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The $\mathbb{L}_2$ structure of harmonizable random fields

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ABSTRACT: In this paper, we aim to study a large class of harmonizable random fields ($h.r.f.$, for short) when the indexing set is $\mathbb{Z}^d$, $d \geq 2$ which contains several classes of stochastic processes. So, we present an appropriate spectral theory approach in order to characterize certain subclass of $h.r.f.$ This class of processes is a natural generalization of the so-called wide sense stationary processes to random fields ($r.f.$) that have been investigated in many varieties of subjects and constitute an important class of nonstationary $r.f.$ Moreover, since many phenomena across space in various sciences and industries can be interpreted as realizations of random fields, which may be further assumed to satisfy some invariance properties such as isotropy. So, the definition of isotropy is given and some examples in $\mathbb{R}^d$ are addressed. In the end, necessary and sufficient conditions ensuring the stability of the linear transformation of a particular class are studied.

Key Words: Covariance function, Harmonizable random field, Isotropic covariance, Locally stationary, Spectral representation.

Contents

1. Introduction

Since the seminal works by Whittle [20], there have been significant attempts to extend the well-understood theory of unidimensionally indexed time series to classes of multidimensionally indexed (or spatial) one by many researchers. Such extensions are not limited to the second-order stationary time series, but also to nonstationary time series observed in many real datasets (interested readers are advised to see the monographs by Cressie [4] and Gaetan et al. [7]). A stupefying theoretical development in random fields was observed recently concerning the class of the harmonizable random fields, so, a readable discussion of their properties with many references is given in Chang et al. [2], and Rao ([13], [14]). This class has become an appealing tool for modeling and forecasting, for instance among others, the spatiotemporal and/or spatial processes (see Gaetan et al. [7]) and continue to gain a growing interest of researchers. This interest is due to it encompassing the following hierarchy of classes studied in the literature (see Dehay [6] , Rao [12] for more discussions and details)

$$\text{Stationary} \subset \text{strongly harmonizable} \subset \text{weakly harmonizable} \subset \text{Karhunen class} \subset \text{Cramér class}. \tag{1.1}$$

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All these inclusions are proper (see Section 2). Despite the multiple researches carried out by several authors on the characterizing properties of such subclasses which were proposed (see Rao [15]), in order to obtain more representativity of *h.r.f.*, we refer the reader to the monograph by Rao [14] and the references therein, the statistical and probabilistic properties still what to say especially for non-stationary classes. Additionally, it was always felt by authors working in this area that the completeness of the spectral domain will probably be lost when we goes from stationary to non-stationarity processes (see for example Cambanis [1] and Cramér [3]), this completeness problem stayed open for various decades. Recently, Rao [16] claimed that the spectral domain of any harmonizable process is complete. A short review of main previous researches are summarized in the next subsection.

1.1. Main literature review

The class of harmonizable processes has been initiated by Gramèr [3] which subject to the inclusions (1.1) and their isotropic versions has pointed out by Yaglom [21]. Isotropy is an invariance property under the transformations of rotations and reflections that have been well developed by among others, Yadrenko [22] and recently Rao [17]. This class has been successfully extended to *h.r.f.*, and has been widely employed to model various time series see Cressie [4] and Gaetan et al. [7]. Swift [18] who studied the local behavior of an isotropic *h.r.f.*, and gave the spectral representations for generalized and ordinary isotropic *h.r.f.* Moreover, in separate paper, Swift [19] has extended the classical spectral and covariance representations for stationary isotropic random fields to the harmonizable isotropic one. Rao [17] (chapter 4) has obtain integral representations of Cramér class random fields through the analysis of generalized random functions, using the Schwartz theory of such classes. Dehay et al. [6] introduce a family of strongly harmonizing operators which smooths every suitably weighted continuous random field on a locally compact Abelian (LCA) group G into a strongly harmonizable one. By means of these operators, Dehay et al. [6] prove that the class of strongly harmonizable random fields admitting spectral density whose support is compact and dense in the set of continuous random fields on G endowed with the compact convergence topology. Rao [15] gave several characterizations of *h.r.f.*, on an LCA group as well as for nonlocally compact groups. Also, the resulting concepts of strong and weak harmonizable isotropic fields are given.

In the present paper, we focused on series representation of *h.r.f.*, particularly interested in the orthogonal representation whose covariance functions are isotropic. These representations allow us to give certain characterizations of *h.r.f.* and their properties. So, this paper offers some extensive study of the properties of the *h.r.f.*, and our body of results expands what is known about the *h.r.f.*, both qualitatively and quantitatively. The main scope of our paper is articulated on the following methodology:

1.2. Motivations

In analysis of many spatio-temporal series for which stationarity is an unacceptable assumption, and for which it is of interest to maintain a convenient and easy-to-interpret spectral analysis (e.g., econometrics, signal processing, vibroacoustics, ...). Sometimes nonstationarity can be modeled by assuming that stationarity holds locally. This can be done, for example, using piecewise stationary or locally stationary processes. $ARMA$ models with periodic coefficients (Miamee [11]), and/or time-varying coefficients (Dahlhaus [5]), provide a very powerful and convenient parametric class flexible enough to describe many real data. Nevertheless, those models do not allow for capturing many complex dependencies that characterize some nonstationary spatial data such as dependence in the frequency domain without a harmonizability assumption. The main motivations for studying the subclasses of *h.r.f.*, appear in several reasons. First, it is raised in the diagram given by the inclusion (1.1) showing however, that *h.r.f.*, constitutes certain form a wide class of nonstationary processes, which admit a convenient Fourier transform and have spectral distributions characterized by correlated components. Their analysis is linked with some algebraic theory concept. On the other hand, they are proved to be useful in many fields of application, for instance, it may be applied in the analysis of electroencephalogram (EEG) in several locations for studying the brain connectivity. Second, for the second-order stationary *h.r.f.*, the correlation between two locations simply depends on the usual Euclidean distance and is invariant under rotation. Third, an integral representation is well established for the continuous isotropic positive definite function.

1.3. Contribution

The structural analysis here is thus to study certain discrete time *h.r.f.*, according to their Fourier integral (resp. series) representation. So, we are extending the results of the concept spectral analysis of unidimensional processes to a multidimensional one preserving the potential use of Fourier analysis techniques. Various generalizations in that direction have been presented, among others, the classes of stationary and non-stationary harmonizable random fields as well as their isotropic versions are studied. Moreover, this line of investigation is pursued, and some new series representation of non-stationary *h.r.f.*, are introduced and then characterized. Numerous illustrative examples aimed to clarify the expositions are addressed.

1.4. Notations and conventions

Before we proceed, let us introduce some notations and definitions. Throughout the paper, let $\mathbb{L}_2 = \mathbb{L}_2(\Omega, \Im, P)$ be the Hilbert space of equivalence classes of complex random variables with zero mean and finite mean square defined on the probability space $(\Omega, \Im, P)$ and $\mathfrak{L}_1 = \left\{ a \in \mathbb{R}^{\mathbb{Z}^d} : \sum\limits_{\mathbf{n}\in\mathbb{Z}^d} |a_{\mathbf{n}}| < +\infty \right\}$. We consider a second order *r.f.*, $X = \left(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\right) \in \mathbb{L}_2$, $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$ where d is a positive integer $\underline{\pi} = [-\pi, \pi]$, $\Pi = \underbrace{\underline{\pi} \times ... \times \underline{\pi}}_{d-times} = \underline{\pi}^d$, $\mathcal{B}_\Pi$ being the Borelian $\sigma-$field of Π, Let $\mathbb{L}_2(X) = \sigma\left(X_{\mathbf{t}}, \mathbf{t} \in \mathbb{Z}^d\right)$ denote the span of the process $X \in \mathbb{L}_2$. $\mathbf{a}.\mathbf{b}$ is the usual inner product of vectors $\mathbf{a} = (a_1, ..., a_d)$ and $\mathbf{b} = (b_1, ..., b_d)$ and $\mathbf{a} \preceq \mathbf{b}$ means that $a_i \leq b_i$, $i = 1, ..., d$. For any $d-$dimensional vector $T = (T_1, ..., T_d)$, we shall set $\|T\|$ the Euclidean distance and $|T|^r = \prod_{i=1}^d T_i^r$, for any positive real r. The notations $T \to \infty$ mean that $T_j \to \infty, j = 1, ..., d$, $\lambda \neq \mu[2\pi]$ where $\lambda = (\lambda_1, ..., \lambda_d), \mu = (\mu_1, ..., \mu_d)$ means that $\lambda_j \neq \mu_j[2\pi]$. In the end, the asterisk $(*)$ denotes the conjugate.

1.5. Content

A brief synopsis of the paper is as follows: In Section 2, a short description of the spectral representation of some subclasses of *h.r.f.*, and their isotropic version is presented, and some examples and remarks are given. In Section 3, the notion of series (resp. integral) representation of *h.r.f.*, is introduced and its spectrum is characterized. In particular, the linear series of *h.r.f*, is characterized and necessary and sufficient conditions for strong harmonizability and the existence of the spectral density are given. Section 4, is aimed to highlight our theory through some examples and properties characterizing these examples. In the end, Section 5, concluding remarks and some discussions on certain future research perspectives conclude the paper.

2. Background of harmonizable isotropic random fields

A general nonstationary class of $\mathbb{L}_2$-valued random fields that extend the main Cramér class (hence also of Karhunen's) for further study and applications, calling it a weak class as it automatically includes the weakly *r.f.*, class is given in the following definition

Definition 2.1 *A r.f., $X = (X_{\mathbf{t}})_{t\in\mathbb{Z}^d} \subset \mathbb{L}_2$ is termed weak class of Cramér r.f., relatively to an integrable family $(g_{\mathbf{t}})_{\mathbf{t}\in\mathbb{Z}^d}$ if there exists a stochastic measure Z_X: $\mathcal{B}_\Pi \to \mathbb{L}_2$, ($Z_X$ does not necessary possess orthogonal increments) such that*

$$\forall \mathbf{t} \in \mathbb{Z}^d : X_{\mathbf{t}} = \int\limits_{\Pi} g_{\mathbf{t}}(\lambda) dZ_X(\lambda),$$

and its covariance function of centered r.f., is $r_X(\mathbf{s}, \mathbf{t}) = \int_{\Pi\times\Pi} g_{\mathbf{t}}(\lambda)\, g_{\mathbf{s}}^*(\mu)\, d^2F_X(\lambda, \mu)$ where F_X is a bimeasure associated with Z_X defined on $\mathcal{B}_\Pi \times \mathcal{B}_\Pi$ by $F_X(A, B) = E\{Z_X(A) Z_X^*(B)\}$ and $\sup\{\sum\limits_{\mathbf{i},\mathbf{j}\in\mathbb{Z}^d} a_{\mathbf{i}} \overline{a_{\mathbf{j}}} \left|F(A_{\mathbf{i}}, A_{\mathbf{j}})\right|, A_{\mathbf{j}} \cap A_{\mathbf{i}} = \emptyset, A_{\mathbf{j}} \in \mathcal{B}_\Pi, |a_{\mathbf{j}}| \leq 1\} < +\infty$. *In particular, when $g_{\mathbf{t}}(\lambda) = e^{i\lambda.\mathbf{t}}$, the* *resulting one is strongly harmonizable r.f. and X is represented as*

$$\forall \mathbf{t} \in \mathbb{Z}^d : X_{\mathbf{t}} = \int\limits_{\Pi} e^{i\lambda.\mathbf{t}} dZ_X(\lambda). \tag{2.1}$$

Moreover, it is called Karhunen r.f., whenever Z_X is orthogonally scattered and hence its covariance reduces to $r_X(\mathbf{s},\mathbf{t}) = \int_{\Pi} g_{\mathbf{t}}(\lambda) g_{\mathbf{s}}^*(\lambda) dF_X(\lambda)$, additionally, if $g_{\mathbf{t}}(\lambda) = e^{i\lambda.\mathbf{t}}$, the resulting one is the classical (wide sense) stationary r.f.

An extensive discussion of the properties of such a class with many recent references is given in Chang and Rao [2]. Note here, that the description of the subclasses of the inclusion (1.1) may be obtained according to the definition 2.1.

Remark 2.1 *For a strong h.r.f., the bimeasure $F_X(.,.)$ can be expressed in terms of the covariance function. Indeed, from the inversion formula*

$$F_X(I, I') = \frac{1}{(2\pi)^{2d}} \lim_{\substack{\mathbf{m}\to+\infty \\ \mathbf{n}\to+\infty}} \sum_{-\mathbf{m}\leq\mathbf{s}\leq\mathbf{m}} \sum_{-\mathbf{n}\leq\mathbf{t}\leq\mathbf{n}} r_X(\mathbf{s},\mathbf{t}) \int_I \exp\{-i\mathbf{x}.\mathbf{s}\} d\mathbf{x} \int_{I'} \exp\{-i\mathbf{y}.\mathbf{t}\} d\mathbf{y} \tag{2.2}$$

where I and I' are continuity intervals for $F_X(.,.)$ with $I = \prod\limits_{i=1}^{d}]\lambda_i, \mu_i[$ and $I' = \prod\limits_{i=1}^{d}]\lambda_i', \mu_i'[$ with $F_X(\pm\lambda, \pm\mu) = F_X(\lambda,\mu)$ and $F_X(\pm\lambda', \pm\mu') = F_X(\lambda', \mu')$ (see Rao [17] pages 72-73).

Some interesting subclasses linked with *r.f.* having a covariance function $r_X(\mathbf{s},\mathbf{t})$ are subjected to the following definition

Definition 2.2 *Consider a r.f., $X \subset \mathbb{L}_2$ having a covariance function $r_X(\mathbf{s},\mathbf{t})$, then the X is called:*

1. *Locally harmonizable: if there exists a positive function $r_1(.)$ and a harmonizable covariance $r_2(.,.)$ such that $r_X(\mathbf{s},\mathbf{t}) = r_1(\frac{\mathbf{s}+\mathbf{t}}{2}) r_2(\mathbf{s},\mathbf{t})$. Moreover, the covariance $r_X(\mathbf{s},\mathbf{t})$ is called locally stationary if $r_2(.,.)$ is a stationary covariance function i.e., $r_2(\mathbf{s},\mathbf{t}) = r_2(\mathbf{t}-\mathbf{s})$.*

2. *Splitting r.f. if $r_X(\mathbf{s},\mathbf{t}) = r(\mathbf{s})\overline{r}(\mathbf{t})$ for some real function r. A study of Mehlman's [10] paper is helpful here.*

3. *Isotropic: if $r_X(\mathbf{s},\mathbf{t})$, admits the representation*

   $$r_X(\mathbf{s},\mathbf{t}) = 2^{\upsilon}\Gamma\left(\frac{d}{2}\right) \int_{\underline{\pi}\times\underline{\pi}} \frac{J_{\upsilon}\left(\|\lambda.\mathbf{s} - \mu.\mathbf{t}\|\right)}{\|\lambda.\mathbf{s} - \mu.\mathbf{t}\|^{\upsilon}} d^2F_X(\lambda,\mu),\ \mathbf{s},\mathbf{t} \in\mathbb{Z}^d$$

   where $\upsilon = \frac{d-2}{2}$, $\Gamma(.)$ is the gamma function, and $J_{\upsilon}(.)$ is the Bessel function of the first kind with series representation i.e., $J_{\upsilon}(x) = \sum\limits_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\upsilon+1)} \left(\frac{x}{2}\right)^{2m+\upsilon}$, $x \in \mathbb{R}$ and $\upsilon > -1$ (see Swift [18] for more discussion). The above representation reduce to

   $$r_X(\mathbf{h}) = r_X(\|\mathbf{h}\|) = 2^{\upsilon}\Gamma\left(\frac{d}{2}\right) \int_{\underline{\pi}} \frac{J_{\upsilon}(\lambda\|\mathbf{h}\|)}{(\lambda\|\mathbf{h}\|)^{\upsilon}} dF_X(\lambda)$$

   whenever the h.r.f. X is second-order stationary where $\mathbf{h} = \mathbf{s} - \mathbf{t}$. So, the above formula reduces to (up to a multiplicative constant) $r_X(\mathbf{h}) = \int_{\underline{\pi}} J_0(\lambda\|\mathbf{h}\|) dF_X(\lambda)$ when $d = 2$ and to $r_X(\mathbf{h}) = \int_{\underline{\pi}} \frac{\sin(\lambda\|\mathbf{h}\|)}{\lambda\|\mathbf{h}\|} dF_X(\lambda)$ when $d = 3$.

Note here that if $\|\mathbf{s}\|^{d-1} r_X(\mathbf{s})$ is absolutely summable on $\mathbb{Z}^d$, then when $dF_X(\lambda)$ is absolutely continuous, and possesses a spectral density function given by

$$f(\lambda) = \frac{\lambda^{d/2}}{2^{\upsilon}\Gamma\left(\frac{d}{2}\right)} \sum_{\mathbf{s}\in\mathbb{Z}^d} \|\mathbf{s}\|^{d/2} J_{\upsilon}(\lambda\|\mathbf{s}\|) r_X(\mathbf{s}). \tag{2.3}$$

The formula (2.3) allows us the construction of the spectral density of the isotropic second-order stationary *h.r.f.* X, given its covariance function. A list of examples of isotropic covariance functions in $\mathbb{R}^d$ was given by Leonenko [9] and Yaglom [21]. The following table contains some examples of isotropic covariance functions in $\mathbb{R}^d$ in first column followed by the isotropic spectral density function in the second column is evaluated via the formula (2.3)

$r_X(\mathbf{s})$	$f_X(\lambda)$	range
$exp\left\{-\alpha\left\|\mathbf{s}\right\|\right\}$	$2\alpha\frac{\Gamma\left(\frac{d+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}\frac{\lambda^{d-1}}{(\alpha^2+\lambda^2)^{(d+1)/2}}$	$\alpha>0$
$exp\left\{-\alpha\left\|\mathbf{s}\right\|^2\right\}$	$\frac{1}{2^{d-1}a^{d/2}\Gamma\left(\frac{d}{2}\right)}\lambda^{d-1}\exp\left\{-\frac{\lambda^2}{4\alpha}\right\}$	$\alpha>0$
$(1+\theta\alpha\left\|\mathbf{s}\right\|)\,exp\left\{-\alpha\left\|\mathbf{s}\right\|\right\}$	$2\alpha\frac{\Gamma\left(\frac{d+1}{2}\right)\left((1+\theta d)\alpha^2+(1-\theta)\lambda^2\right)}{\sqrt{\pi}\Gamma\left(\frac{d}{2}\right)}\frac{\lambda^{d-1}}{(\alpha^2+\lambda^2)^{(d+3)/2}}$	$\begin{cases}\alpha>0\\ -\frac{1}{d}\leq\theta\leq 1\end{cases}$
$\left(1-\theta\alpha\left\|\mathbf{s}\right\|^2\right)exp\left\{-\alpha\left\|\mathbf{s}\right\|^2\right\}$	$2\sqrt{\pi}\frac{\left(4\alpha-2d\alpha\theta+\theta\lambda^2\right)}{\Gamma\left(\frac{d}{2}\right)\alpha^{(d-3)/2}}\lambda^{d-1}\exp\left\{-\frac{\lambda^2}{4\alpha}\right\}$	$\begin{cases}\alpha>0\\ 0\leq\theta\leq\frac{2}{d}\end{cases}$
$\frac{\left(1+\sqrt{(1+\alpha\left\|\mathbf{s}\right\|^2)}\right)^{-\upsilon}}{\sqrt{1+\alpha\left\|\mathbf{s}\right\|^2}}$	$\frac{1}{a^{d/4}\Gamma\left(\frac{d}{2}\right)}\lambda^{d/2-1}\exp\left\{-\frac{\lambda}{\sqrt{\alpha}}\right\}$	$\alpha>0$

Table (1): Some isotropic covariance functions in $\mathbb{R}^d$ and their spectral density functions

Remark 2.2 *If the h.r.f.* X *is locally stationary, then the spectral density* $f_X(\lambda,\mu)$ *(when exists) can be decomposed as* $f_X(\lambda,\mu)=f_1(\frac{\lambda+\mu}{2})f_2(\lambda-\mu)$ *with* $f_1\geq 0$ *and* f_2 *is a spectral density. Conversely, If the spectral density of an h.r.f.* X *maybe decomposed as* $f_X(\lambda,\mu)=f_1(\frac{\lambda+\mu}{2})f_2(\lambda-\mu)$*, so, the covariance can be rewritten as* $r_X(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{s}.\lambda-i\mathbf{t}.\mu}f_1(\frac{\lambda+\mu}{2})f_2(\lambda-\mu)d\lambda d\mu$*, then using the transformation* $\mathbf{u}=\frac{\lambda+\mu}{2}$ *and* $\upsilon=\lambda-\mu$*, we obtain* $r_X(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{u}.(\mathbf{s}-\mathbf{t})}e^{i\upsilon.\frac{1}{2}(\mathbf{s}+\mathbf{t})}f_1(\mathbf{u})f_2(\upsilon)d\mathbf{u}d\upsilon=r_1(\frac{\mathbf{s}+\mathbf{t}}{2})r_2(\mathbf{s}-\mathbf{t})$ *and hence* X *is locally stationary r.f.*

Remark 2.3 *From the inversion formula (2.2), it can be shown that an h.r.f.* X *is splitting iff* $r_X(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{s}.\lambda-i\mathbf{t}.\mu}\,dF_X(\lambda)dF_X^*(\mu)=r(\mathbf{s})r(\mathbf{t})$ *where* r *is a Fourier transform of Borel measure* dF_X *and hence* X *is a strongly harmonizable r.f.*

Remark 2.4 *The fundamental properties of an isotropic stationary covariance function in* $\mathbb{R}^d$ *are*

1. *If* $r_X(\mathbf{s})$ *and* $r_Y(\mathbf{s})$ *are isotropic stationary covariance functions in* $\mathbb{R}^d$*, so is their convex combination* $\lambda r_X(\mathbf{s})+(1-\lambda)r_Y(\mathbf{s})$*,* $\mathbf{s}\in\mathbb{R}^d$ *where* $\lambda\in]0,1[$
2. *If* $r_X(\mathbf{s})$ *and* $r_Y(\mathbf{s})$ *are isotropic stationary covariance functions in* $\mathbb{R}^d$*, so is their product* $r_X(\mathbf{s}).r_Y(\mathbf{s})$*,* $\mathbf{s}\in\mathbb{R}^d$*.*
3. *If* $\left(r_{X_i}(\mathbf{s})\right)_{1\leq i\leq n}$ *is a sequence of isotropic stationary covariance functions in* $\mathbb{R}^d$*, so is their limit* $\lim\limits_{n\to\infty}r_{X_n}(\mathbf{s})$ *provided that the limit exists.*
4. *An isotropic stationary covariance function in* $\mathbb{R}^{d1}$*, is also an isotropic stationary covariance function in* $\mathbb{R}^{d2}$*, if* $d1>d2$ *where* $d1,d2\in\mathbb{N}$*.*

3. Series representation of *h.r.f.*

In this section, we study a competitor representation called series expansions of random fields which are useful in various areas of time series theory, and we provide some insight into the structure of *r.f.*, X. Quite generally, we write

$$X_{\mathbf{t}}=\sum_{\mathbf{n}\in\mathbb{Z}^d}a_{\mathbf{n}}(\mathbf{t})\xi_{\mathbf{n}} \tag{3.1}$$

where $(\xi_{\mathbf{n}})_{\mathbf{n}}$ is some weak white noise, and the convergence is usually taken to be in the stochastic mean. So, various constraints may be imposed on $X_{\mathbf{t}}$, $\xi_{\mathbf{t}}$ and on the coefficients $a_{\mathbf{n}}(\mathbf{t})$. In the Karhunen class, the $(\xi_{\mathbf{n}})_{\mathbf{n}}$ is a set of square integrable and orthogonal random fields and the coefficients $a_{\mathbf{n}}(\mathbf{t})$ are absolutely summable. More precisely, we have

Proposition 3.1 *Let X be a r.f. Then the following are equivalent.*

1. *X is a splitting harmonizable r.f. such that $r_X(\mathbf{s},\mathbf{t})=r(\mathbf{s})r(\mathbf{t})$ where $r(.)$ is the Fourier transform of the some measure.*

2. *$\forall \mathbf{t}\in\mathbb{Z}^d$, almost surely (a.s.), $X_\mathbf{t}=r(\mathbf{t})W$ where W is a centred random variable with $Var(W)=1$.*

3. *$\forall \mathbf{t}\in\mathbb{Z}^d: X_\mathbf{t}=\int\limits_\Pi e^{i\mathbf{t}.\lambda}dZ_X(\lambda)$ where $Z_X(.)=r(.)W$*

Proof: Our proof is cyclical, $(1\Rightarrow 2)$ Define the function $g(.)$ such that $g(\mathbf{s})r(\mathbf{s})=1$ and set $W_\mathbf{t}=g(\mathbf{t})X_\mathbf{t}$. Then for all $t,s\in\mathbb{Z}^d$, we have $E\left\{W_\mathbf{t}W_\mathbf{s}\right\}=1$ and $E\left\{(W_\mathbf{t}-W_\mathbf{s})^2\right\}=0$ which implies that $W_\mathbf{t}=W_\mathbf{s}=W$ a.s. So $X_\mathbf{t}=r(\mathbf{t})W$ a.s. $(2\Rightarrow 3)$ We have $\int\limits_\Pi e^{i\mathbf{t}.\lambda}dZ_X(\lambda)=X_\mathbf{t}=r(\mathbf{t})W=\int\limits_\Pi e^{i\mathbf{t}.\lambda}dF(\lambda)W$, hence $Z_X(.)=F(.)W$ *a.s.* $(3\Rightarrow 1)$ It is not difficult to see that $r_X(\mathbf{s},\mathbf{t})=E\left\{X_\mathbf{s}\overline{X}_\mathbf{t}\right\}=E\left\{\int\limits_\Pi e^{i\mathbf{s}.\lambda}dF(\lambda)W\overline{\int\limits_\Pi e^{i\mathbf{t}.\mu}dF(\mu)W}\right\}=r(\mathbf{s})r(\mathbf{t})$. □

Remark 3.1 *The above proposition shows that the r.f. X is splitting harmonizable such that $r_X(\mathbf{s},\mathbf{t})=r(\mathbf{s})r(\mathbf{t})$ iff $X_\mathbf{t}=r(\mathbf{t})W$ a.s. with $Var(W)=1$.*

An important property of locally harmonizable (resp. stationary) *r.f.* is given in the following proposition

Proposition 3.2

1. *The product XY of two locally harmonizable r.f. X and Y is too locally harmonizable r.f.*

2. *Finite linear combinations of splitting harmonizables r.f. are splitting harmonizable r.f.*

Proof:

1. Let $r_X(\mathbf{s},\mathbf{t})$ and $r_Y(\mathbf{s},\mathbf{t})$ be the covariance functions of X and Y, then by Fubini theorem we have $r_{XY}(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{s}.\lambda-i\mathbf{t}.\mu}d^2G_{XY}(\lambda,\mu)$ where $d^2G_{XY}(\lambda,\mu)=\int_{\Pi\times\Pi}d^2F_X(\lambda-\lambda',\mu-\mu')\,d^2F_Y(\lambda',\mu')$.

2. Let $y_\mathbf{t}=\sum\limits_{j=1}^n a_iX_\mathbf{t}^{(i)}$ where $X_\mathbf{t}^{(i)}$ are splitting harmonizable *r.f.* By proposition 3.1, the results follows.

□

A series representation for general harmonizable *r.f.* is provided by the following:

Proposition 3.3 [*Moving average representation*] *Let X be a r.f., then X has in mean-square sense a moving average (MA for short) representation (3.1) in which where $a_\mathbf{n}(.)\in\mathfrak{L}_1$ and the innovation field $\xi=\left(\xi_\mathbf{t},\mathbf{t}\in\mathbb{Z}^d\right)$ is such that $r_\xi(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{s}.\lambda-i\mathbf{t}.\mu}d^2F_\xi(\lambda,\mu)$ iff X is harmonizable with covariance function*

$$r_X(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{s}.\lambda-i\mathbf{t}.\mu}A_\mathbf{t}(\lambda)\overline{A}_\mathbf{s}(\mu)\,d^2F_\xi(\lambda,\mu) \tag{3.2}$$

where $A_\mathbf{t}(\lambda)=\sum_{\mathbf{n}\in\mathbb{Z}^d}a_\mathbf{n}(\mathbf{t})e^{i\mathbf{n}.\lambda}$, and thus X has spectral representation $X_\mathbf{t}=\int\limits_\Pi e^{i\mathbf{t}.\lambda}A_\mathbf{t}(\lambda)\,dZ_\xi(\lambda)$.

Proof: $(\Rightarrow)$ For any fixed $\mathbf{t} \in \mathbb{Z}^d$

$$\begin{aligned} X_{\mathbf{t}} &= \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}}(\mathbf{t})\xi_{\mathbf{n}} = \sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}}(\mathbf{t}) \int\limits_{\Pi} e^{i\mathbf{n}.\lambda} dZ_\xi(\lambda) \\ &= \lim_{\mathbf{N}\to\infty} \sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{n}=\mathbf{N}} a_{\mathbf{n}}(\mathbf{t}) \int\limits_\Pi e^{i\mathbf{n}.\lambda} dZ_\xi(\lambda) = \lim_{\mathbf{N}\to\infty}\int\limits_\Pi \left\{\sum_{\mathbf{n}=-\mathbf{N}}^{\mathbf{n}=\mathbf{N}} a_{\mathbf{n}}(\mathbf{t})e^{i\mathbf{n}.\lambda}\right\} dZ_\xi(\lambda) \\ &= \int\limits_\Pi e^{i\mathbf{t}.\lambda}\left\{\sum_{\mathbf{n}\in\mathbb{Z}^d} a_{\mathbf{n}}(\mathbf{t}) e^{i\mathbf{n}.\lambda}\right\} dZ_\xi(\lambda) = \int\limits_\Pi e^{i\mathbf{t}.\lambda} A_{\mathbf{t}}(\lambda)\, dZ_\xi(\lambda) \end{aligned}$$

The covariance of $X_{\mathbf{t}}$ is obtained from their spectral representation. $(\Leftarrow)$ Conversely, it is easily seen. □

Corollary 3.1 *If a r.f. X has a stationary (resp. Strong harmonizable, weakly stationary) MA representation* (3.1)*, then it is a stationary (resp. Strong harmonizable, weakly stationary) r.f. with covariance function* (3.2)*. Conversely, if* (3.1) *is a MA representation and X is a strongly harmonizable (resp. stationary) r.f., then MA representation is strongly harmonizable (resp. stationary).*

Proof: If the $r.f.$ X has a MA representation (3.1) with $\xi_{\mathbf{n}} = \int\limits_\Pi e^{i\mathbf{n}.\lambda} dZ_\xi(\lambda)$, then it is not difficult the see that $r_X(\mathbf{s},\mathbf{t}) = \int_{\Pi\times\Pi} e^{i\mathbf{s}.\lambda - i\mathbf{t}.\mu} A_{\mathbf{t}}(\lambda)\, \overline{A}_{\mathbf{s}}(\mu)\, d^2F_\xi(\lambda,\mu)$ where $F_\xi(.,.)$ is the bimeasure of ξ, so, X is harmonizable $r.f.$ Hence (3.1) is strongly harmonizable (stationary)$\Leftrightarrow$ ξ is strongly harmonizable (stationary) $r.f.$ $\Leftrightarrow$ $F_\xi(.,.)$ is a measure concentrated on the diagonal of $\Pi\times\Pi$ $\Leftrightarrow$ The bimeasure $A_{\mathbf{t}}(\lambda)\, \overline{A}_{\mathbf{s}}(\mu)\, F_\xi(\lambda,\mu)$ is a measure concentrated on the diagonal of $\Pi\times\Pi$ $\Leftrightarrow$ X is strongly harmonizable (stationary) $r.f.$ □

Proposition 3.4 [*Stationary case*] *A stationary h.r.f. X has an orthonormal MA representation* (3.1) *iff its covariance function is given by $r_X(\mathbf{s}) = \int_\Pi e^{i\mathbf{s}.\lambda} f(\lambda) d\lambda$ where $f(\lambda)$ is the spectral density.*

Proof: Assume that X has a MA representation (3.1) with an orthonormal $r.f.$ ξ. Then $r_\xi(\mathbf{s},\mathbf{t}) = \int_\Pi e^{i(\mathbf{s}.-\mathbf{t}).\lambda} d\lambda$ and hence $E\{X_s X_t\} = \int_\Pi e^{i(\mathbf{s}.-\mathbf{t}).\lambda} |A(\lambda)|^2 d\lambda$ and the results follows by taking $f(\lambda) = |A(\lambda)|^2$. Conversely, let $A(\lambda)$ be a function on Π such that $f(\lambda) = |A(\lambda)|^2$. Furthermore, let $\mathbf{D} = \{(\lambda,\mu) \in \pi\times\pi : \lambda = \mu\}$ and let $d^2\mu_\xi(\lambda,\mu) = I_{\mathbf{D}}(\lambda,\mu)\, d\lambda$. Then with these notations $r_X(\mathbf{s},\mathbf{t}) = \int_{\Pi\times\Pi} e^{i\mathbf{s}.\lambda - i\mathbf{t}.\mu} A(\lambda)\, \overline{A}(\mu)\, d^2\mu_\xi(\lambda,\mu)$ and by the proposition 3.3 we get a strongly harmonizable MA representation (3.1) of $r.f.$ X. Since $r_\xi(\mathbf{s},\mathbf{t}) = \int_{\Pi\times\Pi} e^{i\mathbf{s}.\lambda - i\mathbf{t}.\mu} d^2\mu_\xi(\lambda,\mu) = \int_\Pi e^{i(\mathbf{s}.-\mathbf{t}).\lambda} d\lambda$ then ξ is an orthonormal $r.f.$ and thus X has an orthonormal MA. □

Proposition 3.5 *Assume that the harmonizable r.f. X is locally stationary and having a spectral representation* (2.1)*. Then for any $\varphi \in \mathbb{L}_2$, the r.f. Y defined by $\forall \mathbf{t} \in \mathbb{Z}^d : Y_{\mathbf{t}} = \int\limits_\Pi e^{i\mathbf{t}.\lambda}\varphi(\lambda) dZ_X(\lambda)$ is locally stationary iff there exist functions f and g such that*

1. $\varphi(\mu + \frac{\lambda}{2})\overline{\varphi}(\mu - \frac{\lambda}{2}) = f(\mu)g(\lambda)$
2. $\int\limits_\Pi e^{i\mathbf{t}.\lambda} f(\lambda) dF_1(\lambda)$ *is a stationary covariance function*
3. $\int\limits_\Pi e^{i\mathbf{s}.\lambda} g(\lambda) dF_2(\lambda) \geq 0$ *for all* $\mathbf{s} \in \mathbb{Z}^d$,

where F_1 and F_2 are as in remark 2.2.

Proof: Suppose that $r.f.$ Y is locally stationary, then following the remark 2.2, we have

$$r_Y(\mathbf{s},\mathbf{t})=\int_{\Pi\times\Pi}e^{i\mathbf{u}.(\mathbf{s}-\mathbf{t})}e^{i\upsilon.\frac{1}{2}(\mathbf{s}+\mathbf{t})}\varphi\left(\mathbf{u}+\frac{\upsilon}{2}\right)\varphi\left(\mathbf{u}-\frac{\upsilon}{2}\right)dF_1(\mathbf{u})dF_2(\upsilon)$$

where F_1 is the a probability distribution and $F_2\geq 0$. However $r_Y(\mathbf{s},\mathbf{t})=r_1(\mathbf{s})\,r_2(\mathbf{0})$ and $r_Y\left(\frac{\mathbf{t}}{2},-\frac{\mathbf{t}}{2}\right)=r_2(\mathbf{t})\,r_1(\mathbf{0})$, so

$$\begin{aligned}r_2(\mathbf{0})\,r_1(\mathbf{s})&=\int_{\Pi\times\Pi}e^{i\upsilon.\mathbf{s}}\varphi\left(\mathbf{u}+\frac{\upsilon}{2}\right)\varphi\left(\mathbf{u}-\frac{\upsilon}{2}\right)dF_1(\mathbf{u})dF_2(\upsilon);\\ r_1(\mathbf{0})\,r_2(\mathbf{t})&=\int_{\Pi\times\Pi}e^{i\mathbf{u}.\mathbf{t}}\varphi\left(\mathbf{u}+\frac{\upsilon}{2}\right)\varphi\left(\mathbf{u}-\frac{\upsilon}{2}\right)dF_1(\mathbf{u})dF_2(\upsilon).\end{aligned}$$

After some tedious computation we obtain

$$\begin{aligned}\varphi\left(\mathbf{x}+\frac{\mathbf{y}}{2}\right)\varphi\left(\mathbf{x}-\frac{\mathbf{y}}{2}\right)&=\frac{1}{r_Y(\mathbf{0},\mathbf{0})}\int_{\Pi}\varphi\left(\mathbf{x}+\frac{\upsilon}{2}\right)\varphi\left(\mathbf{x}-\frac{\upsilon}{2}\right)dF_2(\upsilon)\times\int_{\Pi}\varphi\left(\mathbf{u}+\frac{\mathbf{y}}{2}\right)\varphi\left(\mathbf{u}-\frac{\mathbf{y}}{2}\right)dF_1(\mathbf{u})\\&=f(\mathbf{x})g(\mathbf{y})\end{aligned}$$

and hence $r_Y(\mathbf{s},\mathbf{t})=r_1\left(\frac{\mathbf{s}+\mathbf{t}}{2}\right)r_2(\mathbf{s}-\mathbf{t})$. □

Proposition 3.6 *If Z_X is orthogonal, then under the condition of Proposition 3.3, the innovation field ξ and the coefficients $a_{\mathbf{n}}(\mathbf{t})$ maybe given by*

$$\xi_{\mathbf{t}}=\int\limits_{\Pi}f_{\mathbf{t}}(\lambda)\,dZ_{\xi}(\lambda)\ \text{ and } a_{\mathbf{n}}(\mathbf{t})=\int\limits_{\Pi}e^{i\mathbf{t}.\mu}f_{\mathbf{n}}(\lambda)\,d^2F(\mu,\lambda)$$

where $(f_{\mathbf{n}}(.))_{\mathbf{n}\in\mathbb{Z}^d}$ is an orthonormal and complete sequence in the set of all complex valued, $\mathcal{B}_{\Pi}-$measurable function f such that $\int\limits_{\Pi\times\Pi}f(\lambda)\overline{f}(\mu)d^2F(\lambda,\mu)<+\infty$ denoted $\Lambda_2(F)$.

Proof: The $(f_{\mathbf{t}}(\lambda))$ be an orthonormal basis in $\Lambda_2(F)$. If for each $\mathbf{t}$, $\xi_{\mathbf{t}}$ is the element of $\mathbb{L}_2(X)$ corresponding to $f_{\mathbf{t}}(\lambda)\in\Lambda_2(F)$, then by the isomorphism and the fact that $\mathbb{L}_2(X)=\mathbb{L}_2(Z)$, we have $\xi_{\mathbf{t}}=\int\limits_{\Pi}f_{\mathbf{t}}(\lambda)\,dZ(\lambda)$. Hence, in the mean square sense for all $\mathbf{t}\in\mathbb{Z}^d$, $X_{\mathbf{t}}=\sum_{\mathbf{n}\in\mathbb{Z}^d}a_{\mathbf{n}}(\mathbf{t})\xi_{\mathbf{n}}$ and $a_{\mathbf{n}}(\mathbf{t})=E\left\{X_{\mathbf{t}}\xi_{\mathbf{n}}^*\right\}$, or also, $a_{\mathbf{n}}(\mathbf{t})=\int\limits_{\Pi}\int\limits_{\Pi}\exp\left\{i\mathbf{t}.\lambda\right\}f_{\mathbf{t}}(\mu)\,d^2F(\lambda,\mu)$. □

Remark 3.2 *As a consequence of the proposition 3.6, that the covariance function of a h.r.f., X has an orthogonal series expansion (3.1) is $r_X(\mathbf{t},\mathbf{s})=\sum\limits_{\mathbf{n}\in\mathbb{Z}^d}a_{\mathbf{n}}(\mathbf{t})a_{\mathbf{n}}^*(\mathbf{s})$.*

4. Examples of harmonizable random fields

Before presenting other theoretical results, we give some elementary examples of harmonizable noises.

Example 4.1 Let $\xi=(\xi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{Z}^d}$ consists a white noise $r.f.$ Consider the following $r.f.$

1. $X_{\mathbf{r}}^{(1)}=\xi_{\mathbf{t}}$ for all $\mathbf{t}\in\mathbb{Z}^d$,
2. $X_{\mathbf{r}}^{(2)}=\begin{cases}\xi_{\mathbf{t}}\text{ if }\mathbf{n}_1\preceq\mathbf{t}\preceq\mathbf{n}_2,\ \mathbf{n}_1,\mathbf{n}_2\in\mathbb{Z}^d\\ 0\text{ otherwise,}\end{cases}$
3. $X_{\mathbf{r}}^{(3)}=\begin{cases}\xi_{\mathbf{t}}\text{ if }\mathbf{0}\preceq\mathbf{t}\\ 0\text{ otherwise.}\end{cases}$

Then the series $\left(X_{\mathbf{t}}^{(i)}\right)_{\mathbf{t}\in\mathbb{Z}^d}$, $i=1,2,3$ are $h.r.f.$, white noises.

Example 4.2 Consider the $r.f.$ X defined by $X_{\mathbf{t}} = \sum\limits_{\mathbf{u}\in\mathbb{Z}^d} e^{i\mathbf{t}.\mathbf{u}}\xi_{\mathbf{u}}$ with ξ consists of some centred $r.f.$ with bounded variations spectral measure F_ξ. Then X is harmonizable. Moreover, if ξ is locally stationary and $\sum_{\mathbf{t}} \sqrt{Var(\xi_{\mathbf{t}})} < +\infty$, then X is locally stationary. Furthermore, the convoluted $r.f.$, $y_{\mathbf{t}} = h * X_{\mathbf{t}}$ where $h(\mathbf{u}) = I_{\{\mathbf{0},\ldots,\mathbf{m-1}\}}(\mathbf{u})$ is harmonizable and $y_{\mathbf{t}} = \sum_{\mathbf{v}=\mathbf{0}}^{\mathbf{m-1}} X_{\mathbf{t}-\mathbf{v}} = \sum\limits_{\mathbf{u}\in\mathbb{Z}^d} Q_{\mathbf{u}}\left(\mathbf{m}\right) e^{i\mathbf{t}.\mathbf{u}}\xi_{\mathbf{u}}$ where $Q_{\mathbf{u}}\left(\mathbf{m}\right)$ is the multidimensional Féjer kernel $Q_{\mathbf{u}}\left(\mathbf{m}\right) = \sum\limits_{\mathbf{t}=\mathbf{0}}^{\mathbf{m-1}} e^{i\mathbf{t}.\mathbf{u}} = \frac{\sin(\mathbf{u}.\mathbf{m}/2)}{\sin(\mathbf{u}/2)}$, so by the proposition 3.5 the $r.f.$ Y is not locally stationary nor an isotropic $r.f.$, even when $\left(\xi_{\mathbf{t}}\right)_{\mathbf{t}\in\mathbb{Z}^d}$ is orthogonal field.

Example 4.3 Consider MA representation of $r.f.$ X with time-varying coefficients, i.e.,

$$X_{\mathbf{t}} = \sum_{\mathbf{u}\in\mathbb{Z}^d} a_{\mathbf{u}}\left(\mathbf{t}\right)\xi_{\mathbf{t}-\mathbf{u}} \tag{4.1}$$

where the $a_{\mathbf{u}}\left(.\right)'s$ are non stochastic sequence. If ξ is such that $\sum\limits_{\mathbf{v}\in\mathbb{Z}^d}\sum\limits_{\mathbf{u}\in\mathbb{Z}^d} a_{\mathbf{u}}\left(\mathbf{t}\right)\overline{a_{\mathbf{v}}\left(\mathbf{t}\right)}r_\xi\left(\mathbf{t}-\mathbf{u},\mathbf{t}-\mathbf{v}\right) < +\infty$ for all $\mathbf{t}\in\mathbb{Z}^d$, then X is a second order $r.f.$ belonging in Cramér class. In general, a sufficient condition for that (4.1) to be a second order $r.f.$ for all $r.f.$, ξ having uniformly bounded covariance functions $\left|r_\xi\left(\mathbf{u},\mathbf{v}\right)\right| \leq M < +\infty$ is that $a_{\mathbf{u}}\left(\mathbf{t}\right) \in \mathfrak{L}_1$. However, every stationary $r.f.$ belong to this class. Indeed, since $r_\xi\left(\mathbf{u},\mathbf{v}\right) = \left|r_\xi\left(\mathbf{u}-\mathbf{v}\right)\right| \leq r_\xi\left(\mathbf{0}\right) < +\infty$, for all $\mathbf{u},\mathbf{v}\in\mathbb{Z}^d$ and so do the harmonizable $r.f.$, ξ, since $\left|r_\xi\left(\mathbf{u},\mathbf{v}\right)\right| < +\infty$ for all $\mathbf{u},\mathbf{v}\in\mathbb{Z}^d$. The following theorem provided a set of sufficient conditions which imply the harmonizability of the $r.f.$ having time-varying representation (4.1).

Proposition 4.1 *Let* $\left(a_{\mathbf{u}}\left(\mathbf{t}\right)\right)_{\mathbf{u}\in\mathbb{Z}^d}$ *be the time-varying coefficients of the representation* (4.1) *of* $r.f.$, X*. Assume that* r_ξ *is finite and* $a_{.}\left(\mathbf{t}\right)$ *is the discrete Fourier transform of a function* $A_{\mathbf{t}}\left(.\right)\in\mathbb{L}_1$ *for all* $\mathbf{t}\in\mathbb{Z}^d$ *which satisfies*

$$\sum_{\mathbf{v}\in\mathbb{Z}^d}\sum_{\mathbf{u}\in\mathbb{Z}^d}\int\limits_{\Pi}\int\limits_{\Pi}\left|A_{\mathbf{u}}\left(\lambda\right)\right|\left|\overline{A_{\mathbf{v}}\left(\mu\right)}\right|\left|r_\xi\left(\mathbf{u},\mathbf{v}\right)\right| d\lambda d\mu < +\infty \tag{4.2}$$

then X *is harmonizable.*

Some particular cases we have in mind are:

a. If $a_{\mathbf{u}}(\mathbf{t}) = \prod\limits_{i=1}^{d}\dfrac{\alpha e^{-\beta\left|u_i\right|}}{t_i^2+\alpha^2}$, $\alpha,\beta>0$, then $a_{\mathbf{u}}(\mathbf{t})$ is the Fourier transform of $A_{\mathbf{t}}\left(\lambda\right) = \prod\limits_{i=1}^{d} A_{t_i}\left(\lambda_i\right)$ where $A_{t_i}\left(\lambda_i\right) = \dfrac{\alpha\left(1-e^{-2\beta}\right)}{\left(t_i^2+\alpha^2\right)\left(1-2e^{-\beta}\cos\left(\lambda_i\right)+e^{-2\beta}\right)}$ and $A_{\mathbf{t}}\left(\lambda\right)\in\mathbb{L}_1$. Then the $r.f.$ having time-varying MA representation satisfying the condition (4.2) is harmonizable.

b. If ξ is harmonizable with $\xi_{\mathbf{t}} = \int\limits_{\Pi} e^{i\mathbf{t}.\lambda}dZ_\xi\left(\lambda\right)$, then the $r.f.$ (4.1) has the harmonizable representation given by $X_{\mathbf{t}} = \int\limits_{\Pi} e^{i\mathbf{t}.\lambda}A_{\mathbf{t}}(\lambda)dZ_\xi\left(\lambda\right)$.

Example 4.4 Suppose now that $a_{\mathbf{u}}\left(\mathbf{t}\right)$ is a second order $r.f.$ independent of $\xi_{\mathbf{t}}$ for all $\mathbf{u}\in\mathbb{Z}^d$ with covariance function $r_{\mathbf{u},\mathbf{v}}^{(a)}\left(\mathbf{t},\mathbf{s}\right) = E\left\{a_{\mathbf{u}}\left(\mathbf{t}\right)\overline{a_{\mathbf{v}}\left(\mathbf{s}\right)}\right\}$ and such that

$$\sum_{\mathbf{v}\in\mathbb{Z}^d}\sum_{\mathbf{u}\in\mathbb{Z}^d} r_{\mathbf{u},\mathbf{v}}^{(a)}\left(\mathbf{t},\mathbf{t}\right)\; r_\xi\left(\mathbf{t}-\mathbf{u},\mathbf{t}-\mathbf{v}\right) < +\infty \text{ for all } \mathbf{t}\in\mathbb{Z}^d \tag{4.3}$$

so (4.1) is second order $r.f.$. Then a sufficient condition for (4.3) to hold true for all $r.f.$, ξ with uniformly bounded covariance functions is clearly $\sum\limits_{\mathbf{v}\in\mathbb{Z}^d}\sum\limits_{\mathbf{u}\in\mathbb{Z}^d}\left|r_{\mathbf{u},\mathbf{v}}^{(a)}\left(\mathbf{t}\right)\right| < +\infty$ for all $\mathbf{t}\in\mathbb{Z}^d$. Moreover, a sufficient conditions for the harmonizability of linear $r.f.$ coefficients are given in the following theorem.

Theorem 4.1 *Suppose that the r.f. coefficient* $a_{\mathbf{u}}(\mathbf{t})$ *admits the representation* $a_{\mathbf{u}}(\mathbf{t}) = \int\limits_{\Pi} A_{\mathbf{u}}(\lambda) e^{i\mathbf{t}.\lambda} d\lambda$ *where* $A_{\mathbf{u}}(\lambda)$ *are some second−order r.f. with covariance function* $r_{\mathbf{u},\mathbf{v}}(\lambda,\mu) = E\{A_{\mathbf{u}}(\lambda)A_{\mathbf{u}}(\mu)\}$ *and such that*

a. $a_{\mathbf{u}}(\mathbf{t})$ *is independent of* $\xi_{\mathbf{t}}$ *for all* $\mathbf{u} \in \mathbb{Z}^d$,

b. $\sum\limits_{\mathbf{v}\in\mathbb{Z}^d} \sum\limits_{\mathbf{u}\in\mathbb{Z}^d} \int\limits_{\Pi} \int\limits_{\Pi} \left| r^{(A)}_{\mathbf{u},\mathbf{v}}(\lambda,\mu) \right| \left| r_{\xi}(\mathbf{t}-\mathbf{u},\mathbf{t}-\mathbf{v}) \right| d\lambda d\mu < +\infty$ *for all* $\mathbf{t} \in \mathbb{Z}^d$.

then the r.f. having the representation (4.2) *with r.f. coefficients is harmonizable.*

Proof: Straightforward and hence omitted. □

5. Conclusion

This work represents an attempt at presenting a unified theory of discrete (or continuous)-time stationary and/or nonstationary *h.r.f*, for some subclass given in the inclusion (1.1). The isotropic stationary covariance function in $\mathbb{R}^d$ and their properties for some subclasses are also addressed and illustrated by some examples. The basic quantities of this theory are the constriction of spectral density for the isotropic *h.r.f.* These basic results were interrelated through the Fourier transforms which maybe allow us to investigate as a future perspective, the $n-th$ order moments for the isotropic *h.r.f.*, which demonstrated that the stationary (resp. nonstationary) limit is readily recoverable and that the Fourier transform pair formed by the $n-th$ order moments function and their associated $n-th$ order spectral (polyspectral) are only relevant functions for *h.r.f.*, (see Kimouche et al. [8]). Moreover, the analysis of isotropic *h.r.f*,, allows us to understand as Hilbert space inner products, we were led to suggest a novel and powerful definition of polycoherence *h.r.f.*, which maybe generalize the familiar polycoherence of stationary theory to *h.r.f.* In this paper, we did not discuss the issue of estimators for the various polyspectral representations of *h.r.f.* It is natural, however, to expect that kernel based estimators involving; local smoothing will play a central role when constructing consistent estimators. Alternatively, computer-intensive resembling techniques may be applied to build consistent polyspectral estimates for *h.r.f.* Various estimators for the polyspectral densities will be presented in forthcoming papers. Note that it is possible to generalize the results of this paper to the more general Karhunen and Cramér class of *h.r.f.* The theory of higher-order moments and their associated polyspectral functions for the Karhunen and the Cramér classes will be presented in forthcoming publications.

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