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Some Aspects of Fuzzy (b, θ)-open sets in Fuzzy Topology

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ABSTRACT: The purpose of this article is to define fuzzy (b, θ)-quasi neighbourhood using the concept of fuzzy (b, θ)-open sets introduced by Dutta and Tripathy with some structural properties by defining the (b, θ)-cluster point in a different way. Moreover, our second objective is to introduce the notion of fuzzy (b, θ)-continuous functions and fuzzy strongly (b, θ)-continuous functions using this type of nearly open sets and to establish several characterizations.

Key Words: Fuzzy topological space, fuzzy b-open set, fuzzy θ-open set, fuzzy (b, θ)-open set.

Contents

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1. Introduction

The concept of fuzzy set was introduced and studied by Zadeh [8] for the first time in 1965. This provides a natural framework for generalizing many topological concepts in various directions. Accordingly, Chang [2] has developed the theory of fuzzy topology and after that several eminent mathematicians all over the world turned their attention in the fuzzy setting of various types of nearly open sets of general topology. As a result, Benchalli and Kernel [1] has introduced fuzzy b-open sets and investigated the inter-relationship with generalized fuzzy b-open sets. The concepts of fuzzy θ-closure operator were established by Salleh and Wahab [6] and since then it has been studied intensively by several authors. Also, Dutta and Tripathy [4] has defined the notion of fuzzy (b, θ)-open sets with the help of fuzzy b-open sets and fuzzy θ-closure operator and established several properties.

In this paper, our main purpose is to study the fuzzy (b, θ)-open sets by introducing the fuzzy (b, θ)-quasi neighbourhood structure with some of their properties. Moreover, we have defined fuzzy (b, θ)-continuous functions and fuzzy strongly (b, θ)-continuous functions with some structural properties by showing their inter-relationship.

2. Preliminaries

In this section, various concepts and results from the theories of fuzzy sets and fuzzy topological spaces are considered which are collected from different research papers for ready references.

Throughout this article, X or Y denotes the fuzzy topological space (X, τ) or (Y, τ). Also FTS denotes the fuzzy topological space.

According to Zadeh [8], a fuzzy set F in a non-empty set X is a function $\lambda_F : X \to I$, where λ_F denotes the membership function of F and $\lambda_F(a)$ is the membership grade of a in F. By $F^c = 1 - F$, we mean the complement of a fuzzy subset F whose membership function is defined as $\lambda_{F^c} = 1 - \lambda_F$ and if a fuzzy subset F is contained in another fuzzy subset G, then we can write it as $F \leq G$ if $\lambda_F \leq \lambda_G$.

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The empty set is represented by $\lambda_\emptyset(a) = 0$, for every $a \in X$ and is denoted by O_X and the whole set is represented by $\lambda_X(a) = 1$, for every $a \in X$ and is denoted by 1_X.

For a family $\{F_i : i \in \Delta, \Delta$ an index set$\}$ of fuzzy sets in X, the union and intersection are written as $V_{i=1}F_i$ and $\Lambda_{i=1}F_i$ respectively and are defined as $\lambda_{V_{i\in\Delta}}F_i(a) = Sup\{\lambda_{F_i}(a) : i \in \Delta\}$ and $\lambda_{\Lambda_{i\in\Delta}}F_i(a) = Inf\{\lambda_{F_i})(a) : i \in \Delta\}$

Definition 2.1. *[5] A fuzzy point in X is a fuzzy set in X which is zero everywhere except at one point say a where it takes the value $\varepsilon \in (0, 1)$, i.e. $0 < \varepsilon < 1$ and is denoted by a_ε. If F be a fuzzy set then $a_\varepsilon \in F$ means $\varepsilon < F(a)$, where $0 < \varepsilon < 1$.*

Definition 2.2. *[5] A fuzzy point a_ε is said to be quasi-coincident (in short, q-coincident) with a fuzzy set F denoted by $a_\varepsilon qF$ if $\varepsilon + F(a) > 1$. In particular, a fuzzy set F is q-coincident with another fuzzy set G if there exists an $a \in X$ such that $F(a) + G(a) > 1$ and is denoted by F_qG. Also, if F and G are not q-coincident, then we can write it as $F_{\bar{q}}G$ or $F_q(1 - G)$ if $F \le G$.*

Definition 2.3. *[5] A fuzzy set F is said to be quasi-neighbourhood (in short, q-nbd) of a fuzzy point a_ε if there exists a fuzzy open set G in X such that $a_\varepsilon qG \le F$.*

Definition 2.4. *[1] A fuzzy set F in a FTS X is called fuzzy b-open if $F \le int(cl(F)) \vee cl(int(F))$. The complement of fuzzy b-open set is fuzzy b-closed.*

Definition 2.5. *[1] The fuzzy b-closure and fuzzy b-interior of a fuzzy set F in a FTS X are denoted by b-$cl(F)$ and b-$int(F)$ respectively and are defined as b-$cl(F) = \wedge\{G \ge F : G$ is fuzzy b-closed$\}$ and b-$int(F) = \vee\{G \le F : G$ is fuzzy b-open$\}$.*

Definition 2.6. *[1] A fuzzy set F is called fuzzy b-regular if F is both fuzzy b-open and fuzzy b-closed.*

Definition 2.7. *[6] A fuzzy point a_ε in X is said to be fuzzy θ-cluster point of a fuzzy subset F of X if for every fuzzy open set G containing a_ε, $cl(G) \wedge F \ne O_X$. The set of all fuzzy θ-cluster points of F is said to be fuzzy θ-closure of F and can be denoted by θ-$cl(F)$. The fuzzy set F is fuzzy θ-closed if $F = \theta$-$cl(F)$ and F is fuzzy θ-open if $F = \theta$-$int(F)$.*

Lemma 2.8. *[1] In a FTS X*
(a) Arbitrary union of fuzzy b-open sets is fuzzy b-open.
(b) Arbitrary intersection of fuzzy b-closed sets is fuzzy b-closed.

Lemma 2.9. *[1] Let F be a fuzzy subset of a FTS X. Then*
(a) b-$cl(F)$ is fuzzy b-closed.
(b) b-$int(F)$ is fuzzy b-open.
(c) F is fuzzy b-open $\Leftrightarrow F =$ b-$int(F)$.
(d) F is fuzzy b-closed $\Leftrightarrow F =$ b-$cl(F)$.

Definition 2.10. *[4] A fuzzy point a_ε of X is called a fuzzy (b, θ)-cluster point of a fuzzy set F if b-$cl(G) \wedge F \ne O_X$ for every fuzzy b-open set in X containing a_ε. The set of all fuzzy (b, θ)-cluster points of F is called fuzzy (b, θ)-closure of F and it is denoted by (b, θ)-$cl(F)$. A fuzzy subset F is said to be fuzzy (b, θ)-closed if $F = (b, \theta)$-$cl(F)$. The complement of a fuzzy (b, θ)-closed set is said to be fuzzy (b, θ)-open.*

3. Fuzzy (b, θ)-open sets

Definition 3.1. *A fuzzy set A in a FTS (X, τ) is called fuzzy (b, θ)-neighbourhood (in short, (b, θ)-nbd) of a fuzzy point a_ε if there exists a fuzzy (b, θ)-open set B in X such that $a_\varepsilon \in B \le A$.*

Definition 3.2. *A fuzzy set A in a FTS (X, τ) is called fuzzy (b, θ)-quasi neighbourhood (in short, (b, θ)-q-nbd) of a fuzzy point a_ε if there exists a fuzzy (b, θ)-open set B in X such that $a_\varepsilon qB \le A$.*

Definition 3.3. *A fuzzy point a_ε in a FTS (X, τ) is said to be fuzzy (b, θ)-cluster point of a fuzzy set U of X if the b-closure of every fuzzy (b, θ)-q-nbd) of a_ε is q-coincident with U. The union of all fuzzy (b, θ)-cluster points of U is called fuzzy (b, θ)-closure of U and it is denoted by (b, θ)-$cl(U)$. A fuzzy set U is said to be fuzzy (b, θ)-closed if $U = (b, \theta)$-$cl(U)$.*

Theorem 3.4. *For a fuzzy subset U in a FTS (X,τ), (b,θ)-$cl(U)$ is the intersection of every fuzzy (b,θ)-closed sets containing U*

Proof. Suppose that $V = \wedge\{W \in I^X : U \leq W$ and W is fuzzy (b,θ)-closed set$\}$. To show that $V = (b,\theta)$-$cl(U)$. Let $a_\varepsilon \in V$. Suppose if possible $a_\varepsilon \notin (b,\theta)$-$cl(U)$. Then there exists a fuzzy (b,θ)-q-nbd A of a_ε such that b-$cl(A)\overline{q}U$. Since, A is fuzzy (b,θ)-q-nbd of a_ε, so there exists a fuzzy (b,θ)-open set B such that $a_\varepsilon\ qB \leq A \leq b$-$cl(A)$. Now, b-$cl(A)\overline{q}U \Rightarrow B\overline{q}U \Rightarrow B(a)+U(a) \leq 1 \Rightarrow U(a) \leq 1-B(a)$ Since, $1-B$ is fuzzy (b,θ)-closed, so $V \leq 1-B$. As $a_\varepsilon \notin 1-B$, so $a_\varepsilon \notin V$, which is a contradiction. Therefore, $a_\varepsilon \in (b,\theta)$-$cl(U)$. Hence $V \leq (b,\theta)$-$cl(U)$. Conversely, let $a_\varepsilon \notin V$. So, there exists a fuzzy (b,θ)-closed set W containing U such that $a_\varepsilon \notin W$. This implies that $a_\varepsilon q(1-W)$. Also, $U \leq W \Rightarrow U(a) \leq W(a) \Rightarrow U(a)-W(a) \leq 0 \Rightarrow U(a)+(1-W)(a) \leq 1 \Rightarrow U\overline{q}(1-W)$. Since $1-W$ is fuzzy (b,θ)-open, so $a_\varepsilon \notin (b,\theta)$-$cl(U)$. Thus (b,θ)-$cl(U) \leq V$. Hence $V = (b,\theta)$-$cl(U)$.

Theorem 3.5. *Let U be a fuzzy subset in a FTS(X,τ). Then a fuzzy point $a_\varepsilon \in (b,\theta)$-$cl(U)$ if and only if every fuzzy (b,θ)-q-nbd of a_ε is quasi-coincident with U.*

Proof. Let $a_\varepsilon \in (b,\theta)$-$cl(U)$ and V be a fuzzy (b,θ)-q-nbd of a_ε such that $V\overline{q}U$. Then there exists a fuzzy (b,θ)-open set W in (X,τ) such that $a_\varepsilon qW \leq V$ and $W\overline{q}U$. Also, since $1-W$ is fuzzy (b,θ)-closed set such that $U \leq 1-W$, so (b,θ)-$cl(U) \leq 1-W$. Again we have $a_\varepsilon \notin 1-W$, so $a_\varepsilon \notin (b,\theta)$-$cl(U)$, a contradiction. Conversely, let $a_\varepsilon \notin (b,\theta)$-$cl(U)$. Therefore there exists a fuzzy (b,θ)-closed set V in X such that $a_\varepsilon \notin V$ and $U \leq V$. Hence $1-V$ is fuzzy (b,θ)-open set such that $a_\varepsilon q(1-V)$ and $U\overline{q}(1-V)$, a contradiction.

Theorem 3.6. *If U is a fuzzy subset and V be a fuzzy (b,θ)-open set in a FTS (X,τ) such that $U\overline{q}V$, then (b,θ)-$cl(U)\overline{q}V$.*

Proof. Let (b,θ)-$cl(U)_qV$. So for every $a \in X$ we have (b,θ)-$cl(U(a))+V(a) > 1$. Let (b,θ)-$cl(U(a)) = \varepsilon$. Then $\varepsilon + V(a) > 1$ and so $a_\varepsilon qV$. Since V is fuzzy (b,θ)-open set and $a_\varepsilon\ qV$, so V is a fuzzy (b,θ)-q-nbd of a_ε such that $U\overline{q}V$. Hence by theorem 3.5, we have $a_\varepsilon \notin (b,\theta)$-$cl(U)$, a contradiction. Consequently, (b,θ)-$cl(U)\overline{q}V$.

Theorem 3.7. *Let U be a fuzzy subset in a FTS (X,τ). Then U is fuzzy (b,θ)-open if and only if for every fuzzy point $a_\varepsilon qU$, U is fuzzy (b,θ)-q-nbd of a_ε.*

Proof. Let U be fuzzy (b,θ)-open in X and a_ε be a fuzzy point such that $a_\varepsilon qU$. Then clearly U is fuzzy (b,θ)-q-nbd of a_ε. Conversely, let U be a fuzzy (b,θ)-q-nbd of a_ε and $a_\varepsilon \in U$. Then $\varepsilon < U(a)$, where $0 < \varepsilon < 1$. Since $0 \neq \varepsilon < U(a)$, we see that $a_{1-\varepsilon}qU$. Now, by hypotheses U is fuzzy (b,θ)-q-nbd of $a_{1-\varepsilon})$. So there exists a fuzzy (b,θ)-open set V in X such that $a_{1-\varepsilon}qV \leq U$. Thus $a_\varepsilon \in V \leq U$. Since V is fuzzy (b,θ)-open so U is fuzzy (b,θ)-open. Hence the proof.

Definition 3.8. *The fuzzy (b,θ)-interior of a fuzzy subset F in a FTS X is denoted by (b,θ)-$int(F)$ and is defined as (b,θ)-$int(F) = \vee\{G \leq F : G$ is fuzzy (b,θ)-open$\}$.*

Theorem 3.9. *The following properties hold for a fuzzy subset F in a FTS (X,τ).*
(a) (b,θ)-$cl(O_X) = O_X$
(b) (b,θ)-$int(O_X) = O_X$
(c) (b,θ)-$cl(1_X) = 1_X$
(d) (b,θ)-$int(1_X) = 1_X$
(e) (b,θ)-$int(F)$ *is fuzzy* (b,θ)*-open*
(f) (b,θ)-$cl(F)$ *is fuzzy* (b,θ)*-closed*
(g) F *is fuzzy* (b,θ)*-open* $\Leftrightarrow F = (b,\theta)$-$int(F)$
(h) F *is fuzzy* (b,θ)*-closed* $\Leftrightarrow F = (b,\theta)$-$cl(F)$.

Proof. Proofs of (a) to (f) are obvious, so omitted.
(g) Let F be fuzzy (b,θ)-open. Since $F \leq F$, so $F \in \{G \leq F : G$ is fuzzy (b,θ)-open$\}$. Therefore, $F = \vee\{G \leq F : G$ is fuzzy (b,θ)-open$\} \Rightarrow F = (b,\theta)$-$int(F)$. Conversely, let $F = (b,\theta)$-$int(F)$. Then $F = \vee\{G \leq F : G$ is fuzzy (b,θ)-open$\}$. So, we have $F \in \{G \leq F : G$ is fuzzy (b,θ)-open$\}$. Hence F is fuzzy (b,θ)-open.
Similarly, we can prove (h) also.

Theorem 3.10. *Let F and G be two fuzzy subsets in a FTS (X,τ).*
(a) If $F<G$, then (b,θ)-$cl(F)<(b,\theta)$-$cl(G)$ and (b,θ)-$int(F)<(b,\theta)$-$int(G)$
(b) (b,θ)-$cl(1-F)=1-(b,\theta)$-$int(F)$
(c) (b,θ)-$int(1-F)=1-(b,\theta)$-$cl(F)$.

Proof. It is obvious.

4. Fuzzy (b,θ)-continuous Functions

Definition 4.1. *A function $f:(X,\tau)\rightarrow(Y,\tau)$ is said to be fuzzy (b,θ)-continuous if for every fuzzy point $a_\varepsilon\in X$ and every fuzzy open set F of Y containing $f(a_\varepsilon)$, there exists a fuzzy b-open set G in X containing a_ε such that $f(b\text{-}cl(G))\leq cl(F)$.*

Theorem 4.2. *The following results are equivalent for the function $f:(X,\tau)\rightarrow(Y,\tau)$*
(a) f is fuzzy (b,θ)-continuous
(b) (b,θ)-$cl(f^{-1}(G))\leq f^{-1}(\theta-cl(G))$ for every fuzzy subset G of Y
(c) $f((b,\theta)$-$cl(K))\leq\theta$-$cl(f(K))$, for every fuzzy subset K of X.

Proof. (a) $\Rightarrow$ (b) Let G be a fuzzy subset of Y and a_ε be a fuzzy point in X. Suppose if possible, $a_\varepsilon\notin f^{-1}(\theta\text{-}cl(G))$. This implies $f(a_\varepsilon)\notin\theta\text{-}cl(G)$ and so there exists a fuzzy open set E containing $f(a_\varepsilon)$ such that $cl(E)\wedge G=O$. Since f is fuzzy (b,θ)-continuous, so there exists a fuzzy b-open set F in X containing a_ε such that $f(b\text{-}cl(F))\leq cl(E)$. Thus we get $f(b\text{-}cl(F))\wedge G=0$ and so $b\text{-}cl(F)\wedge f^{-1}(G)=0$, which shows that $a_\varepsilon\notin(b,\theta)\text{-}cl(f^{-1}(G))$. Hence, $(b,\theta)\text{-}cl(f^{-1}(G))\leq f^{-1}(\theta\text{-}cl(G))$.
(b) $\Rightarrow$ (c) Let K be any fuzzy subset of X. Then $f(K)$ is a fuzzy subset of Y. Now, $(b,\theta)\text{-}cl(K)\leq(b,\theta)\text{-}cl(f^{-1}(f(K)))\leq f^{-1}(\theta\text{-}cl(f(K)))$. Hence, $f((b,\theta)\text{-}cl(K))\leq\theta\text{-}cl(f(K))$.
(c) $\Rightarrow$ (b) Let G be any fuzzy subset of Y. By (c), $f((b,\theta)\text{-}cl(f^{-1})(G)))\leq\theta\text{-}cl(f(f^{-1}(G)))\leq\theta\text{-}\ cl(G)$. Hence, $(b,\theta)\text{-}cl(f^{-1}(G))\leq f^{-1}(\theta\text{-}cl(G))$.
(b) $\Rightarrow$ (a) Let E be any fuzzy open set of Y containing $f(a_\varepsilon)$. Then we see that $cl(E)\wedge$ (1-$cl(E)$) $=0$ and so $f(a_\varepsilon)\notin\theta\text{-}cl(1-cl(E))$. This implies $a_\varepsilon\notin f^{-1}(\theta\text{-}cl(1-cl(E)))$. By hypothesis, we get $a_\varepsilon\notin(b,\theta)\text{-}cl(f^{-1}(1-cl(E)))$. Thus there exists a fuzzy b-open set F in X containing a_ε such that $b\text{-}cl(F)\wedge f^{-1}(1-cl(E))=0$. Consequently, $f(b\text{-}cl(F))\leq cl(E)$. Hence f is fuzzy (b,θ)-continuous.

Theorem 4.3. *The following results are equivalent for the function $f:(X,\tau)\rightarrow(Y,\tau)$*
(a) f is fuzzy (b,θ)-continuous.
(b) $f^{-1}(G)\leq(b,\theta)$-$int(f^{-1}(cl(G)))$, for every fuzzy open set G of Y.
(c) (b,θ)-$cl(f^{-1}(G))\leq f^{-1}(cl(G))$, for every fuzzy open set G of Y.

Proof. (a) $\Rightarrow$ (b) Let G be a fuzzy open set in Y and a_ε be a fuzzy point. Suppose $a_\varepsilon\in f^{-1}(G)$. Then $f(a_\varepsilon)\in G$. Since f is fuzzy (b,θ)-continuous, so there exists a fuzzy b-open set K in X containing a_ε such that $f(b\text{-}cl(K))\leq cl(G)$. Now $a_\varepsilon\in K\leq b\text{-}cl(K)\leq f^{-1}(cl(G))$. This shows that $a_\varepsilon\in(b,\theta)\text{-}int(f(cl(G)))$. Hence $f^{-1}(G)\leq(b,\theta)\text{-}int(f^{-1}(cl(G)))$.
(b) $\Rightarrow$ (c) Let G be a fuzzy open set in Y and $a_\varepsilon\notin f^{-1}(cl(G))$. Then we have $f(a_\varepsilon)\notin cl(G)$. So, there exists a fuzzy open set K containing $f(a_\varepsilon)$ such that $K\wedge G=0$. Thus $cl(K)\wedge G=0$ and so $f^{-1}(cl(K))\wedge f^{-1}(G)=0$. But $a_\varepsilon\in f^{-1}(K)$. So by (b), we have $a_\varepsilon\in(b,\theta)\text{-}int(f^{-1}(cl(K)))$. Therefore, there exists a fuzzy b-open set F in X containing a_ε such that $b\text{-}cl(F)\leq f^{-1}(cl(K))$. Thus we see that, $b\text{-}cl(F)\wedge f^{-1}(G)=0$. Consequently, $a_\varepsilon\notin(b,\theta)\text{-}cl(f^{-1}(G))$. Hence, $(b,\theta)\text{-}cl(f^{-1}(G))\leq f^{-1}(cl(G))$.
(c) $\Rightarrow$ (a) Let $a_\varepsilon\in X$ and G be any fuzzy open set in Y containing $f(a_\varepsilon)$. Then we have $G\wedge(1-cl(G))=0$. Therefore, $f(a_\varepsilon)\notin cl(1-cl(G))$ and so $a_\varepsilon\notin f^{-1}(cl(1-cl(G)))$. Now by (c), we have $a_\varepsilon\notin(b,\theta)\text{-}cl(f^{-1}(1-cl(G)))$. Therefore there exists a fuzzy b-open set in X containing a_ε such that $b\text{-}cl(G)\wedge f^{-1}(1-cl(G))=0$. Thus we get $f(b\text{-}cl(G))\leq cl(G)$. Hence f is fuzzy (b,θ)-continuous.

5. Fuzzy strongly (b,θ)-continuous function

Definition 5.1. *A function $f:(X,\tau)\rightarrow(Y,\tau)$ is said to be fuzzy strongly (b,θ)-continuous if for every fuzzy point a_ε in X and every fuzzy open set F of Y containing $f(a_\varepsilon)$, there exists a fuzzy b-open set G in X containing a_ε such that $f(b\text{-}cl(G)\leq F$.*

Lemma 5.2. *If F be a fuzzy subset in X then F is fuzzy b-open in X if and only if b-$cl(F)$ is fuzzy b-regular in X.*

Lemma 5.3. *([4]). A fuzzy subset F is fuzzy (b, θ)-open if and only if for every $a_\varepsilon \in F$ there exists a fuzzy b-regular set G in X containing a_ε such that $a_\varepsilon \in G < F$.*

Theorem 5.4. *The following results are equivalent for the function $f : (X, \tau) \to (Y, \tau)$*
(a) f is fuzzy strongly (b, θ)-continuous
(b) for every fuzzy point a_ε in X and for every fuzzy open set F of Y containing $f(a_\varepsilon)$, there exists a fuzzy b-regular set G in X containing a_ε such that $f(G) \leq F$.
(c) $f^{-1}(F)$ is fuzzy (b, θ)-open in X , for every fuzzy open set F of Y.
(d) $f^{-1}(F)$ is fuzzy (b, θ)-closed in X , for every fuzzy closed set F of Y.
(e) $f((b, \theta)$-$cl(M)) \leq cl(f(M))$, for every fuzzy subset M of X.
(f) (b, θ)-$cl(f^{-1}(N)) \leq f^{-1}(cl(N))$, for every fuzzy subset N of Y.

Proof. (a) ⇒(b) Let a_ε be a fuzzy point in X and F be a fuzzy open set in Y containing $f(a_\varepsilon)$. Since f is fuzzy strongly (b, θ)-continuous, so there exists a fuzzy b-open set G in X containing a_ε such that $f(b\text{-}cl(G) \leq F$. By lemma 5.2 , b-$cl(G)$ is fuzzy b-regular and so b-$cl(G) = G$. Hence, there exists a fuzzy b-regular set G in X containing a_ε such that $f(G) \leq F$.
(b) ⇒(c) Let F be any fuzzy open set in Y and let $a_\varepsilon \in f^{-1}(F)$. This implies $f(a_\varepsilon) \in F$. By (b), there exists a fuzzy b-regular set G in X containing a_ε such that $f(G) \leq F$. Therefore, $a_\varepsilon \in G \leq f^{-1}(F)$. Hence by lemma 5.3, we have $f^{-1}(F)$ is fuzzy (b, θ)-open in X.
(c) ⇒ (d) The result follows by taking the complement.
(d) ⇒ (e) Let M be any fuzzy subset in X. So, $f(M)$ is a fuzzy subset in Y and hence $cl(f(M))$ is fuzzy closed set in Y. By (d), $f^{-1}(cl(f(M)))$ is fuzzy (b, θ)-closed in X. Now (b, θ)-$cl(M) \leq (b, \theta)$-$cl(f^{-1}(f(M))) \leq (b, \theta)$-$cl(f^{-1}(cl(f(M)))) = f^{-1}(cl(f(M)))$. $\Rightarrow f((b, \theta)$-$cl(M) \leq cl(f(M))$.
(e) ⇒ (f) Let N be any fuzzy subset in Y. By (e) we have $f((b, \theta)$-$cl(f^{-1}(N))) \leq cl(f(f^{-1}(N))) \leq cl(N)$ $\Rightarrow (b, \theta)$-$cl(f^{-1}(N)) \leq f^{-1}(cl(N))$.
(f) ⇒(a) Let a_ε be a fuzzy point in X and G be a fuzzy open set in Y containing $f(a_\varepsilon)$. Now, $1 - G$ is fuzzy closed in Y. So, by (f), we have (b, θ)-$cl(f^{-1}(1 - G)) \leq f^{-1}(cl(1 - G)) = f^{-1}(1 - G)$. This shows that $f^{-1}(1-G)$ is fuzzy (b, θ)-closed in X and so $f^{-1}(G)$ is fuzzy (b, θ)-open in X such that $a_\varepsilon \in f^{-1}(G)$. Thus by lemma 5.3, there exists a fuzzy b-regular set F in X containing a_ε such that $a_\varepsilon \in F \leq f^{-1}(G)$. This implies that $a_\varepsilon \in b$-$cl(F) \leq f^{-1}(G)$, since F is fuzzy b-regular. Hence, there exists a fuzzy b-open set F in X containing a_ε such that $f(b\text{-}cl(F)) \leq G$. Consequently, f is fuzzy strongly (b, θ)-continuous.

Definition 5.5. *A function $f : (X, \tau) \to (Y, \tau)$ is called fuzzy b-continuous if for every fuzzy point a_ε in X and for every fuzzy open set F of Y containing $f(a_\varepsilon)$, there exists a fuzzy b-open set G in X containing a_ε such that $f(G) \leq F$.*

Definition 5.6. *([3]). A fuzzy FTS X is said to be fuzzy regular if for every $a_\varepsilon \in X$ and every fuzzy open set G in X there exists a fuzzy open set K in X such that $a_\varepsilon \in K \leq cl(K) \leq G$.*

Theorem 5.7. *Let a FTS Y be fuzzy regular. Then a function $f : (X, \tau) \to (Y, \tau)$ is fuzzy strongly (b, θ)-continuous if and only if f is fuzzy b-continuous.*

Proof. Let a_ε be a fuzzy point in X and G be a fuzzy open set in Y containing $f(a_\varepsilon)$. Since Y is fuzzy regular, so there exists a fuzzy open set K in Y such that $f(a_\varepsilon) \in K \leq cl(K) \leq G$. Suppose that, f is fuzzy b-continuous, so there exists a fuzzy b-open set F in X containing a_ε such that $f(F) \leq K$. Let $b_\varepsilon \notin cl(K)$. Then there exists a fuzzy open set N containing b_ε such that $N \wedge K = 0$. Also, since f is fuzzy b-continuous so $f^{-1}(N)$ is fuzzy b-open in X and hence $f^{-1}(N) \wedge F = 0$. This implies that $f^{-1}(N) \wedge b$-$cl(F) = 0$ and which again implies $N \wedge f(b\text{-}cl(F)) = 0$. Thus $b_\varepsilon \notin f(b\text{- }cl(F))$.Consequently, $f(b\text{-}cl(F)) \leq cl(K) \leq G$. Hence, f is fuzzy strongly (b, θ)-continuous. Conversely, let f be fuzzy strongly (b, θ)-continuous. Then by theorem 5.4 and definition 2.6, we have f is fuzzy b-continuous.

Definition 5.8. *A FTS X is called fuzzy b-regular if for every fuzzy closed set G and every fuzzy point $a_\varepsilon \in 1 - G$, there exists two fuzzy b-open set F_1 and F_2 such that $a_\varepsilon \in F_1, G \leq F_2$ and $F_1 \wedge F_2 = 0$.*

Lemma 5.9. *A FTS X is fuzzy b-regular if and only if for every fuzzy point $a_\varepsilon \in X$ and every fuzzy open set F in X containing a_ε , there exists a fuzzy b-open set G in X such that $a_\varepsilon \in G \leq b$- $cl(G) \leq F$.*

Theorem 5.10. *A fuzzy continuous function $f : (X, \tau) \to (Y, \tau)$ is fuzzy strongly (b, θ)-continuous if X is fuzzy b-regular.*

Proof. Let X be fuzzy b-regular. Since f is fuzzy continuous, so for every $a_\varepsilon \in X$ and every fuzzy open set G in X containing $f(a_\varepsilon)$, $f^{-1}(G)$ is fuzzy open in Y containing a_ε. Also, X is fuzzy b-regular, so by lemma 5.9, there exists a fuzzy b-open set F in X such that $a_\varepsilon \in F \leq b\text{-}cl(F) \leq f^{-1}(G)$. This implies $f(b\text{-}cl(F)) \leq G$ and hence f is fuzzy strongly (b, θ)-continuous.

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Declaration

The authors declare that the article does not have any competing interest involved.

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