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Subordination results for certain admissible classes of multivalent analytic functions involving Sălăgean operator

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ABSTRACT: In this paper, we obtain some subordination results as an application of the third-order differential subordination and superordination for some admissible classes defined for multivalent analytic functions involving the Sălăgean operator. Some third order differential sandwich-type results are obtained for these classes. Applications of our main results are also given for some special classes.

Key Words: Third order Differential subordination, superordination, multivalent analytic functions, admissible function, sandwich-type result, Sălăgean operator.

Contents

1. Introduction, Definitions and Preliminaries

Let $\mathbb{C}$ be a set of complex numbers and $H(\mathbb{D})$ be a class of functions which are analytic in the open unit disk $\mathbb{D} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}$. For $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and for some $a, a_n, a_{n+1}, ... \in \mathbb{C}$, consider a class

$$H[a, n] = \{f : f \in H(\mathbb{D}) \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots\}.$$

We denote $H_n = H[1, n]$.

Let $\mathcal{A}_m$ denotes a class of analytic m-valent functions $f \in H(\mathbb{D})$ of the form:

$$f(z) = z^m + \sum_{n=1}^{\infty} a_{m+n} z^{m+n} \quad (m \in \mathbb{N}, a_{m+n} \in \mathbb{C}). \tag{1.1}$$

Let f and F be in the class $H(\mathbb{D})$. Then the function f is said to be subordinate to F or, F is said to be superordinate to f, if there exist an analytic function $w(z)$ in $\mathbb{D}$ satisfying

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{D}),$$

such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{D})$$

and written as

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{D}).$$

Further, if F is univalent in $\mathbb{D}$, then (see for details [4])

$$f \prec F \Leftrightarrow f(0) = F(0) \quad \text{and} \quad f(\mathbb{D}) \subset F(\mathbb{D}).$$

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For $f \in \mathcal{A}_m$, the Sălăgean derivative operator $\mathcal{D}_m^\lambda$ of order λ $(\lambda \in \mathbb{N} \cup \{0\})$ [8] is defined as

$$\mathcal{D}_m^0 f(z) = f(z) \qquad \text{and} \qquad \mathcal{D}_m^1 f(z) = \mathcal{D}_m f(z) = \frac{zf'(z)}{m}$$

and in general

$$\mathcal{D}_m^\lambda f(z) = \mathcal{D}_m(\mathcal{D}_m^{\lambda-1} f(z)) \qquad (\lambda \in \mathbb{N}). \tag{1.2}$$

For $f \in \mathcal{A}_m$ of the form (1.1),

$$\mathcal{D}_m^\lambda f(z) = z^m + \sum_{n=1}^{\infty} \left(\frac{n+m}{m}\right)^\lambda a_{m+n} z^{m+n}. \tag{1.3}$$

In the past several years there are many articles in the literature dealing with the first order, second order and third-order differential subordinations for various linear, nonlinear operators and derived many useful results. One may find the earlier work on third-order differential subordinations in [1,7,10,12,13] etc.. It is worth to mention that the theory of differential subordination was introduced by Miller and Mocanu [5] and several pioneer work may be found in the monograph [4], where several significant results obtained earlier are compiled. Antonino and Miller [1] introduced the concept of third order differential subordination and superordination for analytic functions. Motivated with the work of Aouf et al. [2], in this paper, we consider the function $\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu$ for $f \in \mathcal{A}_m$ and obtain new subordination, superordination and sandwich type results as a solutions of the third-order differential subordination and superordination.

We now recall some definitions and Lemmas from the work of Antonino and Miller [1] and Tang et al. [10] (see [11]) for third-order differential subordination and superordination, respectively, as follows:

Definition 1.1 *(see [1, Definition 2, p. 441]) Let Q denote the set of functions q that are analytic and univalent in $\overline{\mathbb{D}} \setminus E(q)$, where*

$$E(q) = \{\xi : \xi \in \partial\mathbb{D} \quad \textit{and} \quad \lim_{z\to\xi} q(z) = \infty\}$$

and are such that

$$\min |q'(\xi)| > 0 \quad (\xi \in \partial\mathbb{D} \setminus E(q)).$$

The subclass of Q for which $q(0) = a$ is denoted by $Q(a)$. In particular, we denote $Q(1)$ by Q_1.

Definition 1.2 *[1] (see [9,10]) Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ and $h(z)$ be univalent in $\mathbb{D}$. If $p(z)$ is analytic in $\mathbb{D}$ and satisfies the third-order differential subordination*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \prec h(z), \tag{1.4}$$

then $p(z)$ is called a solution of the differential subordination (1.4). Furthermore, a univalent function $q(z)$ is called a dominant of the solutions of the differential subordination (1.4) or more simply a dominant if $p(z) \prec q(z)$ $\forall$ $p(z)$ satisfying (1.4). A dominant $\tilde{q}(z)$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$ of (1.4) is said to be the best dominant. Note that the best dominant is unique upto a rotation of $\mathbb{D}$.

Definition 1.3 *[1, p. 449] (see [9]) Let Ω be a set in the complex plane $\mathbb{C}$, $q \in Q$ and $n \in \mathbb{N}\setminus\{1\}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ that satisfy the admissibility condition*

$$\psi(r, s, t, u; z) \notin \Omega$$

whenever

$$r = q(w), \quad s = kwq'(w), \quad \mathfrak{Re}\left(\frac{t}{s} + 1\right) \geq k\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right)$$

and

$$\mathfrak{Re}\left(\frac{u}{s}\right) \geq k^2 \mathfrak{Re}\left(\frac{w^2q'''(w)}{q'(w)}\right),$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D}\setminus E(q)$ and $k \geq n$.

Definition 1.4 *Let Ω be a set in the complex plane $\mathbb{C}$, $q \in Q$. The class of admissible functions $\Phi_n[\Omega, q]$ $(n \in \mathbb{N})$ consists of functions $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$, that satisfy the admissibility condition*

$$\psi(r, s, t; z) \notin \Omega,$$

where

$$r = q(w), \quad s = kwq'(w), \quad \mathfrak{Re}\left(\frac{t}{s} + 1\right) \geq k\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right)$$

$$(z \in \mathbb{D}, w \in \partial\mathbb{D}\backslash E(q) \text{ and } k \geq n).$$

Lemma 1.1 *[1, Theorem 1, p. 449] Let $p \in \mathcal{H}[a, n]$ with $n \in \mathbb{N}\backslash\{1\}$. Also let $q \in \mathcal{Q}(a)$ satisfying*

$$\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(w)}\right| \leq k,$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D}\backslash E(q)$ and $k \geq n$. If Ω is a subset in $\mathbb{C}$, $\psi \in \Psi_n[\Omega, q]$ and

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z).$$

Lemma 1.2 *Let $p \in \mathcal{H}[a, n]$ with $n \in \mathbb{N}$ and let $q \in \mathcal{Q}(a)$. If Ω is a subset in $\mathbb{C}$, $\psi \in \Phi_n[\Omega, q]$ and*

$$\psi(p(z), zp'(z), z^2p''(z); z) \in \Omega,$$

then

$$p(z) \prec q(z).$$

Definition 1.5 *[10, Definition 5, p. 3] Let $\psi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ and $h(z)$ be analytic in $\mathbb{D}$. If the function $p(z)$ and*

$$\psi\{p(z), zp'(z), z^2p''(z), z^3p'''(z); z\}$$

are univalent in $\mathbb{D}$ and satisfy the third-order differential superordination

$$h(z) \prec \psi\{p(z), zp'(z), z^2p''(z), z^3p'''(z); z\}, \tag{1.5}$$

then $p(z)$ is called a solution of the differential superordination (1.5). An analytic function $q(z)$ is called a subordinant of the solutions of the differential subordination or more simply a subordinant if $q(z) \prec p(z)$ $\forall$ $p(z)$ satisfying (1.5). A univalent subordinant $\tilde{q}(z)$ that satisfies $q(z) \prec \tilde{q}(z)$ for all subordinants $q(z)$ of (1.5) is said to be the best subordinant. We note that the best subordinant is unique upto a rotation of $\mathbb{D}$.

Definition 1.6 *[10, Definition 7, p. 4] (see [11]) Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a, n]$ and $q'(z) \neq 0$. The class of admissible function $\Psi'_n[\Omega, q]$ consists of those functions $\psi : \mathbb{C}^4 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfies the following admissibility condition*

$$\psi(r, s, t, u; w) \in \Omega$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{k}, \quad \mathfrak{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{k}\mathfrak{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right)$$

and

$$\mathfrak{Re}\left(\frac{u}{s}\right) \leq \frac{1}{k^2}\mathfrak{Re}\left(\frac{z^2q'''(z)}{q'(z)}\right),$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D}$ and $k \geq n \geq 2$.

Definition 1.7 *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}[a,n]$ and $q'(z) \neq 0$. The class of admissible functions $\Phi'_n[\Omega, q]$ $(n \in \mathbb{N})$ consists of functions $\psi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$, that satisfy the admissibility condition*

$$\psi(r, s, t; w) \in \Omega,$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{k}, \quad \mathfrak{Re}\left(\frac{t}{s} + 1\right) \leq \frac{1}{k}\mathfrak{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right)$$
$$(z \in \mathbb{D}, w \in \partial\mathbb{D} \text{ and } k \geq n \geq 1).$$

Lemma 1.3 *[10, Theorem 8, p. 4] (see [11]) Let $q \in \mathcal{H}[a,n]$ and $\psi \in \Psi'_n[\Omega, q]$. If*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$

is univalent in $\mathbb{D}$ and $p \in \mathcal{Q}(a)$ satisfying the conditions

$$\mathfrak{Re}\left(\frac{zq''(z)}{q'(z)}\right) \geq 0 \quad \text{and} \quad \left|\frac{zp'(z)}{q'(z)}\right| \leq k,$$

where $z \in \mathbb{D}$ and $k \geq n \geq 2$, then

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{D}\right\}$$

implies that

$$q(z) \prec p(z) \quad (z \in \mathbb{D}).$$

Lemma 1.4 *Let $q \in \mathcal{H}[a,n]$ and $\psi \in \Phi'_n[\Omega, q]$. If*

$$\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z)$$

is univalent in $\mathbb{D}$ and $p \in \mathcal{Q}(a)$ satisfying

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z), z^3p'''(z); z) : z \in \mathbb{D}\right\}$$

then

$$q(z) \prec p(z) \quad (z \in \mathbb{D}).$$

2. Subordination Results for admissible class $\mathcal{P}_m[\Omega, q, \mu]$

Here, we first define the class $\mathcal{P}_m[\Omega, q, \mu]$ of admissible functions that will be used in proving our upcoming results.

Definition 2.1 *Let Ω be any set in $\mathbb{C}$ and $q \in Q_1 \cap H_n$ $(n \in \mathbb{N} \setminus \{1\})$. Define a class $\mathcal{P}_m[\Omega, q, \mu]$ of admissible functions $\phi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ that satisfy the condition*

$$\phi(\alpha, \beta, \gamma, \delta; z) \notin \Omega,$$

whenever

$$\alpha = q(w), \quad \beta = \frac{kwq'(w) + \mu m q(w)}{\mu m},$$

.

$$\mathfrak{Re}\left(\frac{\gamma - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}\right) \geq k\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right)$$

and

$$\mathfrak{Re}\left(\frac{\delta - 3(\mu m + 1)\gamma + (3\mu^2m^2 + 6\mu m + 2)\beta - (\mu m + 1)(\mu m + 2)\alpha}{\beta - \alpha}\right)$$
$$\geq \quad k^2\mathfrak{Re}\left(\frac{w^2q'''(w)}{q'(w)}\right),$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D} \setminus E(q)$ and $0 < \mu \leq 1, k \geq n$.

Definition 2.2 *Let Ω be any set in $\mathbb{C}$ and $q \in Q_1 \cap H_n$. Denote a class by $Q_m[\Omega, q, \mu]$ of those admissible functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfy the condition*

$$\phi(\alpha, \beta, \gamma; z) \notin \Omega,$$

whenever

$$\alpha = q(w), \quad \beta = \frac{kwq'(w) + \mu m q(w)}{\mu m}$$

and

$$\mathfrak{Re}\left(\frac{\gamma - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}\right) \geq k\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right)$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D} \setminus E(q)$ and $0 < \mu \leq 1, k \geq n$.

Theorem 2.1 *Let for $\lambda \in \mathbb{N} \cup \{0\}$, $\mathcal{D}_m^\lambda$ be the Sălăgean operator of order λ defined by (1.3) and for $0 < \mu \leq 1$, $\phi \in \mathcal{P}_m[\Omega, q, \mu]$. If $f \in \mathcal{A}_m$ satisfies the conditions*

$$\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0, \quad \left|\frac{z\left(\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu\right)'}{q'(w)}\right| \leq k \tag{2.1}$$

$$(z \in \mathbb{D}, \quad k \geq 2; \quad w \in \partial\mathbb{D} \setminus E(q))$$

and

$$\{\phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right) : z \in \mathbb{D}\} \subset \Omega, \tag{2.2}$$

where

$$p(z) = \left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu, \tag{2.3}$$

$$p_1(z) = p(z)\frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)}, \tag{2.4}$$

$$p_2(z) = p(z)\left\{\frac{m\mathcal{D}_m^{\lambda+2} f(z)}{\mathcal{D}_m^\lambda f(z)} + (\mu - 1)m\left(\frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)}\right)^2\right\}, \tag{2.5}$$

and

$$p_3(z) = p(z)\left\{\frac{m^2\mathcal{D}_m^{\lambda+3} f(z)}{\mathcal{D}_m^\lambda f(z)} + 3m^2(\mu - 1)\frac{\mathcal{D}_m^{\lambda+1} f(z)\mathcal{D}_m^{\lambda+2} f(z)}{(\mathcal{D}_m^\lambda f(z))^2} + m^2(\mu - 1)(\mu - 2)\left(\frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)}\right)^3\right\}, \tag{2.6}$$

then

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec q(z) \quad (z \in \mathbb{D}). \tag{2.7}$$

Proof: Let

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu = p(z). \tag{2.8}$$

Then on differentiating (2.8) we get

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)} = \frac{zp'(z) + \mu m\, p(z)}{\mu m}. \tag{2.9}$$

Further computation shows that

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \left\{\frac{m\mathcal{D}_m^{\lambda+2} f(z)}{\mathcal{D}_m^\lambda f(z)} + (\mu-1)m\left(\frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)}\right)^2\right\}$$
$$= \frac{z^2p''(z) + (2\mu m+1)zp'(z) + \mu^2m^2\ p(z)}{\mu m} \tag{2.10}$$

and

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \left\{\frac{m^2\mathcal{D}_m^{\lambda+3} f(z)}{\mathcal{D}_m^\lambda f(z)} + 3m^2(\mu-1)\frac{\mathcal{D}_m^{\lambda+1} f(z)\mathcal{D}_m^{\lambda+2} f(z)}{(\mathcal{D}_m^\lambda f(z))^2}\right.$$
$$\left. + m^2(\mu-1)(\mu-2)\left(\frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)}\right)^3\right\}$$
$$= \frac{z^3p'''(z) + 3(\mu m+1)z^2p''(z) + (3\mu^2m^2+3\mu m+1)zp'(z) + \mu^3m^3\ p(z)}{\mu m}. \tag{2.11}$$

Now we define the transformation from $\mathbb{C}^4$ to $\mathbb{C}$ by

$$\psi(r,s,t,u;z) = \phi(\alpha,\beta,\gamma,\delta;z), \tag{2.12}$$

where

$$\alpha = \alpha(r,s,t,u) = r, \qquad \beta = \beta(r,s,t,u) = \frac{s+\mu m r}{\mu m},$$
$$\gamma = \gamma(r,s,t,u) = \frac{t+(2\mu m+1)s+\mu^2m^2r}{\mu m}$$

and

$$\delta = \delta(r,s,t,u) = \frac{u+3(\mu m+1)t+(3\mu^2m^2+3\mu m+1)s+\mu^3m^3r}{\mu m}.$$

Hence, on using equations (2.8) to (2.11), we obtain

$$\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) = \phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right).$$

So clearly on using (2.2), 2.12 yield that $\psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right) \in \Omega$. We also note from above transformation that

$$\frac{t}{s} + 1 = \frac{\gamma - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}$$

and

$$\frac{u}{s} = \frac{\delta - 3(\mu m+1)\gamma + (3\mu^2m^2+6\mu m+2)\beta - (\mu m+1)(\mu m+2)\alpha}{\beta-\alpha}.$$

Thus the admissibility condition for $\phi \in \mathcal{P}_m[\Omega, q, \mu]$ in Definition 2.1 is equivalent to admissibility condition for $\psi \in \Psi_n[\Omega, q]$ as given in Definition 1.3. Hence, by the Lemma 1.1, we obtain

$$p(z) \prec q(z) \quad (z \in \mathbb{D}).$$

or

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec q(z).$$

which completes the proof of Theorem 2.1. □

When the behaviour of $q(z)$ on $\partial\mathbb{D}$ is not known, we easily get the extension of above Theorem 2.1 in the form of following corollary.

Corollary 2.1 *Under the hypothesis of Theorem 2.1, assume that $\phi \in \mathcal{P}_m[\Omega, q_\rho, \mu]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}_m$, and q_ρ satisfies the conditions*

$$\mathfrak{Re}\left(\frac{wq_\rho''(w)}{q_\rho'(w)}\right) \geq 0 \quad \text{and} \quad \left|\frac{z\left(\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu\right)'}{q_\rho'(w)}\right| \leq k \tag{2.13}$$

$$(z \in \mathbb{D}, \quad k \geq 2; \quad w \in \partial\mathbb{D} \setminus E(q_\rho))$$

and ϕ satisfies (2.2), then the result (2.7) holds.

Proof: Since we have

$$q_\rho(z) \prec q(z) \qquad (z \in \mathbb{D})$$

and hence, by using Theorem 2.1 for $q_\rho(z)$, we have

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec q_\rho(z) \prec q(z) \quad (z \in \mathbb{D})$$

which completes the proof. □

If Ω ($\neq \mathbb{C}$) is simply-connected domain, then $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω. In this case the class $\mathcal{P}_m[h(\mathbb{D}), q, \mu]$ is written as $\mathcal{P}_m[h, q, \mu]$. The following two results are easily obtained by using Theorem 2.1 and Corollary 2.1.

Corollary 2.2 *Under the hypothesis of Theorem 2.1, let $\phi \in \mathcal{P}_m[h, q, \mu]$, where h is univalent in $\mathbb{D}$. If $f \in \mathcal{A}_m$ satisfies the conditions* (2.1) *and*

$$\phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right) \prec h(z), \tag{2.14}$$

where $p(z), p_1(z), p_2(z)$ and $p_3(z)$ are given by (2.3)-(2.6), respectively, then the result (2.7) holds.

Corollary 2.3 *Under the hypothesis of Theorem 2.1, let $\phi \in \mathcal{P}_m[h, q_\rho, \mu]$, where h is univalent in $\mathbb{D}$ and for some $\rho \in (0,1)$, $q_\rho(z) = q(\rho z)$. If the function $f \in \mathcal{A}_m$ and q_ρ satisfies the conditions (2.13) and (2.14), then the result (2.7) holds.*

The next result yields the best dominant of the subordination (2.14) .

Corollary 2.4 *Let h be univalent in $\mathbb{D}$. Also let $\phi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ be analytic in $\mathbb{D}$ and ψ be defined by the transformation (2.12). Suppose that*

$$\psi\left(q(z), zq'(z), z^2q''(z), z^3q'''(z); z\right) = h(z) \tag{2.15}$$

has a solution $q(z)$ with $q(0) = 1$. If $f \in \mathcal{A}_m$ satisfies the conditions (2.1) *for this solution $q(z)$, and for $0 < \mu \leq 1$, and $\lambda \in \mathbb{N} \cup \{0\}$, ϕ satisfy (2.14), then the result (2.7) holds and $q(z)$ is the best dominant of (2.14).*

Proof: By using Corollary 2.2, we deduce that q is a dominant of (2.14). Since $q(z)$ satisfies (2.15), so q is also a solution of (2.14). Therefore $q(z)$ will be dominated by all dominants. Hence $q(z)$ is the best dominant. □

Putting $\mu = 1$ in Theorem 2.1, we obtain the following corollary.

Corollary 2.5 *Let for $\lambda \in \mathbb{N} \cup \{0\}$, $\mathcal{D}_m^\lambda$ be defined by (1.3) and $\phi \in \mathcal{P}_m[\Omega, q, 1]$. If $f \in \mathcal{A}_m$ satisfies the conditions*

$$\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0, \quad \left|\frac{z\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)'}{q'(w)}\right| \leq k$$

$$(z \in \mathbb{D}, \quad k \geq 2; \quad w \in \partial\mathbb{D} \setminus E(q))$$

and

$$\left\{\phi\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}, \frac{\mathcal{D}_m^{\lambda+1} f(z)}{z^m}, \frac{m\mathcal{D}_m^{\lambda+2} f(z)}{z^m}, \frac{m^2\mathcal{D}_m^{\lambda+3} f(z)}{z^m}; z\right) : z \in \mathbb{D}\right\} \subset \Omega,$$

then

$$\frac{\mathcal{D}_m^\lambda f(z)}{z^m} \prec q(z) \qquad (z \in \mathbb{D}).$$

If we put $\lambda = 0$ in Theorem 2.1, we get our next result.

Corollary 2.6 *Let for $0 < \mu \leq 1$, $\phi \in \mathcal{P}_m[\Omega, q, \mu]$. If $f \in \mathcal{A}_m$ satisfies the conditions*

$$\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)}\right) \geq 0, \quad \left|\frac{z\left(\left(\frac{f(z)}{z^m}\right)^\mu\right)'}{q'(w)}\right| \leq k$$

$$(z \in \mathbb{D}, \quad k \geq 2; \quad w \in \partial\mathbb{D} \setminus E(q))$$

and

$$\{\phi\left(r(z), r_1(z), r_2(z), r_3(z); z\right) : z \in \mathbb{D}\} \subset \Omega,$$

where

$$\begin{aligned}
r(z) &= \left(\frac{f(z)}{z^m}\right)^\mu, \\
r_1(z) &= \left(\frac{f(z)}{z^m}\right)^\mu \frac{\mathcal{D}_m f(z)}{f(z)}, \\
r_2(z) &= \left(\frac{f(z)}{z^m}\right)^\mu \left\{\frac{m\mathcal{D}_m^2 f(z)}{f(z)} + (\mu - 1)m\left(\frac{\mathcal{D}_m f(z)}{f(z)}\right)^2\right\},
\end{aligned}$$

and

$$\begin{aligned}
r_3(z) = &\left(\frac{f(z)}{z^m}\right)^\mu \left\{\frac{m^2\mathcal{D}_m^3 f(z)}{f(z)} + 3m^2(\mu - 1)\frac{\mathcal{D}_m f(z)\mathcal{D}_m^2 f(z)}{(f(z))^2}\right. \\
&\left. + m^2(\mu - 1)(\mu - 2)\left(\frac{\mathcal{D}_m f(z)}{f(z)}\right)^3\right\},
\end{aligned}$$

then

$$\left(\frac{f(z)}{z^m}\right)^\mu \prec q(z) \quad (z \in \mathbb{D}).$$

Similar to the proof of Theorem 2.1, on using Lemma 1.2 for the class defined by Definition 1.4, we get following result for the class $Q_m[\Omega, q, \mu]$ defined in the Definition 2.2.

Corollary 2.7 *Let $\phi \in Q_m[\Omega, q, \mu]$. If $f \in \mathcal{A}_m$ and $p(z), p_1(z)$ and $p_2(z)$ are given by (2.3)-(2.5), respectively, then*

$$\{\phi\left(p(z), p_1(z), p_2(z); z\right) : z \in \mathbb{D}\} \subset \Omega \tag{2.16}$$

implies

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec q(z) \quad (z \in \mathbb{D}).$$

If we consider $\lambda = 1$ in the above corollary (2.7), we get following result of Aouf et al. [2, Theorem 6, p. 2]:

Corollary 2.8 *Let $\phi \in Q_m[\Omega, q, \mu]$. If $f \in \mathcal{A}_m$ satisfies the condition*

$$\{\phi\left(r(z), r_1(z), r_2(z); z\right) : z \in \mathbb{D}\} \subset \Omega,$$

where

$$r(z) = \left(\frac{f(z)}{z^m}\right)^{\mu}, \tag{2.17}$$

$$r_1(z) = \left(\frac{f(z)}{z^m}\right)^{\mu} \frac{zf'(z)}{mf(z)}, \tag{2.18}$$

$$r_2(z) = \left(\frac{f(z)}{z^m}\right)^{\mu} \left\{\frac{m\mathcal{D}_m^2 f(z)}{f(z)} + (\mu - 1)m\left(\frac{zf'(z)}{mf(z)}\right)^2\right\}, \tag{2.19}$$

then

$$\left(\frac{f(z)}{z^m}\right)^{\mu} \prec q(z) \quad (z \in \mathbb{D}).$$

3. Superordination and Sandwich-type Results

In this section, we obtain superordination results for the admissible class $\mathcal{P}'_m[\Omega, q, \mu]$ using third-order differential superordination, and sandwich-type result for multivalent functions. For this aim, we first define the class $\mathcal{P}'_m[\Omega, q, \mu]$ as follows:.

Definition 3.1 *Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap H_n$ with $q'(z) \neq 0$. Then function class $\mathcal{P}'_m[\Omega, q, \mu]$ consists of those function $\phi : \mathbb{C}^4 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the following conditions;*

$$\phi(\alpha, \beta, \gamma, \mu; w) \in \Omega,$$

whenever

$$\alpha = q(w), \quad \beta = \frac{kwq'(w) + \mu m q(w)}{\mu m},$$

$$\mathfrak{Re}\left(\frac{\gamma - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}\right) \leq \frac{1}{k}\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right)$$

and

$$\mathfrak{Re}\left(\frac{\delta - 3(\mu m + 1)\gamma + (3\mu^2 m^2 + 6\mu m + 2)\beta - (\mu m + 1)(\mu m + 2)\alpha}{\beta - \alpha}\right) \leq \frac{1}{k^2}\mathfrak{Re}\left(\frac{w^2 q'''(w)}{q'(w)}\right),$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D}$ and $k \geq n \geq 2$.

Definition 3.2 *Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap H_n$ with $q'(z) \neq 0$. Then function class $Q'_m[\Omega, q, \mu]$ consists of those function $\phi : \mathbb{C}^3 \times \overline{\mathbb{D}} \to \mathbb{C}$ that satisfy the following conditions;*

$$\phi(\alpha, \beta, \gamma; w) \in \Omega,$$

whenever

$$\alpha = q(w), \quad \beta = \frac{kwq'(w) + \mu m q(w)}{\mu m},$$

and

$$\mathfrak{Re}\left(\frac{\gamma - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}\right) \leq \frac{1}{k}\mathfrak{Re}\left(\frac{wq''(w)}{q'(w)} + 1\right)$$

where $z \in \mathbb{D}, w \in \partial\mathbb{D}$ and $k \geq n \geq 1$.

Theorem 3.1 *Let $\phi \in \mathcal{P}'_m[\Omega, q, \mu]$. If $f \in \mathcal{A}_m$ and $\left(\frac{\mathcal{D}^{\lambda}_m f(z)}{z^m}\right)^{\mu} \in Q_1$ satisfy the conditions*

$$\mathfrak{Re}\left(\frac{zq''(z)}{q'(z)}\right) \geq 0 \quad \textit{and} \quad \left|\frac{z\left(\left(\frac{\mathcal{D}^{\lambda}_m f(z)}{z^m}\right)^{\mu}\right)'}{q'(z)}\right| \leq k, \tag{3.1}$$

and

$$\phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right) \quad (z \in \mathbb{D})$$

is univalent, where $p(z), p_1(z), p_2(z)$ and $p_3(z)$ are given by (2.3)-(2.6), respectively, then

$$\Omega \subset \{\phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right) \ : \ z \in \mathbb{D}\} \tag{3.2}$$

implies that

$$q(z) \prec \left(\frac{\mathcal{D}^{\lambda}_m f(z)}{z^m}\right)^{\mu} \quad (z \in \mathbb{D}).$$

Proof: Let $p(z)$ and ψ be given by (2.8) and (2.12), respectively. Since from the Definition 3.1, $\phi \in \mathcal{P}'_m[\Omega, q, \mu]$ so from (2.12) and (3.2), we obtain

$$\Omega \subset \psi\left(p(z), zp'(z), z^2p''(z), z^3p'''(z); z\right)$$

and the admissible condition for ϕ in the Definition 3.1 is equivalent to the admissible condition for ψ as given in Definition 1.6. Hence by the Lemma 1.3, we get

$$q(z) \prec p(z) = \left(\frac{\mathcal{D}^{\lambda}_m f(z)}{z^m}\right)^{\mu}.$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, and $\Omega = h(\mathbb{D})$ for some conformal mapping $h(z)$ of $\mathbb{D}$ onto Ω, then we write the class $\mathcal{P}'_m[h(\mathbb{D}), q, \mu]$ as $\mathcal{P}'_m[h, q, \mu]$. The next theorem is the direct consequence of Theorem 3.1.

Theorem 3.2 *Let $\phi \in \mathcal{P}'_m[h, q, \mu]$ and $h(z)$ be analytic in $\mathbb{D}$. If $f \in \mathcal{A}_m$ satisfy the same conditions as in Theorem 3.1, then*

$$h(z) \prec \phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right)$$

implies that

$$q(z) \prec \left(\frac{\mathcal{D}^{\lambda}_m f(z)}{z^m}\right)^{\mu} \quad (z \in \mathbb{D}).$$

Putting $\mu = 1$ in Theorem 3.2, we obtain the following corollary.

Corollary 3.1 *Let $\phi \in \mathcal{P}'_m[h, q, 1]$ and $h(z)$ be analytic in $\mathbb{D}$. If $f \in \mathcal{A}_m$ satisfy the same conditions as in Theorem 3.1 if $\mu = 1$, then*

$$h(z) \prec \phi\left(\frac{\mathcal{D}^{\lambda}_m f(z)}{z^m}, \frac{\mathcal{D}^{\lambda+1}_m f(z)}{z^m}, \frac{m\mathcal{D}^{\lambda+2}_m f(z)}{z^m}, \frac{m^2\mathcal{D}^{\lambda+3}_m f(z)}{z^m}\right)$$

implies that

$$q(z) \prec \frac{\mathcal{D}^{\lambda}_m f(z)}{z^m} \qquad (z \in \mathbb{D}).$$

Similar to the proof of Theorem 3.1, on using Lemma 1.4 for the class defined by Definition 1.7, we get following result for the class $Q'_m[\Omega, q, \mu]$ defined in Definition 3.2.

Corollary 3.2 *Let $\phi \in Q'_m[\Omega, q, \mu]$. If $f \in \mathcal{A}_m$ and $\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \in Q_1$,*

$$\phi\left(p(z), p_1(z), p_2(z); z\right) \quad (z \in \mathbb{D})$$

is univalent, where $p(z), p_1(z)$ and $p_2(z)$ are given by (2.3)-(2.5), respectively, then

$$\Omega \subset \{\phi\left(p(z), p_1(z), p_2(z); z\right) \ : \ z \in \mathbb{D}\}$$

implies that

$$q(z) \prec \left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \quad (z \in \mathbb{D}).$$

Remark 3.1 *If we let $\lambda = 0$, in the above Corollary 3.2, we get the result of Aouf et al. [2, Theorem 17, p. 5].*

On combining Theorems 2.2 and 3.2, we obtain the following desired sandwich-type result.

Corollary 3.3 *Let h_1, q_1 be analytic in $\mathbb{D}$ and let h_2 be univalent in $\mathbb{D}$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \mathcal{P}_m[h_2, q_2, \mu] \cap \mathcal{P}'_m[h_1, q_1, \mu]$. If $f \in \mathcal{A}_m$, $\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \in Q_1 \cap \mathcal{H}_n$ and*

$$\phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right),$$

where $p(z), p_1(z), p_2(z)$ and $p_3(z)$ are given by (2.3)-(2.6), respectively, is univalent in $\mathbb{D}$ and the conditions (2.1) and (3.1) are satisfied, then

$$h_1(z) \prec \phi\left(p(z), p_1(z), p_2(z), p_3(z); z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec q_2(z) \quad (z \in \mathbb{D}).$$

Corollary 3.4 *Let h_1, q_1 be analytic in $\mathbb{D}$ and let h_2 be univalent in $\mathbb{D}$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in Q_m[h_2, q_2, \mu] \cap Q'_m[h_1, q_1, \mu]$. If $f \in \mathcal{A}_m$, $\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \in Q_1 \cap \mathcal{H}_n$ and*

$$\phi\left(p(z), p_1(z), p_2(z); z\right),$$

where $p(z), p_1(z)$ and $p_2(z)$ are given by (2.3)-(2.5), respectively, is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \phi\left(p(z), p_1(z), p_2(z); z\right) \prec h_2(z)$$

implies

$$q_1(z) \prec \left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec q_2(z) \quad (z \in \mathbb{D}).$$

Remark 3.2 *If we let $\lambda = 0$ in the above Corollary 3.4, we get the result obtained in [2, Theorem 21, p. 6]*

Putting $\mu = 1$ in the above Corollary 3.3, we get following simple result:

Corollary 3.5 *Let h_1, q_1 be analytic in $\mathbb{D}$ and let h_2 be univalent in $\mathbb{D}$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \mathcal{P}_m[h_2, q_2, \mu] \cap \mathcal{P}'_m[h_1, q_1, \mu]$. If $f \in \mathcal{A}_m$, $\frac{\mathcal{D}_m^\lambda f(z)}{z^m} \in Q_1 \cap \mathcal{H}_n$ and*

$$\phi\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}, \frac{\mathcal{D}_m^{\lambda+1} f(z)}{z^m}, \frac{m\mathcal{D}_m^{\lambda+2} f(z)}{z^m}, \frac{m^2\mathcal{D}_m^{\lambda+3} f(z)}{z^m}\right).$$

is univalent in $\mathbb{D}$, then

$$h_1(z) \prec \phi\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}, \frac{\mathcal{D}_m^{\lambda+1} f(z)}{z^m}, \frac{m\mathcal{D}_m^{\lambda+2} f(z)}{z^m}, \frac{m^2\mathcal{D}_m^{\lambda+3} f(z)}{z^m}\right) \prec h_2(z).$$

implies that

$$q_1(z) \prec \frac{\mathcal{D}_m^\lambda f(z)}{z^m} \prec q_2(z).$$

4. Some Applications of the Main Results

In this section, as some applications of our main results, we discuss some important particular cases of Theorem 2.1 and Corollary 2.7. In these special cases, we denote the class $\mathcal{P}_m[\Omega, q, \mu]$ by $\mathcal{P}_m[\Omega, M, \mu]$, $Q_m[\Omega, q, \mu]$ by $Q_m[\Omega, M, \mu]$ when $q(z) = 1 + Mz$ $(M > 0;\ z \in \mathbb{D})$ and we denote $Q_m[\Omega, q, \mu]$ by $Q_m[\mu]$ when $q(z) = \frac{1+z}{1-z}$ $(z \in \mathbb{D})$. These special classes are defined as follows:

Definition 4.1 *The class $\mathcal{P}_m[\Omega, M, \mu]$ of admissible functions consists of those functions $\phi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ that satisfies the following admissibility condition*

$$\phi(\alpha, \beta, L, N, z) \notin \Omega, \tag{4.1}$$

whenever

$$\alpha = 1 + Me^{i\theta}, \quad \beta = \frac{(k + \mu m)Me^{i\theta} + \mu m}{\mu m},$$

$$\mathfrak{Re}\left(\frac{L - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}\right) \geq k$$

and

$$\mathfrak{Re}\left(\frac{N - 3(\mu m + 1)L + (3\mu^2 m^2 + 6\mu m + 2)\beta - (\mu m + 1)(\mu m + 2)\alpha}{\beta - \alpha}\right) \geq 0$$

where $\mathfrak{Re}(Le^{-i\theta}) \geq k(k-1)M$, and $\mathfrak{Re}(Ne^{-i\theta}) \geq 0$ for all $\theta \in [0, 2\pi]$ and $k \in \mathbb{N} \backslash \{1\}$.

Definition 4.2 *([6, Case 1, p. 33]) The class $Q_m[\Omega, M, \mu]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfies the following admissibility condition*

$$\phi(\alpha, \beta, L; z) \notin \Omega,$$

whenever

$$\alpha = 1 + Me^{i\theta}, \quad \beta = \frac{(k + \mu m)Me^{i\theta} + \mu m}{\mu m}$$

and

$$\mathfrak{Re}\left(\frac{L - 2\mu m\beta + \mu m\alpha}{\beta - \alpha}\right) \geq k$$

where $\mathfrak{Re}(Le^{-i\theta}) \geq k(k-1)M$ for all $\theta \in [0, 2\pi]$ and $k \in \mathbb{N}$.

Definition 4.3 *([4, Case 2, p. 34])The class $Q_m[\mu]$ of admissible functions consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ that satisfies the following admissibility condition*

$$\mathfrak{Re}\ \psi\left(\rho i, \frac{k}{\mu m}\sigma + \rho i, \eta + i\zeta; z\right) \leq 0 \ \ (z \in \mathbb{D}),$$

whenever$\rho, \sigma, \eta, \zeta \in \mathbb{R}$, such that

$$\sigma \leq -\frac{k}{2}\left(1 + \rho^2\right), 2k\sigma \leq \eta \leq -\sigma, k \in \mathbb{N}.$$

For these above special classes, we get following results:

Corollary 4.1 *Let $\phi \in \mathcal{P}_m[\Omega, M, \mu]$ and $f \in \mathcal{A}_m$ satisfies*

$$\left|\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \left(\frac{\mathcal{D}_m^{\lambda+1} f(z)}{\mathcal{D}_m^\lambda f(z)} - 1\right)\right| < \frac{kM}{\mu m} \qquad (z \in \mathbb{D}). \tag{4.2}$$

If $\phi : \mathbb{C}^4 \times \mathbb{D} \to \mathbb{C}$ satisfies (2.2), then

$$\left(\frac{\mathcal{D}_m^\lambda f(z)}{z^m}\right)^\mu \prec 1 + Mz.$$

In case when $\Omega = q(\mathbb{D}) = \{w \in \mathbb{C} : |w-1| < M\}$, then above corollary reduces to the following.

Corollary 4.2 *Let $\phi \in \mathcal{P}_m[q(\mathbb{D}), M, \mu]$ and $f \in \mathcal{A}_m$ satisfies (4.2), then*

$$|\phi\left(p(z), p_1(z), p_2(z), p_3(z)\right) - 1| < M$$

implies that

$$|p(z) - 1| < M,$$

where $p(z), p_1(z), p_2(z)$ and $p_3(z)$ are given by (2.3)-(2.6), respectively.

Corollary 4.3 *Let $\phi \in Q_m[\Omega, M, \mu]$ and $f \in \mathcal{A}_m$.If $\phi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ satisfies*

$$|\phi\left(p(z), p_1(z), p_2(z)\right) - 1| < M$$

then

$$|p(z) - 1| < M,$$

where $p(z), p_1(z)$ and $p_2(z)$ are given by (2.3)-(2.5), respectively.

Corollary 4.4 *Let $\phi \in Q_m[\mu]$. Then*

$$\mathfrak{Re}\ \phi\left(p(z), p_1(z), p_2(z); z\right) > 0 \ \Rightarrow \mathfrak{Re}\ p(z) > 0,$$

where $p(z), p_1(z)$ and $p_2(z)$ are given by (2.3)-(2.5), respectively.

If we take $\lambda = 0$ in the above Corollary 4.4, we get following result:

Corollary 4.5 *Let $\phi \in Q_m[\mu]$. Then*

$$\mathfrak{Re}\ \phi\left(r(z), r_1(z), r_2(z); z\right) > 0 \ \Rightarrow \mathfrak{Re}\ r(z) > 0,$$

where $r(z), r_1(z)$ and $r_2(z)$ are given by (2.17)-(2.19), respectively.

Further, we may get following simple result for the class $Q_m[1]$:

Corollary 4.6 *Let $\phi \in Q_m[1]$. Then*

$$\mathfrak{Re}\ \phi\left(\frac{f(z)}{z^m}, \frac{zf'(z)}{mz^m}, \frac{zf'(z)}{mz^m} + \frac{z^2 f''(z)}{mz^m}; z\right) > 0 \ \Rightarrow \mathfrak{Re}\left(\frac{f(z)}{z^m}\right) > 0.$$

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