to \sum_{\mathrm{k}=0}^{\imath-2}(-1)\mathrm{k}\binom{\imath-2}{\mathrm{k}}a = 0.$$

This shows that $(-1)^{\imath-1}\beta_{\imath-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$. Thus $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-hyperexpansive □

Proposition 3.3 (i) *If $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$, $\forall \tau \in \boldsymbol{S}_{\mathcal{H}}$, then for $n \geq \imath$,*

$$\|\mathfrak{G}^n\tau\|^2_{\mathfrak{A}} \leq \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}}, \qquad \tau \in \boldsymbol{S}_{\mathcal{H}}. \tag{3.3}$$

(ii) *If $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0$, $\forall \tau \in \boldsymbol{S}_{\mathcal{H}}$, then for $n \geq \imath$,*

$$\|\mathfrak{G}^n\tau\|^2_{\mathfrak{A}} \geq \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}}, \qquad \tau \in \boldsymbol{S}_{\mathcal{H}}. \tag{3.4}$$

Proof: These inequalities are proved by induction on n. The inequality is proved easily for $n = \imath$ by (3.2). Assume now Eq. (3.3) holds for some $n > \imath$. Then

$$\begin{aligned}
\left\|\mathfrak{G}^{n+1}\tau\right\|^2_{\mathfrak{A}} &= \|\mathfrak{G}^n(\mathfrak{G}\tau)\|^2_{\mathfrak{A}} \leq \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \mathfrak{G}\tau)_{\mathfrak{A}} \\
&= \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} + \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}+1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \\
&= \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} + \sum_{\mathrm{k}=1}^{\imath}\binom{n}{\mathrm{k}-1}\beta_{\mathrm{k}}(\mathfrak{G}\mathfrak{G}, \tau)_{\mathfrak{A}} \\
&= \sum_{\mathrm{k}=0}^{\imath-1}\left[\binom{n}{\mathrm{k}} + \binom{n}{\mathrm{k}-1}\right]\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} + \binom{n}{\imath-1}\beta_\imath(\mathfrak{G}, \tau) \\
&= \sum_{\mathrm{k}=0}^{\imath-1}\binom{n+1}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} + \binom{n}{\imath-1}\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq \sum_{\mathrm{k}=0}^{\imath-1}\binom{n+1}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}}.
\end{aligned}$$

Note that, the second inequality is obtained by induction hypothesis and the last inequality is followed from the condition "$\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$". Hence, (i) is proved. The proof of (b) is similar. □

Corollary 3.3 *Let $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$. If $\beta_\imath(\mathfrak{G})_{\mathfrak{A}} \leq 0$, then for $n \geq \imath$,*

$$\mathfrak{G}^{*^n}\mathfrak{A}\mathfrak{G}^n \leq \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G})_{\mathfrak{A}}.$$

If $\beta_\imath(\mathfrak{G})_{\mathfrak{A}} \geq 0$, then the above inequality with $\geq$ holds.

Theorem 3.2 *If $\beta_\imath(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$, $\forall \tau \in \boldsymbol{S}_{\mathcal{H}}$, then $\beta_{\imath-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq 0, \forall \tau \in \boldsymbol{S}_{\mathcal{H}}$.*

Proof: By proposition 3.3 part (i), $\forall \tau \in \boldsymbol{S}_{\mathcal{H}}$,

$$\binom{n}{\imath-1}\beta_{\imath-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \geq \|\mathfrak{G}^n\tau\|^2_{\mathfrak{A}} - \sum_{\mathrm{k}=0}^{\imath-2}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}}.$$

Dividing both sides by $n^{\imath-1}$ results

$$\tfrac{1}{n^{\imath-1}}\binom{n}{\imath-1}\beta_{\imath-1}(\mathfrak{G},\tau)_{\mathfrak{A}} \geq \tfrac{1}{n^{\imath-1}}\|\mathfrak{G}^n\tau\|^2_{\mathfrak{A}} - \sum_{\mathrm{k}=0}^{\imath-2}\tfrac{1}{n^{\imath-1}}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}}.$$

Upon taking the limit as $n\to\infty$ and nothing that $\frac{1}{n^{\imath-1}}\binom{n}{\mathrm{k}}\to 0$, for $0\leq \mathrm{k}\leq \imath-2$ and

$$\tfrac{1}{n^{\imath-1}}\|\mathfrak{G}^n\tau\|^2_{\mathfrak{A}}\geq 0,$$

we have the desired result. □

Theorem 3.3 *Each positive integer power of an $(\mathfrak{A},\imath)$-expansive on a $\boldsymbol{S}_{\mathcal{H}}$, is $(\mathfrak{A},\imath)$-expansive.*

Proof: Fix a positive integer n. Consider the arbitrary positive real numbers τ_i $(0\leq i\leq \imath(n-1))$, so that $\tau_0=1$ and

$$\left[\sum_{j=0}^{n-1}\mathrm{t}^j\right]^{\imath} = \sum_{i=0}^{\imath(n-1)}\tau_i\mathrm{t}^i, \qquad \forall\,\mathrm{t}\in\mathbb{R}.$$

Furthermore, define τ_i to be zero for $i>\imath(n-1)$ and take $\binom{\imath}{i}=0$ for $i>\imath$. A simple compute shows that if

$$s_{\mathrm{k}} = \sum_{i=1}^{\mathrm{k}}(-1)^i\binom{\mathrm{k}}{i}\tau_{\mathrm{k}-i}, \qquad 0\leq \mathrm{k}\leq \imath n,$$

then $s_{\mathrm{k}}=(-1)^{\mathrm{k}_1}\binom{\imath}{\mathrm{k}_1}$, whenever $\mathrm{k}=n\mathrm{k}_1$ for some positive integer k_1, and otherwise $s_{\mathrm{k}}=0$. Indeed

$$\sum_{\mathrm{k}=0}^{\imath n}s_{\mathrm{k}}\mathrm{t}^{\mathrm{k}} = \left[\sum_{\mathrm{k}=0}^{\imath}(-1)^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\mathrm{t}^{\mathrm{k}}\right]\left[\sum_{\mathrm{k}=0}^{\imath(n-1)}\tau_{\mathrm{k}}\mathrm{t}^{\mathrm{k}}\right] = (1-\mathrm{t})^{\imath}\left(\sum_{i=0}^{n-1}\mathrm{t}^i\right)^{\imath} = (1-\mathrm{t}^n)^{\imath} = \sum_{\mathrm{k}=0}^{\imath}(-1)^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\mathrm{t}^{\mathrm{k}n}.$$

$\mathfrak{G}$ is $(\mathfrak{A},\imath)$-expansive, thus

$$(-1)^{\imath}\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}} = \sum_{j=0}^{\imath}(-1)^j\binom{\imath}{j}\left\|\mathfrak{G}^j\tau\right\|^2_{\mathfrak{A}}\leq 0.$$

Consequently,

$$\begin{aligned}0 &\geq \sum_{i=0}^{\imath(n-1)}\tau_i\sum_{j=0}^{\imath}(-1)^j\binom{\imath}{j}\left\|\mathfrak{G}^{i+j}\tau\right\|^2_{\mathfrak{A}}\\ &= \sum_{k=0}^{\imath n}s_{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|^2_{\mathfrak{A}} = \sum_{\mathrm{k}=0}^{\imath}s_{\mathrm{k}n}\left\|\mathfrak{G}^{\mathrm{k}n}\tau\right\|^2_{\mathfrak{A}} = (-1)^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}n}\tau\right\|^2_{\mathfrak{A}}.\end{aligned}$$

Hence, $\mathfrak{G}^n$ is an $(\mathfrak{A},\imath)$-expansive. □

Lemma 3.2 *Let $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$. If $\mathfrak{G}$ is invertible, then $\beta_{\imath}(\mathfrak{G}^{-1},\tau)_{\mathfrak{A}} = (-1)^{\imath}\beta_{\imath}(\mathfrak{G},\mathfrak{G}^{-\imath}\tau)$.*

Proof: We have

$$\begin{aligned}\beta_{\imath}(\mathfrak{G}^{-1},\tau)_{\mathfrak{A}} &= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|(\mathfrak{G}^{-1})^{\mathrm{k}}\tau\right\|^2_{\mathfrak{A}}\\ &= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\imath-\mathrm{k}}\left\|(\mathfrak{G}^{-\mathrm{k}}\tau\right\|^2_{\mathfrak{A}} = \sum_{\mathrm{k}=0}^{\imath}(-1)^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{-(\imath-\mathrm{k})}\tau\right\|^2_{\mathfrak{A}}\\ &= (-1)^{\imath}\sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}(\mathfrak{G}^{-\imath}\tau)\right\|^2_{\mathfrak{A}} = (-1)^{\imath}\beta_{\imath}(\mathfrak{G},\mathfrak{G}^{-\imath}\tau)_{\mathfrak{A}}.\end{aligned}$$

□

Corollary 3.4 *Let $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ be an invertible $(\mathfrak{A}, \grave{\imath})$-expansive operator,* (i) *If $\grave{\imath}$ is even, then $\mathfrak{G}^{-1}$ is $(\mathfrak{A}, \grave{\imath})$-expansive;* (ii) *If $\grave{\imath}$ is odd, then $\mathfrak{G}^{-1}$ is $(\mathfrak{A}, \grave{\imath})$-contractive*

Corollary 3.5 *Assume $\mathfrak{G}$ is invertible. If $\mathfrak{G}$ is $(\mathfrak{A}, \grave{\imath})$-expansive for some even $\grave{\imath}$, then $\mathfrak{G}$ is an $(\mathfrak{A}, \grave{\imath}-1)$-isometry*

Proof: Since $\mathfrak{G}$ is $(\mathfrak{A}, \grave{\imath})$-expansive and $\grave{\imath}$ is even thus $\beta_{\grave{\imath}}(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0$, $\forall \tau \in \boldsymbol{S}_{\mathcal{H}}$, then by Lemma 3.2,

$$\beta_{\grave{\imath}}(\mathfrak{G}^{-1}, \tau)_{\mathfrak{A}} = (-1)^{\grave{\imath}} \beta_{\grave{\imath}}(\mathfrak{G}, \tau)_{\mathfrak{A}} \leq 0.$$

This implies that $\mathfrak{G}^{-1}$ is also $(\mathfrak{A}, \grave{\imath})$-expansive and then with respect to $\grave{\imath}$, we have $\beta_{\grave{\imath}-1}(\mathfrak{G}^{-1}, \tau)_{\mathfrak{A}} \geq 0$. On the other hand

$$\beta_{\grave{\imath}-1}(\mathfrak{G}^{-1}, \tau)_{\mathfrak{A}} = (-1)^{\grave{\imath}-1} \beta_{\grave{\imath}-1}(\mathfrak{G}, \mathfrak{G}^{-(\grave{\imath}-1)}\tau)_{\mathfrak{A}} = -\beta_{\grave{\imath}-1}(\mathfrak{G}, \mathfrak{G}^{-(\grave{\imath}-1)}\tau)_{\mathfrak{A}} \leq 0.$$

Therefore $\beta_{\grave{\imath}-1}(\mathfrak{G}^{-1}, \tau)_{\mathfrak{A}} = 0$, which it follows that $\beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} = 0$, $\forall \tau \in \boldsymbol{S}_{\mathcal{H}}$. In other words, $\mathfrak{G}$ is an $(\mathfrak{A}, \grave{\imath}-1)$-isometry. □

Theorem 3.4 *Let $\mathfrak{G} \in \mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ be an $(\mathfrak{A}, \grave{\imath})$-expansive, then the following properties hold*

1. $\frac{\|\mathfrak{G}^n \tau\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}}$ *converge uniformly to* $\frac{1}{(\grave{\imath}-1)!}\beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}}$ *on the unit ball of* $\boldsymbol{S}_{\mathcal{H}}$.

2. $\frac{\|\mathfrak{G}^n\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}}$ *converge to* $\sup_\tau \beta_{\grave{\imath}-1}(\mathfrak{G})_{\mathfrak{A}}$.

Proof: Apply Proposition 3.3, we have

$$\begin{aligned}
\frac{\|\mathfrak{G}^n \tau\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}} - \frac{1}{(\grave{\imath}-1)!}\beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} &\leq \frac{\binom{n}{\grave{\imath}-1}}{n^{\grave{\imath}-1}}\beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \\
&+ \frac{1}{n^{\grave{\imath}-1}} \sum_{\mathrm{k}=0}^{\grave{\imath}-2} \binom{\grave{\imath}}{\mathrm{k}} \beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}} - \frac{1}{(\grave{\imath}-1)!}\beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \\
&= \left(\frac{\binom{n}{\grave{\imath}-1}}{n^{\grave{\imath}-1}} - \frac{1}{(\grave{\imath}-1)!} \right) \beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} + \frac{1}{n^{\grave{\imath}-1}} \sum_{\mathrm{k}=0}^{\grave{\imath}-2} \binom{\grave{\imath}}{\mathrm{k}} \beta_{\mathrm{k}}(\mathfrak{G}, \tau)_{\mathfrak{A}}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\left| \frac{\|\mathfrak{G}^n \tau\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}} - \frac{1}{(\grave{\imath}-1)!}\beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}} \right| &\leq \left(\frac{n!}{n^{\grave{\imath}-1}(n-\grave{\imath}+1)!} - 1 \right) \sum_{\mathrm{k}=0}^{\grave{\imath}-1} \frac{1}{\mathrm{k}!(\grave{\imath}-1-\mathrm{k})!} \left\| \mathfrak{G}^{\mathrm{k}} \tau \right\|_{\mathfrak{A}}^2 \\
+ \sum_{\mathrm{k}=0}^{\grave{\imath}-2} \frac{\binom{n}{\mathrm{k}}}{n^{\grave{\imath}-1}} \sum_{j=0}^{\mathrm{k}} \binom{\mathrm{k}}{j} \left\| \mathfrak{G}^j \tau \right\|_{\mathfrak{A}}^2 &\leq \left(\frac{n!}{n^{\grave{\imath}-1}(n-\grave{\imath}+1)!} - 1 \right) \frac{2^{\grave{\imath}-1}\breve{M}}{(\grave{\imath}-1)!} + \sum_{\mathrm{k}=0}^{\grave{\imath}-2} \frac{n!}{(n-\mathrm{k})! n^{\grave{\imath}-1}} \frac{2^{\mathrm{k}}\breve{M}}{\mathrm{k}!} \\
= \left(\frac{n!}{n^{\grave{\imath}-1}(n-\grave{\imath}+1)!} - 1 \right) \frac{2^{\grave{\imath}-1}\breve{M}}{(\grave{\imath}-1)!} + \frac{\breve{M} 3^{\grave{\imath}-2}}{n^{\grave{\imath}-1}} &\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

where $\breve{M} = \sum_{0 \leq \mathrm{k} \leq \grave{\imath}-1} \left\| \mathfrak{G}^{\mathrm{k}} \right\|_{\mathfrak{A}}$. (2) Since $\frac{\|\mathfrak{G}^n\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}} = \sup_{\|\tau\| \leq 1} \frac{\|\mathfrak{G}^n \tau\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}}$, we deduce from (1) that

$$\lim_{n \to \infty} \frac{\|\mathfrak{G}^n\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}} = \lim_{n \to \infty} \sup_{\|\tau\| \leq 1} \frac{\|\mathfrak{G}^n \tau\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}} = \sup_{\|\tau\| \leq 1} \lim_{n \to \infty} \frac{\|\mathfrak{G}^n \tau\|_{\mathfrak{A}}^2}{n^{\grave{\imath}-1}} = \sup_{\|\tau\| \leq 1} \beta_{\grave{\imath}-1}(\mathfrak{G}, \tau)_{\mathfrak{A}}.$$

□

4. Dynamic properties

In this section we are going to investigate the supercyclicity of $(\mathfrak{A},\grave{\imath})-\mathbb{E}\mathbb{O}s$. The supercyclicity of 2-expansive (concave) operators has been studied by Karimi *et al.* in [21]. Recall that an operator $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is supercyclic if there exists an element $\tau\in\boldsymbol{S}_{\mathcal{H}}$ such that

$$\mathbb{C}\mathrm{orb}(\mathfrak{G},\tau):=\Big\{\alpha\tau,\alpha\mathfrak{G}\tau,\alpha\mathfrak{G}^2\tau,\cdots\Big\},\qquad\forall\alpha\in\mathbb{C},$$

is dense in $\boldsymbol{S}_{\mathcal{H}}$ and τ is called a supercyclic vector for $\mathfrak{G}$. We start with the following result.

Theorem 4.1 *A power bounded $\mathfrak{A}$-expansive operator cannot be supercyclic.*

Proof: Assume on contrary, $\mathfrak{G}$ is supercyclic operator and $\tau\in\boldsymbol{S}_{\mathcal{H}}$ is a supercyclic vector for $\mathfrak{G}$. Let $0\neq\acute{\tau}\in\boldsymbol{S}_{\mathcal{H}}$ and $\acute{\tau}\notin\ker(\mathfrak{A})$. Then there exists a sequence $(\lambda_i)_i\subset\mathbb{C}$ and a strictly increasing sequence $(n_i)\subset\mathbb{N}$ such that $\lim_{i\to\infty}\lambda_i\mathfrak{G}^{n_i}\tau=\acute{\tau}$, and hence

$$\lim_{i\to\infty}|\lambda_i|\,\|\mathfrak{G}^{n_i}\tau\|_{\mathfrak{A}}=\|\acute{\tau}\|_{\mathfrak{A}}.\tag{4.1}$$

Since $\mathfrak{G}$ is A-expansive thus $\|\acute{\tau}\|_{\mathfrak{A}}\geq\|\tau\|_{\mathfrak{A}}\lim_{i\to\infty}|\lambda_i|$, if $\lim_{i\to\infty}|\lambda_i|=0$, then relation 4.1 shows $\lim_{i\to\infty}\|\mathfrak{G}^{n_i}\tau\|=\infty$, but this is contradiction, because $\mathfrak{G}$ is power bounded. And if $\lim_{i\to\infty}|\lambda_i|=a\in(0,\infty)$, then

$$\|\mathfrak{G}\acute{\tau}\|_{\mathfrak{A}}=\lim_{i\to\infty}|\lambda_i|\,\left\|\mathfrak{G}^{n_i+1}\tau\right\|_{\mathfrak{A}}=\|\acute{\tau}\|_{\mathfrak{A}}.$$

This shows that $\mathfrak{G}$ is A-isometries and is't supercyclic [23]. □

The following example gives a non power bounded $\mathfrak{A}$-expansive which is supercyclic.

Example 4.1 *Take $w_n=3$ for $n\leq 0$, and $w_n=2$ for $n\geq 1$. Let $\mathfrak{G}\in\ell^2(\mathbb{Z})$ be the bilateral weighted shift defined by $\mathfrak{G}e_n=w_ne_{n+1}$ and $\mathfrak{A}\in\ell^2(\mathbb{Z})^+$ be the positive linear operator defined by $\mathfrak{A}e_n=w_ne_n$. Since the condition*

$$w_n^2w_{n+1}^2=\|\mathfrak{G}e_n\|_{\mathfrak{A}}^2\geq\|e_n\|_{\mathfrak{A}}^2=w_n,$$

only involves consecutive n, and since the condition is trivial if $w_n=w_{n+1}$ one only needs to check the condition for $n=0$, where it holds indeed. Thus, the corresponding forward bilateral shift $\mathfrak{G}$ is $\mathfrak{A}$-expansive and invertible. Furthermore, since

$$\lim_{n\to\infty}\prod_{j=1}^{n}\tfrac{w_j}{w_{-j}}=\lim_{n\to\infty}\left(\tfrac{2}{3}\right)^n=0,$$

in light of [16] the operator $\mathfrak{G}$ is supercyclic.

Suppose that B_1 and B_2 are two Banach spaces and $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are two bounded linear operators acting on these spaces, respectively. Recall that the operator $\mathfrak{G}_2$ is said to be a quasi-factor of $\mathfrak{G}_1$, if there exists a continuous map $\boldsymbol{J}:\mathrm{B}_1\to\mathrm{B}_2$, dense range such that $\mathfrak{G}_2\boldsymbol{J}=\boldsymbol{J}\mathfrak{G}_1$.

Lemma 4.1 *For every $(\mathfrak{A},\grave{\imath})$-expansive operator $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ there are a Banach space B_2 and an $\grave{\imath}$-expansive operator $\hat{\mathfrak{G}}$ on B_2 which is quasi-factor of $\mathfrak{G}$.*

Proof: Consider $(\mathfrak{A},\grave{\imath})$-expansive operator $\mathfrak{G}$ on $\boldsymbol{S}_{\mathcal{H}}$ and the quotient space $\frac{\boldsymbol{S}_{\mathcal{H}}}{\ker(\mathfrak{A})}$. Define

$$\|(\tau+\ker\mathfrak{A})\|_{\mathfrak{A}}=\inf\Big\{\|\tau+\acute{\tau}\|_{\mathfrak{A}}\;:\;\acute{\tau}\in\ker(\mathfrak{A})\Big\}.$$

Let (τ_α) be a net in $\ker(\mathfrak{A})$ and $\tau_\alpha\to\tau$, then $\|\tau\|_{\mathfrak{A}}=\lim_\alpha\|\tau_\alpha\|_{\mathfrak{A}}=0$. So the positivity of $\mathfrak{A}$ implies that $\mathfrak{A}\tau=0$. Now, if $\|\tau+\ker(\mathfrak{A})\|_{\mathfrak{A}}=0$, then there is a net $(\tau_\alpha)_\alpha$ in $\ker(\mathfrak{A})$ so that $\|\tau+\tau_\alpha\|\to 0$. Hence, $\|\tau\|_{\mathfrak{A}}=0$ which implies that $\tau\in\ker(\mathfrak{A})$. Define

$$\begin{cases}\hat{\mathfrak{G}}:\left(\frac{\boldsymbol{S}_{\mathcal{H}}}{\ker(\mathfrak{A})},\|\cdot\|_{\mathfrak{A}}\right)\to\left(\frac{\boldsymbol{S}_{\mathcal{H}}}{\ker(\mathfrak{A})},\|\cdot\|_{\mathfrak{A}}\right),\\[2mm] \hat{\mathfrak{G}}(\tau+\ker(\mathfrak{A}))=\mathfrak{G}\tau+\ker(\mathfrak{A}).\end{cases}$$

Then

$$\|\hat{\mathfrak{G}}(\tau+\ker(\mathfrak{A}))\|_{\mathfrak{A}} = \|\mathfrak{G}\tau+\ker(\mathfrak{A})\|_{\mathfrak{A}} = \inf\left\{\|\mathfrak{G}\tau+\acute{\tau}\|_{\mathfrak{A}} \,:\, \acute{\tau}\in\ker(\mathfrak{A})\right\} = \|\mathfrak{G}\tau\|_{\mathfrak{A}}.$$

Thus,

$$\begin{aligned}(-1)^{\imath}\beta_{\imath}\left(\hat{\mathfrak{G}},\tau\right)_{\mathfrak{A}} &= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\hat{\mathfrak{G}}^{\mathrm{k}}(\tau+\ker(\mathfrak{A}))\right\|_{\mathfrak{A}}^{2}\\ &= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\|\mathfrak{G}^{\mathrm{k}}\tau\right\|_{\mathfrak{A}}^{2} = (-1)^{\imath}\beta_{\imath}(\mathfrak{G},\tau)_{\mathfrak{A}} \le 0.\end{aligned}$$

and hence, $\hat{\mathfrak{G}}$ is $\imath$-expansive operator. Now let $\mathcal{K}$ be the completion of the normed space $\boldsymbol{S}_{\mathcal{H}}/\ker\mathfrak{A}$ and let $\tilde{\mathfrak{G}}$ be the extension of $\hat{\mathfrak{G}}$ on the Hilbert space $\mathcal{K}$. Then define the operator

$$\begin{cases} Q:\boldsymbol{S}_{\mathcal{H}}\to\frac{\boldsymbol{S}_{\mathcal{H}}}{\ker(\mathfrak{A})},\\ Q\tau=\tau+\ker(\mathfrak{A}).\end{cases}$$

Since $\|Q\tau\|_{\mathfrak{A}} = \|\tau+\ker(\mathfrak{A})\|_{\mathfrak{A}} = \|\tau\|_{\mathfrak{A}}$, which implies that Q is continuous. Consider diagram,

$$\begin{array}{ccc} \boldsymbol{S}_{\mathcal{H}} & \xrightarrow{\mathfrak{G}} & \boldsymbol{S}_{\mathcal{H}}\\ {\scriptstyle Q}\big\downarrow & & \big\downarrow{\scriptstyle Q}\\ \boldsymbol{S}_{\mathcal{H}}/\ker(\mathfrak{A}) & \xrightarrow{\hat{\mathfrak{G}}} & \boldsymbol{S}_{\mathcal{H}}/\ker(\mathfrak{A})\\ {\scriptstyle I}\big\downarrow & & \big\downarrow{\scriptstyle I}\\ \mathcal{K} & \xrightarrow{\tilde{\mathfrak{G}}} & \mathcal{K}\end{array}$$

Thus, Q has dense range and $\tilde{\mathfrak{G}}\ o\ Q = Q\ o\ \mathfrak{G}$. □

Theorem 4.2 *Every $(\mathfrak{A},2)$-expansive operator is not supercyclic.*

Proof: Suppose that $\mathfrak{G}$ is an $(\mathfrak{A},2)$-expansive operator in $\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$. By last lemma there exist Banach space B_2 and 2-expansive operator $\tilde{\mathfrak{G}}$ on B_2 which is quasi-factor of $\mathfrak{G}$. The comparison principle [9] states that if $\mathfrak{G}$ is supercyclic then so is $\tilde{\mathfrak{G}}$. But the operator $\tilde{\mathfrak{G}}$, being an 2-expansive (concave operator) on a Hilbert space $\mathcal{K}$ cannot be supercyclic [21], which leads us to a contradiction. □

In general, the category of $\mathfrak{A}-\mathbb{EO}s$ is a strict subcaegory of $(\mathfrak{A},\imath)$-expansive [29]. The following results is obtained (see also [23, Page 83]).

Theorem 4.3 *Suppose that $\mathfrak{G}\in\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is an $(\mathfrak{A},2\imath-1)$-expansive and there is a sequence $(n_i)_i$ of positive integers so that $\sup_i\|\mathfrak{G}^{n_i}\|_{\mathfrak{A}}<\infty$. Then $\mathfrak{G}$ is an $\mathfrak{A}$-expansive.*

Proof: If $\imath=1$, the result is obvious, thus assume that $\imath>1$. For every τ in the closure of $\mathrm{R}(\mathfrak{A})$, which is denoted by $\overline{(\mathfrak{A})}$, and every non-negative integer n,

$$\|\mathfrak{G}^{n}\tau\|_{\mathfrak{A}}^{2} \ge \sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}}. \tag{4.2}$$

Let k_0 be the largest integer number such that $1\le\mathrm{k}_0\le\imath-1$ and $\beta_{\mathrm{k}}(\mathfrak{G},\tau)\neq 0$. Then taking for guaranteed that

$$\lim_{n\to\infty}\frac{n^{(\mathrm{k})}}{n^{(\mathrm{k}_0)}}=0, \qquad \mathrm{k}=1,2,\ldots,\mathrm{k}_0-1,$$

we see that

$$\lim_{n\to\infty}\|\mathfrak{G}^n\tau\|^2_{\mathfrak{A}} \geq \lim_{n\to\infty}\sum_{\mathrm{k}=0}^{\imath-1}\binom{n}{\mathrm{k}}\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}} = \lim_{n\to\infty}\tfrac{1}{\mathrm{k}!}\sum_{\mathrm{k}=0}^{\imath-1}n^{(\mathrm{k})}\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}}$$
$$= \lim_{n\to\infty}\tfrac{n^{(\mathrm{k}_0)}}{\mathrm{k}!}\sum_{\mathrm{k}=0}^{\imath-1}\tfrac{n^{(\mathrm{k})}}{n^{(\mathrm{k}_0)}}\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}} = +\infty.$$

On the other hand, since $\sup_i\|\mathfrak{G}^{n_i}\|_{\mathfrak{A}} < \infty$, there is a real number $\breve{M} > 0$ such that for each i and $\tau \in \overline{\mathrm{R}(\mathfrak{A})}$, $\|\mathfrak{G}^{n_i}\tau\|_{\mathfrak{A}} \leq \breve{M}\|\tau\|_{\mathfrak{A}}$, consequently $\lim_{n\to\infty}\|\mathfrak{G}^{n_i}\tau\|_{\mathfrak{A}}$ cannot be $+\infty$, which is a contradiction. Thus, $\beta_{\mathrm{k}}(\mathfrak{G},\tau)_{\mathfrak{A}} = 0$ for $\mathrm{k} = 1, 2, \cdots, \imath-1$. This coupled with (4.1) for $n = 1$ imply that

$$\|\mathfrak{G}\tau\|_{\mathfrak{A}} \geq \|\tau\|_{\mathfrak{A}}, \qquad \tau \in \overline{\mathrm{R}(\mathfrak{A})}.$$

Now, let τ be an arbitrary element in $\boldsymbol{S}_{\mathcal{H}}$ and be written as $\tau = \tau_1 + \tau_2$ for some $\tau_1 \in \ker(\mathfrak{A})^{1/2}$ and $\tau_2 \in \overline{\mathrm{R}(\mathfrak{A})}$. Taking into account that $\|\tau\|_{\mathfrak{A}} = \|\mathfrak{A}^{1/2}\tau\| = \|h_2\|_{\mathfrak{A}}$, and $\|\mathfrak{G}\tau\|_{\mathfrak{A}} = \|\mathfrak{G}\tau_2\|_{\mathfrak{A}}$, we conclude that $\|\mathfrak{G}\tau\|_{\mathfrak{A}} \geq \|\tau\|_{\mathfrak{A}}$, $\forall\tau \in \boldsymbol{S}_{\mathcal{H}}$. Thus, $\mathfrak{G}$ is an $\mathfrak{A}$-expansive. □

Corollary 4.1 *Every power bounded $(\mathfrak{A}, 2\imath-1)$-expansive operator on $\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$ is not supercyclic.*

Theorem 4.4 *If $\mathfrak{G}$ is a power bounded $(\mathfrak{A}, 2\imath)$-expansive in $\mathfrak{L}(\boldsymbol{S}_{\mathcal{H}})$, then $\mathfrak{G}$ is not supercyclic.*

Proof: Since $\beta_{2\imath}(\mathfrak{G})_{\mathfrak{A}} \leq 0$, by Theorem 3.2 we have $\beta_{2\imath-1}(\mathfrak{G})_{\mathfrak{A}} \geq 0$ and this shows that $\mathfrak{G}$ is $(2\imath-1, \mathfrak{A})$-expansive operator. So by the last corollary $\mathfrak{G}$ is not supercyclic. □

In sequence, we describe some spectral properties of an $(\mathfrak{A}, \imath)$-expansive operator.

Theorem 4.5 (A) *If $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive for some even $\imath$, then $\sigma_{ap}(\mathfrak{G}) \subseteq \partial\mathbb{D}$. Therefore either $\sigma(\mathfrak{G}) = \bar{\mathbb{D}}$ or $\sigma(\mathfrak{G}) \subseteq \partial\mathbb{D}$.* (B) *If $\mathfrak{G}$ is $(\mathfrak{A}, \imath)$-expansive for some odd $\imath$, then $\sigma_{ap}(\mathfrak{G}) \subseteq \{z : |z| \geq 1\}$. Moreover, if $\mathfrak{G}$ is non-invertible then $\bar{\mathbb{D}} \subseteq \sigma(\mathfrak{G})$ and if $\mathfrak{G}$ is invertible then $\sigma(\mathfrak{G}) \subseteq \{z : |z| \geq 1\}$.*

Proof: Let $\lambda \in \sigma_{ap}(\mathfrak{G})$ and $\{\tau_i\}_i$ be a sequence of unit vectors such that $\|(\mathfrak{G}-\lambda I)(\tau_i)\| \to 0$, as $i \to \infty$. It's clear that $\|\mathfrak{G}^k\tau_i\|^2 \to |\lambda|^{2\mathrm{k}}$ as $i \to \infty$. Therefore

$$\lim_{i\to\infty}\beta_\imath(\mathfrak{G},\tau)_{\mathfrak{A}} = \lim_{i\to\infty}\sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\|\mathfrak{G}^k\tau_i\|^2_{\mathfrak{A}} = \lim_{i\to\infty}\sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}\left\langle\mathfrak{A}\mathfrak{G}^{\mathrm{k}}\tau_i, \mathfrak{G}^{\mathrm{k}}\tau_i\right\rangle$$
$$= \sum_{\mathrm{k}=0}^{\imath}(-1)^{\imath-\mathrm{k}}\binom{\imath}{\mathrm{k}}|\lambda|^{2\mathrm{k}}\langle\mathfrak{A}\tau_i, \tau_i\rangle = (1-|\lambda|^2)^\imath\|\tau_i\|^2_{\mathfrak{A}} \leq 0.$$

So $|\lambda| = 1$. Therefore $\sigma_{ap}(\mathfrak{G}) \subseteq \partial\mathbb{D}$. On the other hand $\partial\sigma(\mathfrak{G}) \subseteq \sigma_{ap}(\mathfrak{G}) \subseteq \partial\mathbb{D}$ thus $\partial\sigma(\mathfrak{G}) \subseteq \partial\mathbb{D}$. So $\sigma(\mathfrak{G}) = \bar{\mathbb{D}}$ or $\sigma(\mathfrak{G}) \subseteq \partial\mathbb{D}$. Now we prove (B). Similarly, we can show that

$$(-1)^\imath\beta_i(\mathfrak{G},\tau)_{\mathfrak{A}} = -(1-|\lambda|^2)^\imath\|\tau_i\|^2_{\mathfrak{A}} \leq 0.$$

So $|\lambda| \geq 1$. Also if $\mathfrak{G}$ is not invertible then $0 \in \sigma(\mathfrak{G})$; but since $0 \notin \sigma_{ap}(\mathfrak{G})$, it is an interior point of $\sigma(\mathfrak{G})$. Let r be a largest positive number such that $\{z : |z| \leq r\} \subseteq \sigma(\mathfrak{G})$. Then there is a number z_0 such that $|z_0| = r$ and $z_0 \in \partial\sigma(\mathfrak{G}) \subseteq \sigma_{ap}(\mathfrak{G})$ [11, page 210]. This shows that $r \geq 1$, and consequently, $\bar{\mathbb{D}} \subseteq \sigma(\mathfrak{G})$. Now suppose that $\mathfrak{G}$ is invertible and let r be the largest positive number such that

$$\{z : |z| < r\} \subseteq \mathcal{C} - \sigma(\mathfrak{G}).$$

Since $\partial\sigma(\mathfrak{G}) \subseteq \sigma_{ap}(\mathfrak{G}) \subseteq \{z : |z| \geq 1\}, r \geq 1$. Therefore, $\sigma(\mathfrak{G}) \subseteq \{z : |z| \geq 1\}$. □

Corollary 4.2 *If $0 \notin \sigma_{ap}(\mathfrak{A})$ and both $\mathfrak{G}$, $\mathfrak{G}^*$ are $(\mathfrak{A}, \imath) - \mathbb{EO}s$, then $\sigma(\mathfrak{G}) \subset \partial\mathbb{D}$.*

Proof: We argue with contradiction. Suppose that $\sigma(\mathfrak{G}) \nsubseteq \partial\mathbb{D}$. Last theorem implies that

$$\sigma(\mathfrak{G}) \subset \Big\{z \,:\, \|z\| \geq 1\Big\},$$

if $\imath$ is odd, positive integer or $\sigma(\mathfrak{G}) = \overline{\mathbb{D}}$, if $\imath$ is even respectively. Since $0 \notin \sigma_{ap}(\mathfrak{G})$, so $\overline{\mathrm{R}(\mathfrak{G})} = \mathrm{R}(\mathfrak{G}) \neq \boldsymbol{S}_{\mathcal{H}}$, and therefore $N(\mathfrak{G}^*) \neq \{0\}$. This implies that $0 \in \sigma_p(\mathfrak{G}^*) \subset \sigma_{ap}(\mathfrak{G}^*)$, which contradicts the fact that $\mathfrak{G}^*$ is an $(\mathfrak{A}, \imath)$-expansive. □

Note that if $0 \notin \sigma_{ap}(A)$ and $\mathfrak{G}$ is a invertible $(\mathfrak{A}, \imath)$-expansive, then $r(\mathfrak{G}) = 1$.

Corollary 4.3 *If $\mathfrak{G}$ is a $(\mathfrak{A}, \imath)$-expansive operator, then $\mathfrak{G}$ is not compact.*

Proof: If $\mathfrak{G}$ is compact operator then $0 \in \sigma(\mathfrak{G})$. Thus, the last theorem implies that $\bar{\mathbb{D}} \subseteq \sigma(\mathfrak{G})$ which contradicts the most countable property of spectrum of compact operators. □

5. Conclusion

The concept of $(\mathfrak{A}, \imath)$-expansive operator on an infinite dimensional $\boldsymbol{S}_{\mathcal{H}}$ is introduced and some preliminaries and basic properties of such operators are given. We supplied some spectral properties of $(\mathfrak{A}, \imath)$-expansive operators. Finally, we are going to investigate the supercyclicity of $(\mathfrak{A}, \imath) - \mathbb{E}\mathbb{O}s$.

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