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# On Relative Uniform Convergence of Triple Sequence of Functions

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ABSTRACT: This paper discusses relative uniform convergence of triple sequence of functions that are defined on a compact domain. Another central idea that is discussed is the regular relative uniform convergence and the Cauchy relative uniform convergence of triple sequence of functions. The idea that a continuous triple sequence defined on a compact domain is relative uniform convergent if and only if it is relative uniform Cauchy has been discussed and established. Then we have introduced the Cesaŕo summability of triple sequences and a theorem regarding triple Cesaŕo summability of bounded relative uniform triple sequence of functions.

Key Words: Triple Sequence, Regular Convergence, Relative uniform convergence, Scale function.

## Contents



## 1. Introduction

Convergence of sequences play an important role in sequence space studies. Various authors have worked on different types of convergences like ideal convergence, statistical convergence, relative uniform convergence, relative modular convergence, relative statistical convergence etc., ([13],[16]) which had direct applications in approximation theory. It was Hardy [11] who first came up with the idea of double sequences that undergo regular convergence. But before Hardy, in 1900, it was Pringsheim [14], who introduced the concept of convergence of double sequences. Sahner et al. [15] took this idea forward in their works on triple sequences. In 1910, Moore [12] introduced the idea of uniform convergence of sequences in functions relative to a scale function. Later on, relative to a scale function, the uniform convergence of double sequence of functions and its properties were studied in detail by Devi and Tripathy [8] in 2021-22.

## 2. Preliminaries

We begin this paper by presenting some basic definitions regarding triple sequences and relative uniform convergence.

**Definition 2.1.** *[15] A function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}(\text{ or } \mathbb{C})$ is called a real (or complex) triple sequence, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, real numbers and complex numbers respectively.*

**Definition 2.2.** *[15] A triple sequence $(x_{nkl})$ is said to be convergent to $L$ in Pringsheim's sense if for every $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that*

$$|x_{nkl} - L| < \varepsilon \quad \forall \quad n \geq N, k \geq N, l \geq N.$$

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**Example 2.3.** *Let*

$$\begin{aligned} x_{nkl} &= kl \quad n=3,\\ &= nl \quad k=5,\\ &= nk \quad l=7,\\ &= 8 \quad otherwise. \end{aligned}$$

*Then* $(x_{nkl}) \to 8$ *in Pringsheim's sense.*

**Definition 2.4.** *[15] A triple sequence* $(x_{nkl})$ *is said to be a Cauchy sequence if for every* $\varepsilon > 0$ *there exists* $N(\varepsilon) \in \mathbb{N}$ *such that,*

$$|x_{pqr} - x_{nkl}| < \varepsilon, \quad \forall\, p \geq n \geq N, q \geq k \geq N, r \geq l \geq N.$$

**Definition 2.5.** *[6] A triple sequence* $(x_{lmn})$ *is said to converge regularly if it is convergent in Pringsheim's sense, in addition the following limits hold:*

$$\begin{aligned} \lim_{n\to\infty} x_{lmn} &= L_{lm}\ (l,m\in\mathbb{N}),\\ \lim_{m\to\infty} x_{lmn} &= L_{ln}\ (l,n\in\mathbb{N}),\\ \lim_{l\to\infty} x_{lmn} &= L_{mn}\ (m,n\in\mathbb{N}). \end{aligned}$$

**Definition 2.6.** *[8] A sequence* $(f_n)$ *of real,single valued functions* $f_n$ *of a real variable* $x$*, ranging over a compact set* $D$ *of real numbers,converges relatively uniformly on* $D$ *in case* $\exists$ *functions* $g$ *and* $\sigma$*, and* $\forall\varepsilon > 0$*,* $\exists$ *an integer* $n_0$ *(dependent on* $\varepsilon$*) such that* $\forall n \geq n_0$*, the inequality,*

$$|g(x) - f_n(x)| < \varepsilon|\sigma(x)|, \quad holds\ \forall\ x \in D.$$

*The function* $\sigma$ *of the above definition is called a scale function.*
*The sequence* $(f_n)$ *is said converge uniformly relative to the scale function* $\sigma$*.*

## 3. Main Results

In this section, we introduce the notion of relative uniform convergence of triple sequence of functions and study their properties.

**Definition 3.1.** *A triple sequence* $(f_{nkl})$ *defined on compact set* $D$ *is said to converge uniformly on* $D$ *to a function* $f$ *if for any* $\varepsilon > 0$ *and for all* $x \in D$*, there exists an integer* $n_0 = n_0(\varepsilon)$ *such that for all* $x \in D$

$$|f_{nkl}(x) - f(x)| < \varepsilon \quad \forall\ \ n,k,l \geq n_0.$$

**Definition 3.2.** *Let* $(f_{nkl})$ *be a triple sequence of single, real valued functions ranging over a compact set* $D$ *of real numbers.* $(f_{nkl})$ *converges relatively uniformly on* $D$*, if there exists functions* $g$ *and* $\sigma$ *defined on* $D$ *and for every* $\varepsilon > 0$*, there exists an integer* $n_0 = n_0(\varepsilon)$ *such that for every* $n,k,l \geq n_0$*, the inequality*

$$|g(x) - f_{nkl}(x)| < \varepsilon|\sigma(x)| \quad holds\ \forall\ \ x \in D.$$

**Lemma 3.3.** *A triple sequence of functions* $(f_{nkl})$ *is uniformly convergent on* $D$ *implies that* $(f_{nkl})$ *is relatively uniformly convergent on* $D$ *with respect to a scale function* $\sigma$*.*

**Remark 3.4.** *The converse is not necessarily true. Consider the following example.*

**Example 3.5.** *Let $0 < a < 1$ and $D = [a, 1]$. Consider the sequence of real valued functions $(f_{nkl})$, $f_{nkl} : [a, 1] \mapsto \mathbb{R}$, $\forall\, n, k, l \in \mathbb{N}$ and $x \in D$,*

$$\begin{aligned} f_{nkl}(x) &= x \quad && \text{for} \quad n = 1, k = 1 \quad \text{and} \quad l \ \text{odd}, \\ & && \text{for} \quad n = 1, l = 1 \quad \text{and} \quad k \ \text{odd}, \\ & && \text{for} \quad l = 1, k = 1 \quad \text{and} \quad n \ \text{odd}, \\ &= 2x && \text{for} \quad n = 1, k = 1 \quad \text{and} \quad l \ \text{even}, \\ & && \text{for} \quad n = 1, l = 1 \quad \text{and} \quad k \ \text{even}, \\ & && \text{for} \quad l = 1, k = 1 \quad \text{and} \quad n \ \text{even}, \\ &= \frac{1}{(n+k+l-3)x} && \qquad\qquad \text{otherwise}. \end{aligned}$$

*where $n, k, l \in \mathbb{N}$.*
*So we find that $(f_{nkl})$ is relatively uniform convergent on $[a, 1]$ with respect to the scale function,*

$$\sigma(x) = x \quad \forall \ \ x \in [a, 1].$$

*However, the function $(f_{nkl})$ is not uniformly convergent on the interval $[a, 1]$.*

**Remark 3.6.**

1. *If $\sigma$ is bounded in a domain zero to infinity then the convergence is equivalent to uniform convergence of triple sequences.*
2. *If a triple sequence is not convergent uniformly but converges relatively with respect to a scale function $\sigma$, then the scale function $\sigma$ is not bounded.*
3. *Uniform convergence of a triple sequence relative to a constant scale function different from zero is equivalent to uniform convergence of triple sequences.*
4. *Uniform convergence of triple sequence relative to a scale function $\sigma$ implies uniform convergence relatively to every function $\tau(x)$ for which $|\tau(x)| > |\sigma(x)|$, $\forall\, x \in D$.*
5. *Uniform convergence of triple sequence relative to a scale function $\sigma$ such that $A \leq |\sigma(x)| \leq B$ where $A$ and $B$ are positive implies uniform convergence of triple sequence.*

**Definition 3.7.** *A triple sequence $(f_{nkl})$ is said to converge regularly uniformly on a compact set $D$ if there exists functions $g$, $f_n$, $g_k$, $h_l$ and for a given $\varepsilon > 0$, there exists an integer $n_0 = n_0(\varepsilon)$ such that $\forall \ x \in D$,*

$$\begin{aligned} |f_{nkl}(x) - g(x)| &< \varepsilon \quad \forall\, n, k, l \geq n_0, \\ |f_{nkl}(x) - f_n(x)| &< \varepsilon \quad \text{for each } n \in \mathbb{N} \text{ and } \forall\, k, l \geq n_0, \\ |f_{nkl}(x) - g_k(x)| &< \varepsilon \quad \text{for each } k \in \mathbb{N} \text{ and } \forall\, n, l \geq n_0, \\ |f_{nkl}(x) - h_l(x)| &< \varepsilon \quad \text{for each } l \in \mathbb{N} \text{ and } \forall\, n, k \geq n_0. \end{aligned}$$

**Definition 3.8.** *A triple sequence $(f_{nkl})$ of single,real valued functions $(f_{nkl})$ ranging over a compact set $D$ of real numbers converges regularly relatively uniformly on $D$ if there exists functions $g$, $f_n$, $g_k$, $h_n$, $\sigma$, $\xi_n$, $\eta_k$, $\chi_l$ defined on $D$ and for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that $\forall \ x \in D$,*

$$\begin{aligned} |f_{nkl}(x) - g(x)| &< \varepsilon|\sigma(x)| \quad \forall\, n, k, l \geq n_0, \\ |f_{nkl}(x) - f_n(x)| &< \varepsilon|\xi_n(x)| \quad \text{for each } n \in \mathbb{N} \text{ and } \forall\, k, l \geq n_0, \\ |f_{nkl}(x) - g_k(x)| &< \varepsilon|\eta_k(x)| \quad \text{for each } k \in \mathbb{N} \text{ and } \forall\, n, l \geq n_0, \\ |f_{nkl}(x) - h_l(x)| &< \varepsilon|\chi_l(x)| \quad \text{for each } l \in \mathbb{N} \text{ and } \forall\, n, k \geq n_0. \end{aligned}$$

When $g = f_n = g_k = h_l = \theta$, the zero function, then we say that the triple sequence $(f_{nkl})$ regular uniformly null and regular relative uniformly null in the above two definitions respectively.

**Definition 3.9.** *A triple sequence of functions* $(f_{nkl})$ *defined on a compact set* $D$ *is said to be Cauchy triple sequence of functions if for a given* $\varepsilon > 0$*, there exists* $n_0 = n_0(\varepsilon) \in \mathbb{N}$ *such that,*

$$|f_{pqr}(x) - f_{nkl}(x)| < \varepsilon \quad \forall \quad p \geq n \geq n_0,\ q \geq k \geq n_0,\ r \geq l \geq n_0.$$

**Definition 3.10.** *A triple sequence of functions* $(f_{nkl})$ *of single,real valued functions ranging over a compact set* $D$ *of real numbers is said to be relatively uniformly cauchy, if there exists a function* $\sigma$ *defined on* $D$ *and for all* $\varepsilon > 0$*, there exists* $n_0 = n_0(\varepsilon) \in \mathbb{N}$ *such that,*

$$|f_{pqr}(x) - f_{nkl}(x)| < \varepsilon|\sigma(x)| \quad \forall \quad p \geq n \geq n_0,\ q \geq k \geq n_0,\ r \geq l \geq n_0.$$

Let $f$ and $f_{nkl}, \forall\, n, k, l \in \mathbb{N}$ belong to $C(D)$, which is the space of all continuous real valued functions on a compact set $D$ of real numbers. Also $||f||$ denotes the usual supremum norm of $f$ in $C(D)$.

**Lemma 3.11.** *A triple sequence of functions* $(f_{nkl})$ *is regularly relatively uniformly convergent on* $D$ *implies that it is regularly convergent on* $D$ *but the converse is not necessarily true.*

The converse part follows from the following example.

**Example 3.12.** *Let* $0 < a < 1$ *be a real number and* $D = [a, 1]$*.Consider the sequence of real valued functions* $(f_{nkl})$*,* $f_{nkl} : [a, 1] \mapsto \mathbb{R}$*, for all* $n, k, l \in \mathbb{N}$*, defined on* $D$ *by*

$$f_{nkl}(x) = \frac{1}{x} \quad \text{for} \quad n = 1, l = 1, k \ \ \text{odd}, k \in \mathbb{N}, \qquad \text{for} \quad k = 1, l = 1, n \ \ \text{odd}, n \in \mathbb{N}, \tag{3.1}$$

$$= -\frac{1}{x} \quad \text{for} \quad n = 1, l = 1, k \ \ \text{even}, k \in \mathbb{N}, \qquad \text{for} \quad k = 1, l = 1, n \ \ \text{even}, n \in \mathbb{N}, \tag{3.2}$$

$$= 0 \quad \text{for} \quad n = 1, k = 1, l \in \mathbb{N},$$

$$= \frac{1}{(n + k + l)x} \qquad \text{otherwise.}$$

$(f_{nkl})$ *is relatively uniformly convergent to* $0$ *with respect to the scale function*

$$\sigma(x) = x, \quad \text{for all } x \in [a, 1].$$

*However one cannot get a scale function that makes (3.1) and (3.2) of* $(f_{nk}(x))$ *convergent.*
*Hence* $(f_{nkl})$ *is not regular relative uniform convergent.*

**Remark 3.13.** *All the results of regular relative uniform convergent sequences will hold true for regular relative uniform null.*

Let ${}_3w(ru)$, ${}_3l_\infty(ru)$, ${}_3c(ru)$, ${}_3c_0(ru)$, ${}_3c^B(ru)$, ${}_3c_0^B(ru)$, ${}_3c^R(ru)$, ${}_3c_0^R(ru)$ denote classes of all relative uniform, bounded relative uniform ,relative uniform convergent, relative uniform null,relative uniform bounded convergent, relative uniform bounded null, regular relative uniform convergent, regular relative uniform null triple sequence of functions defined on a compact domain $D$ .

**Proposition 3.14.** *The classes of triple sequence of functions* ${}_3l_\infty(ru)$*,* ${}_3c(ru)$*,* ${}_3c_0(ru)$*,* ${}_3c^B(ru)$*,* ${}_3c_0^B(ru)$*,* ${}_3c^R(ru)$*,* ${}_3c_0^R(ru)$ *on a compact domain are linear subspaces of* ${}_3w(ru)$*.*

For the following result, we consider the triple sequence of functions $(f_{nkl})$ with elements chosen from the complete subset $B(D)$ of $C(D)$ which is a Banach space with respect to the norm of $C(D)$.

**Theorem 3.15.** *Let* $D$ *be a compact domain and* $f_{nkl} \in B(D)$*. Then the triple sequence of functions* $(f_{nkl})$ *defined on* $D$ *converges relatively uniformly on* $D$ *if and only if it is relatively uniformly cauchy.*

*Proof.* Let the sequence $(f_{nkl})$ converges relatively uniformly on $D$. Then there exists functions $g$ and $\sigma$ defined on $D$ for every $\varepsilon > 0$ , there exists $m_1, m_3 \in \mathbb{N}$ such that

$$|f_{nkl}(x) - g(x)| < \frac{\varepsilon}{2}|\sigma(x)| \quad \forall \ n, k, l \geq m_1,$$
$$|f_{pqr}(x) - g(x)| < \frac{\varepsilon}{2}|\sigma(x)| \quad \forall \ p, q, r \geq m_2,$$

Let $n_0 = max\{\ m_1, m_2\}$ .

$$\begin{aligned} |f_{pqr}(x) - f_{nkl}(x)| \leq |f_{nkl}(x) - g(x)| + |g(x) - f_{pqr}(x)| &< \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| \\ &= \varepsilon|\sigma(x)|. \end{aligned}$$

That is,

$$|f_{pqr}(x) - f_{nkl}(x)| \leq \varepsilon|\sigma(x)| \text{ for all } p \geq n \geq n_0, q \geq k \geq n_0, r \geq l \geq n_0.$$

Hence $(f_{nkl})$ is relatively uniformly cauchy
Conversely, suppose a triple sequence of functions $(f_{nkl})$ is relative uniform Cauchy. Then, there exist a function $\sigma$ defined on $D$ and for every $\varepsilon > 0$ there exists an integer $n_0 = n_0(\varepsilon)$ such that

$$|f_{pqr}(x) - f_{nkl}(x)| < \varepsilon|\sigma(x)| \text{ , for all } p \geq n \geq n_0 \text{ and } q \geq k \geq n_0. \tag{3.3}$$

That is, for all $n, k, l \geq n_0$, $(f_{nkl})$ becomes relative uniform Cauchy and Cauchy implies convergent.

$$|f_{nkl}(x) - f_n(x)| < \varepsilon \quad \text{for each } n \in \mathbb{N} \text{ and } \forall\ k, l \geq n_0. \tag{3.4}$$

$$|f_{nkl}(x) - g_k(x)| < \varepsilon \quad \text{for each } k \in \mathbb{N} \text{ and } \forall\ n, l \geq n_0. \tag{3.5}$$

$$|f_{nkl}(x) - h_l(x)| < \varepsilon \quad \text{for each } l \in \mathbb{N} \text{ and } \forall\ n, k \geq n_0. \tag{3.6}$$

By (3.3 ), the scale function is same for (3.4),(3.5),(3.6), and hence by (3.4)

$$|f_k(x) - f_n(x)| < \varepsilon|\sigma(x)| \quad \forall \ n, k, l \geq n_0.$$

Hence $(f_n)$ is relatively uniformly cauchy.
Every Cauchy sequence of real or complex numbers is convergent. Thus there exist a function $f$ such that

$$|f_n(x) - f(x)| < \frac{\varepsilon}{2}|\sigma(x)| \quad \forall \ n \geq n_0.$$

Hence,

$$\begin{aligned} |f_{nkl}(x) - f(x)| &= |f_{nkl}(x) - f_n(x) + f_n(x) - f(x)| \\ &\leq |f_{nkl}(x) - f_n(x)| + |f_n(x) - f(x)| \\ &< \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| \\ &< \varepsilon|\sigma(x)|. \end{aligned}$$

This implies,

$$|f_{nkl}(x) - f(x)| < \varepsilon|\sigma(x)|, \forall \ n, k, l \geq n_0. \tag{3.7}$$

Similarly we can show that $(g_k)$ and $(h_l)$ is a convergent sequence. That is there exists function $g$ and $h$ on D such that

$$|g_k(x) - g(x)| < \frac{\varepsilon}{2}|\sigma(x)| \quad \forall \ k \geq n_0,$$

$$|h_l(x) - h(x)| < \frac{\varepsilon}{2}|\sigma(x)| \quad \forall \ l \geq n_0,$$

which implies

$$|f_{nkl}(x) - g(x)| < \varepsilon|\sigma(x)|, \forall \ n, k, l \geq n_0, \tag{3.8}$$

and

$$|f_{nkl}(x) - h(x)| < \varepsilon|\sigma(x)|, \forall \ n, k, l \geq n_0. \tag{3.9}$$

From equations (3.7), (3.8) and (3.9), we get $(f_{nkl})$ converges relatively uniformly for all $n, k, l \geq n_0$. Next for all $n, k, l \geq n_0$,

$$\begin{aligned}
|f(x) - g(x)| &= |f(x) - f_n(x) + f_n(x) - f_{nkl}(x) + f_{nkl}(x) - g_k(x) + g_k(x) + g(x)| \\
&\leq |f_n(x) - f(x)| + |f_{nkl}(x) - f_n(x)| + |f_{nkl}(x) - g_k(x)| + |g_k(x) + g(x)| \\
&< \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| + \frac{\varepsilon}{2}|\sigma(x)| \\
&= 2\varepsilon|\sigma(x)|.
\end{aligned}$$

This implies, $f(x) = g(x)$, for all $x \in D$ relative to the scale function $\sigma(x)$.
Similarly we can prove, $f(x) = h(x)$, for all $x \in D$ relative to the scale function $\sigma(x)$.
Hence, $(f_{nkl}(x))$ converges to $f(x) = g(x) = h(x)$ uniformly, for all $x \in D$ relative to the scale function $\sigma(x)$ in Pringsheim's sense. □

**Proposition 3.16.** *Pringsheim's sense convergence does not imply always that it is triple Cesaŕo summable. This is clear from the following example.*

**Example 3.17.** *Consider the sequence $(x_{nkl})$ defined by ,*

$$\begin{aligned}
x_{nkl} &= k^2, \ \textit{for } n = 1, l = 1 \textit{ for all } k, \\
&= l^2, \ \textit{for } n = 1, k = 1 \textit{ for all } l, \\
&= n^2, \ \textit{for } k = 1, l = 1 \textit{ for all } n, \\
&= 0 \ \textit{otherwise} .
\end{aligned}$$

*Then it can be easily verified that $\lim_{n,k,l\mapsto\infty} x_{nkl} = 0$.*
*However,*
$\lim_{p,q,r} \frac{1}{pqr} \sum_{n=1}^{p} \sum_{k=1}^{q} \sum_{l=1}^{r} x_{nkl} = lim_{p,q,r} \frac{1}{pqr} \frac{1}{6}(2p^3 + 2q^3 + 2r^3 + 3p^2 + 3q^2 + 3r^2 + p + q + r - 12)$, *which does not tend to a finite limit.*
*Hence the sequence $(x_{nkl})$ is not Cesaŕo Summable.*

**Theorem 3.18.** *A bounded triple sequence of functions $(f_{nkl})$ is convergent in Pringsheim's sense implies that it is triple Cesaŕo summable to the same limit.*

*Proof.* A triple sequence of functions $(f_{nkl})$ is said to be Cesaŕo summable to $f$ if

$$\lim_{p,q,r\to\infty} \frac{1}{pqr} \sum_{n=1}^{p} \sum_{k=1}^{q} \sum_{l=1}^{r} |f_{nkl}(x) - f(x)| = 0.$$

Let $(f_{nkl})$ be bounded and convergent in Pringsheim's sense. Then, there is a positive integer $M > 0$ such that

$$|f_{nkl}(x)| \leq M \quad \forall n, k, l.$$

Since, $(f_{nkl})$ is convergent in Pringsheim's sense, for a given $\varepsilon > 0$ there exists an integer $n_0$ such that

$$|f_{nkl}(x) - f(x)| < \frac{\varepsilon}{2} \quad n, k, l \geq n_0.$$

Then,

$$\frac{1}{pqr}\sum_{n=1}^{p}\sum_{k=1}^{q}\sum_{l=1}^{r}|f_{nkl}(x)-f(x)| = \frac{1}{pqr}\{\sum_{\substack{n=1\ k=1\ l=1 \\ p,q,r<n_0}}^{p\quad q\quad r}|f_{nkl}(x)-f(x)| + \frac{1}{pqr}\sum_{\substack{n=1\ k=1\ l=1 \\ p,q,r\geq n_0}}^{p\quad q\quad r}|f_{nkl}(x)-f(x)|\}\,.$$

Now, $\forall \ \ p,q,r \geq n_0$ ,

$$\begin{aligned}\frac{1}{pqr}\sum_{n=1}^{p}\sum_{k=1}^{q}\sum_{l=1}^{r}|f_{nkl}(x)-f(x)| &< \frac{1}{pqr}n_0^3M + \frac{1}{pqr}\frac{\varepsilon}{2}(p-n_0)(q-n_0)(r-n_0) \\ &< \frac{1}{pqr}n_0^3M + \frac{\varepsilon}{2}\,.\end{aligned}$$

Let $n_1$ be a positive integer greater than $\frac{\varepsilon pqr}{2M}$ such that,

$$\frac{1}{pqr}n_0^3M < \frac{\varepsilon}{2} \quad \text{where } p,q,r \geq n_1\,.$$

Thus for $p,q,r \geq max(n_0,n_1)$ ,

$$\frac{1}{pqr}\sum_{n=1}^{p}\sum_{k=1}^{q}\sum_{l=1}^{r}|f_{nkl}(x)-f(x)| < \varepsilon\,.$$

This implies,

$$\lim_{p,q,r\to\infty}\frac{1}{pqr}\sum_{n=1}^{p}\sum_{k=1}^{q}\sum_{l=1}^{r}f_{nkl}(x) = f(x)\,.$$

□

**Theorem 3.19.** *A bounded triple sequence of functions is convergent relatively uniformly with respect to a scale function $\sigma$ in Pringsheim's sense implies it is triple Cesaŕo summable relatively uniformly to the same limit with respect to the scale function $\sigma$.*

## 4. Conclusion

In this study, we have expanded Devi and Tripathy's [8] work on relative uniform convergence of double sequences of functions to triple sequences of functions. Relative uniform convergence can be investigated further, which could lead to a new approach in sequence space research.

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