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Common Fixed Point for pairs of Weakly Compatible Mappings in C^*-algebra Valued b-Metric Space

Rishi Dhariwal and Deepak Kumar*

ABSTRACT: In this article, we vouched for some results on common fixed points for two pairs of weakly compatible mappings with a generalized contractive condition satisfying $(E.A.)$ and (CLR) properties in the framework of C^*-algebra valued b-metric spaces. The proven results extend and generalize some of the results in the literature. To verify the confirmed results, we demonstrated some examples.

Key Words: Common fixed points, $(E.A.)$ property, (CLR) property, weakly compatible maps, C^*-algebra valued b-metric space.

Contents

1. Introduction and Preliminaries

Ma et al. [18] introduce the notion of C^*-algebra valued metric space by replacing the set of real numbers with a set of positive elements of C^*-algebra. Later, the same author generalized the concept to C^*-algebra valued b-metric space [19] by weakening the triangle inequality in C^*-algebra valued metric space and proving some fixed point results. They also demonstrated the existence and uniqueness of the operator and integral equation solution in such space. The existence and uniqueness of fixed points and common fixed points in C^*-algebra under various structures have been discussed by many researchers (for reference, see [1] - [15] and references therein).

In this article, we vouched for some results on common fixed points for two pairs of weakly compatible mappings with a generalized contractive condition satisfying $(E.A.)$ and (CLR) properties in the framework of C^*-algebra valued b-metric spaces. The proven results extend and generalize some of the results in the literature. To verify the confirmed results, we demonstrated some examples.

We gave some notations and definitions mentioned in [19], which will be required in the sequel to prove the results. Throughout the paper, by $\mathbb{A}$, we denote a unital C^*- algebra with the unity element $I_{\mathbb{A}}$.

Definition 1.1 *[19] Suppose Γ is a non-empty set. A function $d : \Gamma \times \Gamma \to \mathbb{A}$ satisfies :*

(i) $d(\varphi, \varsigma) \succeq \theta_{\mathbb{A}}$ *and* $d(\varphi, \varsigma) = \theta_{\mathbb{A}}$ *if and only if* $\varphi = \varsigma$;

(ii) $d(\varphi, \varsigma) = d(\varsigma, \varphi)$;

(iii) $d(\varphi, \varsigma) \preceq \chi(d(\varphi, \varepsilon) + d(\varepsilon, \varsigma))$;

for all $\varphi, \varsigma, \varepsilon \in \Gamma$ and $\chi \in \mathbb{A}'$. Then, d is called a C^*-algebra valued b-metric and $(\Gamma, \mathbb{A}, d)$ is called a C^*-algebra valued b-metric space.

* Corresponding author

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Definition 1.2 *[19] A sequence $\{\varphi_n\}$ in $(\Gamma, \mathbb{A}, d)$ is said to be*

1. (i) *convergent with respect to $\mathbb{A}$, if for given $\epsilon > 0$, there exists a positive integer k such that $\|d(\varphi_n, \varphi)\| < \epsilon$ for all $n > k$;*

   (ii) *Cauchy sequence with respect to $\mathbb{A}$, if for any $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $\|d(\varphi_n, \varphi_m)\| < \epsilon$ for all $n, m > k$.*

2. *$(\Gamma, \mathbb{A}, d)$ is called a complete C^*-algebra valued b-metric space if every Cauchy sequence with respect to $\mathbb{A}$ is convergent.*

Definition 1.3 *[8] Let f and g be two self-mappings of a metric space (Γ, d). Then, the pair (f, g) is said to satisfy $(E.A.)$ property if there exists a sequence $\{\varphi_n\}$ in Γ such that*

$$\lim_{n\to\infty} f\varphi_n = \lim_{n\to\infty} g\varphi_n = \tau \quad \textit{for some} \quad \tau \in \Gamma.$$

Definition 1.4 *[16] Let f and g are two self-mappings of a metric space (Γ, d). Then, the pair (f, g) is said to satisfy $(CLR)_f$ property if there exists a sequence $\{\varphi_n\} \in \Gamma$ such that*

$$\lim_{n\to\infty} f\varphi_n = \lim_{n\to\infty} g\varphi_n = f\tau \quad \textit{for some} \quad \tau \in \Gamma.$$

From the above, if $f\Gamma$ is a closed subspace of Γ in $(E.A.)$ property, it must satisfy the $(CLR)_f$ property.

Definition 1.5 *[17] Let f and g be two self-mappings of a metric space (Γ, d). Then, the pair (f, g) is said to be weakly compatible if they commute at coincidence points.*

2. Main Result

This section discussed some theorems on common fixed points in C^*-algebra valued b-metric space.

2.1. Common Fixed Point Theorem.

Theorem 2.1 *Let $(\Gamma, \mathbb{A}, d)$ be C^*-algebra valued b-metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions:*

(i) *$A(\Gamma) \subseteq g(\Gamma)$ and $B(\Gamma) \subseteq f(\Gamma)$;*

(ii) *for every $\varphi, \varsigma \in \Gamma$, $\vartheta, a, b \in \mathbb{A}$ with $\|\vartheta\| < 1$ and $\|a\|, \|b\| < \frac{1}{2}$,*

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi, \varsigma)\vartheta;$$

where,

$$m(\varphi, \varsigma) \in \{d(f\varphi, g\varsigma), a(d(f\varphi, A\varphi) + d(g\varsigma, B\varsigma)), b(d(f\varphi, B\varsigma) + d(g\varsigma, A\varphi))\}.$$

If one of $f(\Gamma)$, $g(\Gamma)$, $A(\Gamma)$ and $B(\Gamma)$ is a complete subspace of Γ, then the pairs (A, f) and (B, g) have a coincidence point. Moreover, if the pairs (A, f) and (B, g) are weakly compatible, then the mappings A, B, f and g have a unique common fixed point in Γ.

Proof: Let $\varphi_0 \in \Gamma$ be an arbitrary point. From (i), we can construct a sequence $\{\varsigma_n\}$ in Γ as follow:

$$\varsigma_{2n+1} = A\varphi_{2n} = g\varphi_{2n+1} \quad \text{and} \quad \varsigma_{2n+2} = B\varphi_{2n+1} = f\varphi_{2n+2}.$$

Define $d_n = d(\varsigma_n, \varsigma_{n+1})$. Suppose that $d_{2n} = \theta_{\mathbb{A}}$ i.e. $d(\varsigma_{2n}, \varsigma_{2n+1}) = \theta_{\mathbb{A}}$ for some n. Then, $A\varphi_{2n} = g\varphi_{2n+1} = B\varphi_{2n-1} = f\varphi_{2n}$. Thus, A and f have coincidence point. Hence, the result proved. Now, suppose that $d_{2n} \succ \theta_{\mathbb{A}}$ for all $n \in \mathbb{N}$. Then, substitute $\varphi = \varphi_{2n}$ and $\varsigma = \varphi_{2n+1}$ in condition (ii), we have

$$d(A\varphi_{2n}, B\varphi_{2n+1}) \preceq \vartheta^* m(\varphi_{2n}, \varphi_{2n+1})\vartheta; \tag{2.1}$$

where,

$$\begin{aligned} m(\varphi_{2n},\varphi_{2n+1}) &\in \{d(f\varphi_{2n},g\varphi_{2n+1}), a(d(f\varphi_{2n},A\varphi_{2n})+d(g\varphi_{2n+1},B\varphi_{2n+1})), \\ &\quad b(d(f\varphi_{2n},B\varphi_{2n+1})+d(g\varphi_{2n+1},A\varphi_{2n}))\} \\ &\in \{d(\varsigma_{2n},\varsigma_{2n+1}), a(d(\varsigma_{2n},\varsigma_{2n+1})+d(\varsigma_{2n+1},\varsigma_{2n+2})), \\ &\quad b(d(\varsigma_{2n},\varsigma_{2n+2})+d(\varsigma_{2n+1},\varsigma_{2n+1}))\} \\ &\in \{d_{2n}, a(d_{2n}+d_{2n+1}), bd(\varsigma_{2n},\varsigma_{2n+2})\}. \end{aligned}$$

Following cases arises:

Case (i) Substitute $m(\varphi_{2n},\varphi_{2n+1}) = d_{2n}$ in (2.1), we get

$$d_{2n+1} \preceq \vartheta^* d_{2n} \vartheta.$$

In general, we have

$$d_n \preceq \vartheta^*(d_{n-1})\vartheta \quad \text{for all} \quad n \in \mathbb{N},$$

i.e

$$\begin{aligned} d(\varsigma_n,\varsigma_{n+1}) &\preceq (\vartheta^*) d(\varsigma_{n-1},\varsigma_n)\vartheta \\ &\preceq (\vartheta^*)^2 d(\varsigma_{n-2},\varsigma_{n-1})\vartheta^2 \\ &\dots \\ &\preceq (\vartheta^*)^n d(\varsigma_0,\varsigma_1)\vartheta^n. \end{aligned}$$

For any $p \in \mathbb{N}$, we get

$$\begin{aligned} d(\varsigma_{m+p},\varsigma_m) &\preceq \chi(d(\varsigma_{m+p},\varsigma_{m+p-1}) + d(\varsigma_{m+p-1},\varsigma_m)) \\ &= \chi d(\varsigma_{m+p},\varsigma_{m+p-1}) + \chi d(\varsigma_{m+p-1},\varsigma_m) \\ &\preceq \chi d(\varsigma_{m+p},\varsigma_{m+p-1}) + \chi^2(d(\varsigma_{m+p-1},\varsigma_{m+p-2}) + d(\varsigma_{m+p-2},\varsigma_m)) \\ &= \chi d(\varsigma_{m+p},\varsigma_{m+p-1}) + \chi^2 d(\varsigma_{m+p-1},\varsigma_{m+p-2}) + \chi^2 d(\varsigma_{m+p-2},\varsigma_m) \\ &\preceq \chi d(\varsigma_{m+p},\varsigma_{m+p-1}) + \chi^2 d(\varsigma_{m+p-1},\varsigma_{m+p-2}) + \dots \\ &\quad + \chi^{p-1} d(\varsigma_{m+2},\varsigma_{m+1}) + \chi^{p-1} d(\varsigma_{m+1},\varsigma_m) \\ &\preceq \chi(\vartheta^*)^{m+p-1}\kappa\vartheta^{m+p-1} + \chi^2(\vartheta^*)^{m+p-2}\kappa\vartheta^{m+p-2} + \dots \\ &\quad + \chi^{p-1}(\vartheta^*)^{m+1}\kappa\vartheta^{m+1} + \chi^{p-1}(\vartheta^*)^m\kappa\vartheta^m \\ &= \sum_{k=1}^{p-1} \chi^k(\vartheta^*)^{m+p-k}\kappa\vartheta^{m+p-k} + \chi^{p-1}(\vartheta^*)^m\kappa\vartheta^m \\ &= \sum_{k=1}^{p-1} \left((\vartheta^*)^{m+p-k}\chi^{\frac{k}{2}}\sqrt{\kappa}\right)\left(\sqrt{\kappa}\chi^{\frac{k}{2}}\vartheta^{m+p-k}\right) \\ &\quad + \left((\vartheta^*)^m\chi^{\frac{p-1}{2}}\sqrt{\kappa}\right)\left(\sqrt{\kappa}\chi^{\frac{p-1}{2}}\vartheta^m\right) \\ &= \sum_{k=1}^{p-1} \left((\vartheta)^{m+p-k}\chi^{\frac{k}{2}}\sqrt{\kappa}\right)^*\left(\sqrt{\kappa}\chi^{\frac{k}{2}}\vartheta^{m+p-k}\right) \\ &\quad + \left((\vartheta)^m\chi^{\frac{p-1}{2}}\sqrt{\kappa}\right)^*\left(\sqrt{\kappa}\chi^{\frac{p-1}{2}}\vartheta^m\right) \\ &= \sum_{k=1}^{p-1} \left|\sqrt{\kappa}\chi^{\frac{k}{2}}\vartheta^{m+p-k}\right|^2 + \left|\sqrt{\kappa}\chi^{\frac{p-1}{2}}\vartheta^m\right|^2 \end{aligned}$$

$$\begin{aligned}
&\preceq \sum_{k=1}^{p-1} \|\sqrt{\kappa}\chi^{\frac{k}{2}}\vartheta^{m+p-k}\|^2 I + \|\sqrt{\kappa}\chi^{\frac{p-1}{2}}\vartheta^{m}\|^2 I \\
&\preceq \|\sqrt{\kappa}\|^2 \sum_{k=1}^{p-1} \|\vartheta\|^{2(m+p-k)}\|\chi\|^k I + \|\sqrt{\kappa}\|^2\|\chi\|^{p-1}\|\vartheta^m\|^2 I \\
&= \|\kappa\|\|\vartheta\|^{2(m+p)} \frac{\|\chi\|((\|\chi\|\|\vartheta\|^{-2})^{p-1}) - 1}{\|\chi\| - \|\vartheta\|^2} I + \|\kappa\|\|\chi\|^{p-1}\|\vartheta^m\|^2 I \\
&\preceq \|\kappa\| \frac{\|\chi\|^p\|\vartheta\|^{2(m+1)}}{\|\chi\| - \|\vartheta\|^2} I + \|\kappa\|\|\chi\|^{p-1}\|\vartheta\|^{2m} I \\
&\to 0 (m \to \infty);
\end{aligned}$$

where $|\kappa| = d(\varsigma_0, \varsigma_1)$ for some $\kappa \in \mathbb{A}_+$ and $I_{\mathbb{A}}$ is the unity element in $\mathbb{A}$. Hence, $\{\varsigma_n\}$ is a Cauchy sequence.

Case (ii) Substitute $m(\varphi_{2n}, \varphi_{2n+1}) = a(d_{2n} + d_{2n+1})$ in (2.1), we have

$$\begin{aligned}
d_{2n+1} &\preceq \vartheta^*(a(d_{2n} + d_{2n+1}))\vartheta \\
&= \vartheta^* a d_{2n}\vartheta + \vartheta^* a d_{2n+1}\vartheta \\
(1 - \vartheta^* a\vartheta) d_{2n+1} &\preceq \vartheta^* a d_{2n}\vartheta \\
d_{2n+1} &\preceq \frac{\vartheta^* a\vartheta}{1 - \vartheta^* a\vartheta} d_{2n} = \eta d_{2n};
\end{aligned}$$

where, $\eta = \frac{\vartheta^* a\vartheta}{1-\vartheta^* a\vartheta}$ with $\|\eta\| < 1$. On the similar argument, we can conclude that $d_{2n} \preceq \eta d_{2n-1}$, $d_{2n-1} \preceq \eta d_{2n-2}$ and so on. In general, we have

$$d_n \preceq \eta d_{n-1} \quad \text{for all} \quad n \in \mathbb{N},$$

On the similar lines as Case (i), we have $\{\varsigma_n\}$ is a Cauchy sequence.

Case (iii) Substitute $m(\varphi_{2n}, \varphi_{2n+1}) = b(d(\varsigma_{2n}, \varsigma_{2n+2})$, we have

$$\begin{aligned}
d_{2n+1} &\preceq \vartheta^* b(d(\varsigma_{2n}, \varsigma_{2n+2})\vartheta \\
&\preceq \vartheta^* b d_{2n}\vartheta + \vartheta^* b d_{2n+1}\vartheta \\
(1 - \vartheta^* b\vartheta) d_{2n+1} &\preceq \vartheta^* b d_{2n}\vartheta \\
d_{2n+1} &\preceq \frac{\vartheta^* a\vartheta}{1 - \vartheta^* a\vartheta} d_{2n} = \eta d_{2n};
\end{aligned}$$

where, $\eta = \frac{\vartheta^* a\vartheta}{1-\vartheta^* a\vartheta}$ with $\|\eta\| < 1$. In general, we have

$$d_n \preceq \eta d_{n-1} \quad \text{for all} \quad n \in \mathbb{N},$$

On the similar lines of Case (i), we have $\{\varsigma_n\}$ is a Cauchy sequence.

Since $f\Gamma$ is a complete subspace of Γ. Therefore, sequence $\{\varsigma_n\}$ is contained in $f\Gamma$ and has a limit in $f\Gamma$, say ϱ. Let $\nu \in f^{-1}\varrho$, then $f\nu = \varrho$.

Next, we shall show that $A\nu = \varrho$. Assume that, $A\nu \neq \varrho$. Substituting $\varphi = \nu$ and $\varsigma = \varphi_{n-1}$ in contractive condition (ii), we get

$$d(A\nu, B\varphi_{n-1}) \preceq \vartheta^* m(\nu, \varphi_{n-1})\vartheta; \tag{2.2}$$

where,

$$\begin{aligned}
m(\nu, \varphi_{n-1}) &\in \{d(f\nu, g\varphi_{n-1}), a(d(f\nu, A\nu) + d(g\varphi_{n-1}, B\varphi_{n-1})), \\
&\quad b(d(f\nu, B\varphi_{n-1}) + d(g\varphi_{n-1}, A\nu))\} \\
&\in \{d(f\nu, \varsigma_{n-1}), a(d(f\nu, A\nu) + d(\varsigma_{n-1}, \varsigma_n)), b(d(f\nu, \varsigma_n) + d(\varsigma_{n-1}, A\nu))\}.
\end{aligned}$$

Taking limit as $n \to \infty$ in (2.2), we get

$$\begin{aligned}\lim_{n\to\infty} m(\nu, \varphi_{n-1}) &\in \{d(\varrho,\varrho), a(d(\varrho, A\nu)+d(\varrho,\varrho)), b(d(\varrho,\varrho)+d(\varrho, A\nu))\}\\ &\in \{\theta_{\mathbb{A}}, ad(\varrho, A\nu), bd(\varrho, A\nu)\}.\end{aligned}$$

Following cases arises:

Case (i) : Substitute $\lim_{n\to\infty} m(\nu, \varphi_{n-1}) = \theta_{\mathbb{A}}$ and taking norm on both side in (2.2), we get $\|d(A\nu, \varrho)\| \leq 0$.

Case (ii) : Substitute $\lim_{n\to\infty} m(\nu, \varphi_{n-1}) = ad(\varrho, A\nu)$ and taking norm both side in (2.2), we get $\|d(A\nu, \varrho)\| \leq \|\vartheta\|^2\|a\|\|d(\varrho, A\nu)\|$; which is a contradiction.

Case (iii) : Substitute $\lim_{n\to\infty} m(\nu, \varphi_{n-1}) = bd(\varrho, A\nu)$ and taking norm on both side in (2.2), we get $\|d(A\nu, \varrho)\| \leq \|\vartheta\|^2\|b\|\|d(\varrho, A\nu)\|$; which is a contradiction. Hence, from all the above cases, $A\nu = \varrho$.

Thus, we have $f\nu = \varrho = A\nu$ where ν is the coincidence point of the pair (A, f). Since, $A\Gamma \subseteq g\Gamma$, $A\nu = \varrho$ implies $\varrho \in g\Gamma$. Let $\omega \in g^{-1}\varrho$, then $g\omega = \varrho$. Using the same argument as above, we can easily verify that $B\omega = g\omega = \varrho$, i.e. ω is the coincidence point of the pair (B, g). The same result can be verified assuming $g\Gamma$ is complete instead of $f\Gamma$. Now, if $B(\Gamma)$ is complete, then by (i), $\varrho \in B(\Gamma) \subseteq f(\Gamma)$. Similarly, if $A(\Gamma)$ is complete $\varrho \in A(\Gamma) \subseteq g(\Gamma)$. Since the pair (A, f) and (B, g) are weakly compatible. Therefore,

$$\varrho = A\nu = f\nu = g\omega = B\omega,$$

then,

$$\begin{aligned} g\varrho &= gB\omega = Bg\omega = B\varrho,\\ f\varrho &= fA\nu = Af\nu = A\varrho.\end{aligned}$$

We claim that $B\varrho = \varrho$. If possible, let $Bu \neq \varrho$.

$$d(\varrho, B\varrho) = d(A\nu, B\varrho) \preceq \vartheta^* m(\nu, \varrho)\vartheta; \tag{2.3}$$

where,

$$\begin{aligned} m(\nu, \varrho) &\in \{d(f\nu, g\varrho), a(d(f\nu, A\nu)+d(g\varrho, B\varrho)), b(d(f\nu, B\varrho)+d(g\varrho, A\nu))\}\\ &\in \{d(\varrho, B\varrho), \theta_{\mathbb{A}}, b(d(\varrho, B\varrho)+d(B\varrho, \varrho))\}.\end{aligned}$$

The following cases arise:

Case (i) : Substitute $m(\nu, \varrho) = \theta_{\mathbb{A}}$ and taking norm on both side in (2.3), we get $\|d(B\varrho, \varrho)\| \leq 0$.

Case (ii) : Substitute $m(\nu, \varrho) = d(\varrho, B\varrho)$ and taking norm on both side in (2.3), we get $\|d(B\varrho, \varrho)\| \leq \|\vartheta\|^2\|d(\varrho, B\varrho)\|$; which is a contradiction.

Case (iii) : Substitute $m(\nu, \varrho) = 2b\, d(\varrho, B\varrho)$ and taking norm on both side in (2.3), we get $\|d(B\varrho, \varrho)\| \leq 2\|\vartheta\|^2\|b\|\|d(\varrho, B\varrho)\|$; which is a contradiction. Hence, from all the above cases, $B\varrho = \varrho$. On the similar lines, we can show that $A\varrho = \varrho$.

Thus, we get $A\varrho = f\varrho = g\varrho = B\varrho = \varrho$. Hence, ϱ is the common fixed point of A, B, f and g.

Next, to prove uniqueness, let ξ be another common fixed point different from ϱ i.e $\xi \neq \varrho$ of A, B, f and g.
Consider,

$$d(\xi, \varrho) = d(A\xi, B\varrho) \preceq \vartheta^* m(\xi, \varrho)\vartheta; \tag{2.4}$$

where,

$$\begin{aligned} m(\xi,\varrho) &\in \{d(f\xi,g\varrho), a(d(f\xi,A\xi)+d(g\varrho,B\varrho)), b(d(f\xi,B\varrho)+d(g\varrho,A\xi))\} \\ &\in \{d(\xi,\varrho), a(d(\xi,\xi)+d(\varrho,\varrho)), b(d(\xi,\varrho)+d(\varrho,\xi))\} \\ &\in \{d(\xi,\varrho), \theta_{\mathbb{A}}, 2bd(\xi,\varrho)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $m(\xi,\varrho)=\theta_{\mathbb{A}}$ in (2.4), we get $\|d(\xi,\varrho)\| \leq 0$.

Case (ii) : Substitute $m(\xi,\varrho)=d(\varrho,\xi)$ and taking norm on both side in (2.4), we get $\|d(\xi,\varrho)\| \leq \|\vartheta\|^2\|d(\varrho,\xi)\|$; which is a contradiction.

Case (iii) : Substitute $m(\xi,\varrho)=2b\,d(\varrho,\xi)$ and taking norm on both side in (2.4), we get $\|d(\xi,\varrho)\| \leq 2\|\vartheta\|^2\|b\|\|d(\varrho,\xi)\|$; which is a contradiction. Hence, from all the above cases, $\xi=\varrho$. □

2.2. Common Fixed Point Theorem Using (E.A.) Property.

Theorem 2.2 *Let $(\Gamma,\mathbb{A},d)$ be C^*-algebra valued b-metric space and A,B,f and g are four self mappings on Γ satisfying the following conditions:*

(i) $A(\Gamma)\subseteq g(\Gamma)$ *and* $B(\Gamma)\subseteq f(\Gamma)$*;*

(ii) for every $\varphi,\varsigma\in\Gamma$, $\vartheta\in\mathbb{A}$ *with* $\|\vartheta\|<1$, *and* $\|a\|,\|b\|<\frac{1}{2}$

$$d(A\varphi,B\varsigma)\preceq\vartheta^* m(\varphi,\varsigma)\vartheta;$$

where,

$$m(\varphi,\varsigma)\in\{d(f\varphi,g\varsigma), a(d(f\varphi,A\varphi)+d(g\varsigma,B\varsigma)), b(d(f\varphi,B\varsigma)+d(g\varsigma,A\varphi))\}.$$

(iii) The pair (A,f) and (B,g) are weakly compatible;

(iv) one of the pair (A,f) or (B,g) satisfy $(E.A.)$ property.

If the range of one of the mapping $f(\Gamma)$ or $g(\Gamma)$ is a closed subspace of Γ, then the mappings A,B,f and g have a unique common fixed point in Γ.

Proof: Firstly, we assume that the pair (B,g) satisfies the $(E.A.)$ property. Then, there exist a sequence $\{\varphi_n\}$ in Γ such that $\lim_{n\to\infty}B(\varphi_n)=\lim_{n\to\infty}g(\varphi_n)=\tau$ for some $\tau\in\Gamma$.

Further, $B(\Gamma)\subseteq f(\Gamma)$ so there exist a sequence $\{\varsigma_n\}$ in Γ such that $B(\varphi_n)=f(\varsigma_n)$. Hence, $\lim_{n\to\infty}f(\varsigma_n)=\tau$. We claim that $\lim_{n\to\infty}A(\varsigma_n)=\tau$. Let if possible $\lim_{n\to\infty}A(\varphi_n)=\tau_1\neq\tau$. Then, substitute $\varphi=\varsigma_n$ and $\varsigma=\varphi_n$ in condition (ii), we have

$$d(A\varsigma_n,B\varphi_n)\preceq\vartheta^* m(\varsigma_n,\varphi_n)\vartheta; \tag{2.5}$$

where,

$$\begin{aligned} m(\varsigma_n,\varphi_n) &\in \{d(f\varsigma_n,g\varphi_n), a(d(f\varsigma_n,A\varsigma_n)+d(g\varphi_n,B\varphi_n)), \\ &\qquad b(d(f\varsigma_n,B\varphi_n)+d(g\varphi_n,A\varsigma_n))\}. \end{aligned}$$

Taking limit as $n\to\infty$ in (2.5), we get

$$\begin{aligned} \lim_{n\to\infty} m(\varsigma_n,\varphi_n) &\in \lim_{n\to\infty}\{d(f\varsigma_n,g\varphi_n), a(d(f\varsigma_n,A\varsigma_n)+d(g\varphi_n,B\varphi_n)), \\ &\qquad b(d(f\varsigma_n,B\varphi_n)+d(g\varphi_n,A\varsigma_n))\} \\ &\in \{\theta_{\mathbb{A}}, ad(\tau,\tau_1), bd(\tau,\tau_1)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $\lim_{n\to\infty} m(\varsigma_n, \varphi_n) = \theta_{\mathbb{A}}$ and taking norm on both side in (2.5), we get $\|d(\tau_1, \tau)\| \leq 0$.

Case (ii) : Substitute $\lim_{n\to\infty} m(\varsigma_n, \varphi_n) = ad(\tau, \tau_1)$ in (2.5) and taking norm on both side, we get $\|d(\tau, \tau_1)\| \leq \|\vartheta\|^2 \|a\| \|d(\tau, \tau_1)\|$; which is a contradiction.

Case (iii) : Substitute $\lim_{n\to\infty} m(\varsigma_n, \varphi_n) = bd(\tau, \tau_1)$ in (2.5) and taking norm on both side, we get $\|d(\tau, \tau_1)\| \leq \|\vartheta\|^2 \|b\| \|d(\tau, \tau_1)\|$; which is a contradiction. Hence, $\tau_1 = \tau$ from all the above cases.

Now, we suppose that $f(\Gamma)$ is closed subspace of Γ and $f\varrho = \tau$ for some $\varrho \in \Gamma$. Subsequently, we get

$$\lim_{n\to\infty} A(\varsigma_n) = \lim_{n\to\infty} B(\varphi_n) = \lim_{n\to\infty} g(\varphi_n) = \lim_{n\to\infty} f(\varsigma_n) = \tau = f\varrho.$$

We claim that $A\varrho = f\varrho$. Substitute $\varphi = \varrho$ and $\varsigma = \varphi_n$ in condition (ii), we have

$$d(A\varrho, B\varphi_n) \preceq \vartheta^* m(\varrho, \varphi_n)\vartheta; \tag{2.6}$$

where,

$$\begin{aligned} m(\varrho, \varphi_n) \in \; & \{d(f\varrho, g\varphi_n), a(d(f\varrho, A\varrho) + d(g\varphi_n, B\varphi_n)), \\ & b(d(f\varrho, g\varphi_n) + d(g\varphi_n, A\varrho))\}. \end{aligned}$$

Taking limit as $n \to \infty$ in (2.6), we get

$$\begin{aligned} \lim_{n\to\infty} m(\varrho, \varphi_n) \in \; & \lim_{n\to\infty} \{d(f\varrho, g\varphi_n), a(d(f\varrho, A\varrho) + d(g\varphi_n, B\varphi_n)), \\ & b(d(f\varrho, g\varphi_n) + d(g\varphi_n, A\varrho))\} \\ \in \; & \{\theta_{\mathbb{A}}, ad(\tau, A\varrho), bd(\tau, A\varrho)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $\lim_{n\to\infty} m(\varrho, \varphi_n) = \theta_{\mathbb{A}}$ and taking norm on both side in (2.6), we get $\|d(A\varrho, \tau)\| \leq 0$.

Case (ii) : Substitute $\lim_{n\to\infty} m(\varrho, \varphi_n) = ad(\tau, A\varrho)$ and taking norm on both side in (2.6), we get $\|d(A\varrho, \tau)\| \leq \|\vartheta\|^2 \|a\| \|d(A\varrho, \tau)\|$; which is a contradiction.

Case (iii) : Substitute $\lim_{n\to\infty} m(\varrho, \varphi_n) = bd(A\varrho, \tau)$ and taking norm on both side in (2.6), we get $\|d(A\varrho, \tau)\| \leq \|\vartheta\|^2 \|b\| \|d(\tau, A\varrho)\|$; which is a contradiction.

Hence, from all above cases, we have $A\varrho = \tau = f\varrho$ i.e ϱ is the coincidence point of the pair (A, f).

Now the weak compatibility of the pair (A, f) implies that $Af\varrho = fA\varrho$ or $A\tau = f\tau$.
Since $A(\Gamma) \subseteq g(\Gamma)$, there exits $\nu \in \Gamma$ such that $A\varrho = g\nu = f\varrho = \tau$. Now, we prove that ν is coincidence point of pair (B, g), i.e $B\nu = g\nu = \tau$. Substitute $\varphi = \varrho$ and $\varsigma = \nu$ in condition (ii), we get

$$d(A\varrho, B\nu) \preceq \vartheta^* m(\varrho, \nu)\vartheta; \tag{2.7}$$

where,

$$\begin{aligned} m(\varrho, \nu) \in \; & \{d(f\varrho, g\nu), a(d(f\varrho, A\varrho) + d(g\nu, B\nu)), b(d(f\varrho, B\nu) + d(g\nu, A\varrho))\} \\ \in \; & \{\theta_{\mathbb{A}}, ad(\tau, B\nu), bd(\tau, B\nu)\}. \end{aligned}$$

The following cases arise:

Case (i) : Substitute $m(\varrho,\nu)=\theta_{\mathbb{A}}$ and taking norm on both side in (2.7), we get $\|d(B\nu,\tau)\|\leq 0$.

Case (ii) : Substitute $m(\varrho,\nu)=ad(\tau,A\varrho)$ and taking norm on both side in (2.6), we get $\|d(B\nu,\tau)\|\leq \|\vartheta\|^2\|a\|\|d(B\nu,\tau)\|$; which is a contradiction.

Case (iii) : Substitute $m(\varrho,\nu)=bd(B\nu,\tau)$ in (2.7), we get $\|d(B\nu,\tau)\|\leq\|\vartheta\|^2\|b\|\|d(\tau,B\nu)\|$; which is a contradiction. Hence, we get $B\nu=\tau$ from all the above cases.

Further, the weak compatibility of pair (B,g) implies that $Bg\nu=gB\nu$, or $B\tau=g\tau$. Therefore, τ is a common coincidence point of A,B,f and g.

Now, we prove that τ is a common fixed point of A,B,f and g. Substitute $\varphi=\varrho$ and $\varsigma=\tau$ in condition (ii), we get

$$d(A\varrho,B\tau)\preceq\vartheta^* m(\varrho,\tau)\vartheta; \tag{2.8}$$

where,

$$\begin{aligned} m(\varrho,\tau) &\in \{d(f\varrho,g\tau),a(d(f\varrho,A\varrho)+d(g\tau,B\tau)),\\ &\quad b(d(f\varrho,B\tau)+d(g\tau,A\varrho))\}\\ &\in \{d(\tau,B\tau),\theta_{\mathbb{A}},2bd(\tau,B\tau)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $m(\varrho,\tau)=\theta_{\mathbb{A}}$ and taking norm on both side in (2.8), we get $\|d(\tau,B\tau)\|\leq 0$.

Case (ii) : Substitute $m(\varrho,\tau)=d(\tau,A\varrho)$ and taking norm on both side in (2.8), we get $\|d(B\tau,\tau)\|\leq \|\vartheta\|^2\|d(B\tau,\tau)\|$; which is a contradiction.

Case (iii) : Substitute $m(\varrho,\tau)=2bd(B\tau,\tau)$ and taking norm on both side in (2.8), we get $\|d(B\tau,\tau)\|\leq 2\|\vartheta\|^2\|b\|\|d(\tau,B\tau)\|$; which is a contradiction. Hence, we get $B\tau=\tau$ from all the above cases.

Thus, $A\tau=B\tau=f\tau=g\tau=\tau$.

Similar arguments arises if we assume that $g(\Gamma)$ is closed subspace of Γ. Similarly, the (E.A.) property of the pair (A,f) will give a similar result.

For uniqueness, let ω is another common fixed point of A,B,f and g. Then, substitute $\varphi=\omega$ and $\varsigma=\tau$ in condition (ii), we get

$$d(\omega,\tau)=d(A\omega,B\tau)\quad\preceq\quad\vartheta^* m(\omega,\tau)\vartheta; \tag{2.9}$$

where,

$$\begin{aligned} m(\omega,\tau) &\in \{d(f\omega,g\tau),a(d(f\omega,A\omega)+d(g\tau,B\tau)),b(d(f\omega,B\tau)+d(g\tau,A\omega))\}\\ &\in \{d(\omega,\tau),\theta_{\mathbb{A}},2bd(\omega,\tau)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $m(\omega,\tau)=\theta_{\mathbb{A}}$ and taking norm on both side in (2.9), we get $\|d(\tau,\omega)\|\leq 0$.

Case (ii) : Substitute $m(\varrho,\tau)=d(\tau,\omega)$ and taking norm on both side in (2.9), we get $\|d(\omega,\tau)\|\leq \|\vartheta\|^2\|d(\omega,\tau)\|$; which is a contradiction.

Case (iii) : Substitute $m(\omega,\tau)=2bd(\omega,\tau)$ and taking norm on both side in (2.8), we get $\|d(\omega,\tau)\|\leq 2\|\vartheta\|^2\|b\|\|d(\tau,\omega)\|$; which is a contradiction. Therefore, we get $\omega=\tau$ from all the above cases. Hence, τ is the unique common fixed point of A,B,f and g. □

2.3. Common Fixed Point Theorem Using (CLR) Property.

Theorem 2.3 *Let $(\Gamma, \mathbb{A}, d)$ be C^*-algebra valued b-metric space and A, B, f and g are four self mappings on Γ satisfying the following conditions:*

(i) *$A(\Gamma) \subseteq g(\Gamma)$ and $B(\Gamma) \subseteq f(\Gamma)$;*

(ii) *for every $\varphi, \varsigma \in \Gamma$ with $\|\vartheta\| < 1$ and $\|a\|, \|b\| < \dfrac{1}{2}$ satisfying*

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi, \varsigma)\vartheta;$$

where,

$$m(\varphi, \varsigma) \in \{d(f\varphi, g\varsigma), a(d(f\varphi, A\varphi) + d(g\varsigma, B\varsigma)), b(d(f\varphi, B\varsigma) + d(g\varsigma, A\varphi))\}.$$

(iii) *The pairs (A, f) and (B, g) are weakly compatible;*

(iv) *one of the pair satisfying (CLR) property.*

Then, the mappings A, B, f and g have a unique common fixed point in Γ.

Proof: Firstly, we suppose that the pair (B, g) satisfies the $(CLR)_B$ property. Then, there exist a sequence $\{\varphi_n\}$ in Γ such that

$$\lim_{n\to\infty} B(\varphi_n) = \lim_{n\to\infty} g(\varphi_n) = B\varphi = \tau,$$

for some $\varphi \in \Gamma$.

Since, $B(\Gamma) \subseteq f(\Gamma)$, we have $B\varphi = f\varrho$, for some $\varrho \in \Gamma$. We claim that $A\varrho = f\varrho = \tau$. Substitute $\varphi = \varrho$ and $\varsigma = \varphi_n$ in condition (ii), we get

$$d(A\varrho, B\varphi_n) \preceq \vartheta^* m(\varrho, \varphi_n)\vartheta; \tag{2.10}$$

where,

$$\begin{aligned} m(\varrho, \varphi_n) \quad &\in \quad \{d(f\varrho, g\varphi_n), a(d(f\varrho, A\varrho) + d(g\varphi_n, B\varphi_n)), \\ &\qquad b(d(f\varrho, B\varphi_n) + d(g\varphi_n, A\varrho))\}. \end{aligned}$$

Taking limit as $n \to \infty$ in (2.10), we get

$$\begin{aligned} \lim_{n\to\infty} m(\varrho, \varphi_n) \quad &\in \quad \lim_{n\to\infty} \{d(f\varrho, g\varphi_n), a(d(f\varrho, A\varrho) + d(g\varphi_n, B\varphi_n)), \\ &\qquad b(d(f\varrho, B\varphi_n) + d(g\varphi_n, A\varrho))\} \\ &\in \quad \{\theta_{\mathbb{A}}, ad(\tau, A\varrho), bd(\tau, A\varrho)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $\lim_{n\to\infty} m(\varrho, \varphi_n) = \theta_{\mathbb{A}}$ and taking norm on both side in (2.10), we get $\|d(A\varrho, \tau)\| \leq 0$.

Case (ii) : Substitute $\lim_{n\to\infty} m(\varrho, \varphi_n) = ad(\tau, A\varrho)$ and taking norm on both side in (2.10), we get $\|d(A\varrho, \tau)\| \leq \|\vartheta\|^2 \|a\| \|d(A\varrho, \tau)\|$; which is a contradiction.

Case (iii) : Substitute $\lim_{n\to\infty} m(\omega, \tau) = bd(A\varrho, \tau)$ in (2.10), we get $\|d(A\varrho, \tau)\| \leq \|\vartheta\|^2 \|b\| \|d(\tau, A\varrho)\|$; which is a contradiction. Hence, we get $A\varrho = \tau$ from all the above cases. Therefore, $A\varrho = f\varrho = B\varphi = \tau$.

Since, $A(\Gamma) \subseteq g(\Gamma)$, there exist $\nu \in \Gamma$ such that $g\nu = A\varrho = f\varrho = \tau$.

Now, we prove that $g\nu = B\nu = \tau$, i.e ν is the coincidence point of (B, g). Substitute $\varphi = \varrho$ and $\varsigma = \nu$ in condition (ii), we get

$$d(A\varrho, B\nu) \preceq \vartheta^* m(\varrho, \nu)\vartheta; \tag{2.11}$$

where,

$$\begin{aligned} m(\varrho,\nu) &\in \{d(f\varrho,g\nu),a(d(f\varrho,A\varrho)+d(g\nu,B\nu)),b(d(f\varrho,B\nu)+d(g\nu,A\varrho))\} \\ &\in \{\theta_{\mathbb{A}},ad(\tau,B\nu),bd(\tau,B\nu)\}. \end{aligned}$$

The following cases arise:

Case (i) : Substitute $m(\tau,B\nu)=\theta_{\mathbb{A}}$ and taking norm on both side in (2.11), we get $\|d(B\nu,\tau)\|\leq 0$

Case (ii) : Substitute $m(\tau,B\nu)=ad(\tau,B\nu)$ and taking norm on both side in (2.11), we get $\|d(B\nu,\tau)\|\leq\|\vartheta\|^2\|a\|\|d(B\nu,\tau)\|$; which is a contradiction.

Case (iii) : Substitute $m(\tau,B\nu)=bd(B\nu,\tau)$ and taking norm on both side in (2.11), we get $\|d(B\nu,\tau)\|\leq\|\vartheta\|^2\|b\|\|d(\tau,B\nu)\|$; which is a contradiction.

Hence, from all the above cases, we get $B\nu=\tau$ i.e $B\nu=g\nu=\tau$, and ν is the coincidence point of B and g.

Further, the weak compatibility of pair (B,g) implies that $Bg\nu=gB\nu$, or $B\tau=g\tau$. Therefore, τ is a common coincidence point of A,B,f and g. Now, We prove that τ is common fixed point of A, B, f and g. Substitute $\varphi=\varrho$ and $\varsigma=\tau$ in condition (ii), we get

$$d(A\varrho,B\tau)\preceq\vartheta^*m(\varrho,\tau)\vartheta; \tag{2.12}$$

where,

$$\begin{aligned} m(\varrho,\tau) &\in \{d(f\varrho,g\tau),a(d(f\varrho,A\varrho)+d(g\tau,B\tau)),b(d(f\varrho,B\tau)+d(g\tau,A\varrho))\} \\ &\in \{d(\tau,B\tau),\theta_{\mathbb{A}},2bd(\tau,B\tau)\}. \end{aligned}$$

Following cases arise:

Case (i) : Substitute $m(\tau,B\tau)=\theta_{\mathbb{A}}$ and taking norm on both side in (2.12), we get $\|d(B\tau,\tau)\|\leq 0$.

Case (ii) : Substitute $m(\tau,B\tau)=d(\tau,B\tau)$ and taking norm on both side in (2.12), we get $\|d(B\tau,\tau)\|\leq\|\vartheta\|^2\|d(B\tau,\tau)\|$; which is a contradiction.

Case (iii) : Substitute $m(\tau,B\tau)=2bd(B\tau,\tau)$ and taking norm on both side in (2.12), we get $\|d(B\tau,\tau)\|\leq 2\|\vartheta\|^2\|b\|\|d(\tau,B\tau)\|$; which is a contradiction. Hence, we get $B\tau=\tau$ from all the above cases.

Thus, $A\tau=B\tau=f\tau=g\tau=\tau$. i.e τ is common fixed point of A,B,f and g.

For uniqueness, let ω is another common fixed point of A,B,f and g. Then, substitute $\varphi=\omega$ and $\varsigma=\tau$ in condition (ii), we get

$$d(\omega,\tau)=d(A\omega,B\tau) \preceq \vartheta^*m(\omega,\tau)\vartheta; \tag{2.13}$$

where,

$$\begin{aligned} m(\omega,\tau) &\in \{d(f\omega,g\tau),a(d(f\omega,A\omega)+d(g\tau,B\tau)),b(d(f\omega,B\tau)+d(g\tau,A\omega))\} \\ &\in \{d(\omega,\tau),\theta_{\mathbb{A}},2bd(\omega,\tau)\}. \end{aligned}$$

The following cases arise:

Case (i) : Substitute $m(\tau,\omega)=\theta_{\mathbb{A}}$ and taking norm on both side in (2.13), we get $\|d(\omega,\tau)\|\leq 0$.

Case (ii) : Substitute $m(\tau,\omega)=d(\tau,\omega)$ and taking norm on both side in (2.13), we get $\|d(\omega,\tau)\|\leq\|\vartheta\|^2\|d(\omega,\tau)\|$; which is a contradiction.

Case (iii) : Substitute $m(\tau,\omega) = 2bd(\omega,\tau)$ and taking norm on both side in (2.13), we get $\|d(\omega,\tau)\| \leq 2\|\vartheta\|^2\|b\|\|d(\tau,\omega)\|$; which is a contradiction. Hence, $\omega = \tau$ from all the above cases.
Hence, τ is the unique common fixed point of A, B, f and g. □

Example 2.1 *Let $\Gamma = [0,2]$ and $\mathbb{A} = \mathbb{C}$. Define $d : \Gamma \times \Gamma \to \mathbb{A}$ by*

$$d(\varphi,\varsigma) = \begin{cases} |\varphi| + |\varsigma| & \text{if } \varphi \neq \varsigma \\ \theta_{\mathbb{A}} & \text{if } \varphi = \varsigma. \end{cases}$$

Then, $(\Gamma, \mathbb{A}, d)$ is C^*-algebra-valued b-metric space.
Define four self maps A, B, f and g on Γ by

$$A(\varphi) = \begin{cases} 0 & \text{if } \varphi \in [0,1] \\ \frac{\varphi}{5} & \text{if } \varphi \in (1,2] \end{cases}, \qquad B(\varphi) = \begin{cases} \varphi & \text{if } \varphi \in [0,1] \\ \frac{\varphi}{2} & \text{if } \varphi \in (1,2], \end{cases}$$

$$g(\varphi) = \begin{cases} 3\varphi & \text{if } \varphi \in [0,1] \\ 7\varphi & \text{if } \varphi \in (1,2] \end{cases}, \quad \text{and} \quad f(\varphi) = \begin{cases} 4\varphi & \text{if } \varphi \in [0,1] \\ 3\varphi & \text{if } \varphi \in (1,2]. \end{cases}$$

Following cases arise:
***Case** (i) : Let $\varphi, \varsigma \in [0,1]$, clearly $A\Gamma \subset g\Gamma$ and $B\Gamma \subset f\Gamma$.*
Now,

$$\begin{aligned} d(A\varphi, B\varsigma) &= \varsigma, \quad d(f\varphi, A\varphi) = 4\varphi, \quad d(f\varphi, B\varsigma) = 4\varphi + \varsigma, \\ d(g\varsigma, B\varsigma) &= 4\varsigma, \quad d(f\varphi, g\varsigma) = 4\varphi + 3\varsigma \quad \text{and} \quad d(g\varsigma, A\varphi) = 3\varsigma. \end{aligned}$$

where,

$$\begin{aligned} m(\varphi,\varsigma) &\in \left(4\varphi + 3\varsigma, \frac{4\varphi + 4\varsigma}{3}, \frac{4\varphi + 4\varsigma}{3}\right) \\ &\in \left(4\varphi + 3\varsigma, \frac{4\varphi + 4\varsigma}{3}\right). \end{aligned}$$

Thus,

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi,\varsigma)\vartheta \quad \forall \quad \varphi,\varsigma \in [0,1] \quad \text{with} \quad \|\vartheta\| \leq 1, a = b = \frac{1}{3}.$$

***Case** (ii) : Let $\varphi, \varsigma \in (1,2]$, clearly $A\Gamma \subset g\Gamma$ and $B\Gamma \subset f\Gamma$.*
Now,

$$\begin{aligned} d(A\varphi, B\varsigma) &= \frac{2\varphi + 5\varsigma}{10}, \quad d(f\varphi, A\varphi) = \frac{16\varphi}{5}, \quad d(f\varphi, B\varsigma) = 3\varphi + \frac{\varsigma}{2}, \\ d(g\varsigma, B\varsigma) &= \frac{15\varsigma}{2}, \quad d(f\varphi, g\varsigma) = 3\varphi + 7\varsigma \, \text{and} \quad d(g\varsigma, A\varphi) = 7\varsigma + \frac{\varphi}{5}. \end{aligned}$$

where,

$$\begin{aligned} m(\varphi,\varsigma) &\in \left(3\varphi + 7\varsigma, \frac{32\varphi + 75\varsigma}{10}, \frac{32\varphi + 75\varsigma}{10}\right) \\ &\in \left(3\varphi + 7\varsigma, \frac{32\varphi + 75\varsigma}{10}\right). \end{aligned}$$

Thus,

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi,\varsigma)\vartheta \quad \forall \quad \varphi,\varsigma \in (1,2] \quad \text{with} \quad \|\vartheta\| \leq 1, a = b = \frac{1}{3}.$$

Also, $f\Gamma$ is a complete subspace of Γ, the pair (A, f) and (B, g) are weakly compatible. Hence, by Theorem (2.1), the mappings A, B, f and g have a unique common fixed point. Indeed, 0 is a common unique fixed point.

Example 2.2 *Let* $\Gamma = [0,2]$ *and* $\mathbb{A} = \mathbb{C}$*. Define* $d : \Gamma \times \Gamma \to \mathbb{A}$ *by*

$$d(\varphi, \varsigma) = \begin{cases} |\varphi| + |\varsigma| & \text{if } \varphi \neq \varsigma \\ \theta_{\mathbb{A}} & \text{if } \varphi = \varsigma. \end{cases}$$

Then, $(\Gamma, \mathbb{A}, d)$ *is* C^**-algebra-valued metric space.*
Define four self maps A, B, f *and* g *on* Γ *by*

$$A(\varphi) = \begin{cases} \varphi & \text{if } \varphi \in [0,1] \\ \frac{\varphi}{5} & \text{if } \varphi \in (1,2] \end{cases}, \qquad B(\varphi) = \begin{cases} 0 & \text{if } \varphi \in [0,1] \\ \frac{\varphi}{2} & \text{if } \varphi \in (1,2] \end{cases}$$

$$g(\varphi) = \begin{cases} 5\varphi & \text{if } \varphi \in [0,1] \\ 4\varphi & \text{if } \varphi \in (1,2] \end{cases}, \qquad f(\varphi) = \begin{cases} 8\varphi & \text{if } \varphi \in [0,1] \\ 7\varphi & \text{if } \varphi \in (1,2]. \end{cases}$$

Firstly, we show that the pair (A, f) *satisfies the* $(E.A.)$ *property. Taking* $\{\varphi_n\}$ *be a sequence in* Γ *such that* $\{\varphi_n\} = \left(\frac{1}{\sqrt{n^3+2n^2+1}}\right)$*. Then,*

$$\lim_{n\to\infty} A\varphi_n = \lim_{n\to\infty} A\left(\frac{1}{\sqrt{n^3+2n^2+1}}\right) = \lim_{n\to\infty}(0) = \theta_{\mathbb{A}}$$

and

$$\lim_{n\to\infty} f\varphi_n = \lim_{n\to\infty} f\left(\frac{1}{\sqrt{n^3+2n^2+1}}\right) = \lim_{n\to\infty}\left(\frac{8}{\sqrt{n^3+2n^2+1}}\right) = \theta_{\mathbb{A}}$$

So, there exist a sequence $\{\varphi_n\}$ *in* Γ *such that* $\lim_{n\to\infty} A\varphi_n = \lim_{n\to\infty} f\varphi_n = \theta_{\mathbb{A}} \in \Gamma$*. Hence, the pair* (A, f) *satisfy* $(E.A.)$ *property. Similarly, we can show that the pair* (B, g) *satisfies the* $(E.A.)$ *property.*
Following cases arise
Case (i) *: Let* $\varphi, \varsigma \in [0,1]$*, clearly* $A\Gamma \subset g\Gamma$ *and* $B\Gamma \subset f\Gamma$*.*
Now,

$$\begin{aligned} d(A\varphi, B\varsigma) &= \varphi, \quad d(f\varphi, A\varphi) = 9\varphi, \quad d(f\varphi, B\varsigma) = 8\varphi, \\ d(g\varsigma, B\varsigma) &= 5\varsigma, \quad d(f\varphi, g\varsigma) = 8\varphi + 5\varsigma \quad \text{and} \quad d(g\varsigma, A\varphi) = \varphi + 5\varsigma. \end{aligned}$$

where,

$$\begin{aligned} m(\varphi, \varsigma) &\in \left(8\varphi + 5\varsigma, \frac{9\varphi + 5\varsigma}{3}, \frac{9\varphi + 5\varsigma}{3}\right) \\ &\in \left(8\varphi + 5\varsigma, \frac{9\varphi + 5\varsigma}{3}\right). \end{aligned}$$

Thus,

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi, \varsigma)\vartheta \quad \forall \quad \varphi, \varsigma \in [0,1] \quad \text{with} \quad \|\vartheta\| = 0.5, a = b = \frac{1}{3}.$$

Case (ii) *: Let* $\varphi, \varsigma \in (1,2]$*, clearly* $A\Gamma \subset g\Gamma$ *and* $B\Gamma \subset f\Gamma$*.*
Now,

$$\begin{aligned} d(A\varphi, B\varsigma) &= \frac{2\varphi + 5\varsigma}{10}, \quad d(f\varphi, A\varphi) = \frac{36\varphi}{5}, \quad d(f\varphi, B\varsigma) = 7\varphi + \frac{\varsigma}{2}, \\ d(g\varsigma, B\varsigma) &= \frac{9\varsigma}{2}, \quad d(f\varphi, g\varsigma) = 7\varphi + 4\varsigma \text{and} \quad d(g\varsigma, A\varphi) = 4\varsigma + \frac{\varphi}{5}. \end{aligned}$$

where,

$$\begin{aligned} m(\varphi, \varsigma) &\in \left(7\varphi + 4\varsigma, \frac{72\varphi + 45\varsigma}{10}, \frac{72\varphi + 45\varsigma}{10}\right) \\ &\in \left(7\varphi + 4\varsigma, \frac{72\varphi + 45\varsigma}{10}\right). \end{aligned}$$

Thus,

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi,\varsigma)\vartheta \quad \forall \quad \varphi,\varsigma \in (1,2] \quad with \quad \|\vartheta\| = .5, a = b = \frac{1}{3}.$$

Also, $f\Gamma$ is a closed subspace of Γ, the pair (A,f) and (B,g) are weakly compatible. Hence, by Theorem (2.2), the mappings A, B, f and g have unique common fixed point. Indeed, 0 is a common unique fixed point.

Example 2.3 *Let $\Gamma = [0,2]$ and $\mathbb{A} = \mathbb{C}$. Define $d : X \times \Gamma \to \mathbb{A}$ by $d(\varphi,\varsigma) = \begin{cases} |\varphi| + |\varsigma| & if \ \varphi \neq \varsigma \\ \theta_{\mathbb{A}} & if \ \varphi = \varsigma. \end{cases}$ Then, $(\Gamma, \mathbb{A}, d)$ is C^*-algebra-valued metric space.*
Define four self maps A, B, f and g on Γ by
$A(\varphi) = \begin{cases} 2\varphi & if \ \varphi \in [0,1] \\ \varphi & if \ \varphi \in (1,2] \end{cases}$, $B(\varphi) = \begin{cases} \varphi & if \ \varphi \in [0,1] \\ .5 & if \ \varphi \in (1,2] \end{cases}$
$g(\varphi) = \begin{cases} 3\varphi & if \ \varphi \in [0,1] \\ 2\varphi & if \ \varphi \in (1,2] \end{cases}$, $f(\varphi) = \begin{cases} 4\varphi & if \ \varphi \in [0,1] \\ 7 & if \ \varphi \in (1,2] \end{cases}$
Firstly, we show that the pair (A,f) satisfies the $(CLR)_A$ property. Taking $\{\varphi_n\}$ be a sequence in Γ such that $\{\varphi_n\} = \left(\frac{1}{n^2+2}\right)$. Then,

$$\lim_{n\to\infty} A\varphi_n = \lim_{n\to\infty} A\left(\frac{1}{n^2+2}\right) = \lim_{n\to\infty}\left(\frac{2}{n^2+2}\right) = \theta_{\mathbb{A}}$$

and

$$\lim_{n\to\infty} f\varphi_n = \lim_{n\to\infty} f\left(\frac{1}{n^2+2}\right) = \lim_{n\to\infty}\left(\frac{4}{n^2+2n+3}\right) = \theta_{\mathbb{A}}$$

So, there exist a sequence $\{\varphi_n\}$ in Γ such that $\lim_{n\to\infty} A\varphi_n = \lim_{n\to\infty} f\varphi_n = A(0) = \theta_{\mathbb{A}}$ for $0 \in \Gamma$. Hence, the pair (A,f) satisfy $(CLR)_A$ property. Similarly, we can show that the pair (B,g) satisfies the $(CLR)_B$ property.
Following cases arise
***Case** (i) : Let $\varphi,\varsigma \in [0,1]$, clearly $A\Gamma \subset g\Gamma$ and $B\Gamma \subset f\Gamma$.*
Now,

$$\begin{aligned} d(A\varphi, B\varsigma) &= 2\varphi + \varsigma, \quad d(f\varphi, A\varphi) = 6\varphi, \quad d(f\varphi, B\varsigma) = 4\varphi + \varsigma, \\ d(g\varsigma, B\varsigma) &= 4\varsigma, \quad d(f\varphi, g\varsigma) = 4\varphi + 3\varsigma \quad and \quad d(g\varsigma, A\varphi) = 2\varphi + 3\varsigma. \end{aligned}$$

where,

$$\begin{aligned} m(\varphi,\varsigma) &\in \left(4\varphi + 3\varsigma, \frac{9(6\varphi + 4\varsigma)}{20}, \frac{9(6\varphi + 4\varsigma)}{20}\right) \\ &\in \left(4\varphi + 3\varsigma, \frac{9(6\varphi + 4\varsigma)}{20}\right). \end{aligned}$$

Thus,

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi,\varsigma)\vartheta \quad \forall \quad \varphi,\varsigma \in [0,1] \quad with \quad \|\vartheta\| = 0.9, a = b = \frac{9}{20}.$$

***Case** (ii) : Let $\varphi,\varsigma \in (1,2]$, clearly $A\Gamma \subset g\Gamma$ and $B\Gamma \subset f\Gamma$.*
Now,

$$\begin{aligned} d(A\varphi, B\varsigma) &= \varphi + 0.5, \quad d(f\varphi, A\varphi) = \varphi + 7, \quad d(f\varphi, B\varsigma) = 7.5, \\ d(g\varsigma, B\varsigma) &= 2\varsigma + .5, \quad d(f\varphi, g\varsigma) = 2\varsigma + 7 and \quad d(g\varsigma, A\varphi) = \varphi + 2\varsigma. \end{aligned}$$

where,

$$\begin{aligned} m(\varphi,\varsigma) &\in (7+2\varsigma, \varphi+2\varsigma+7.5, \varphi+2\varsigma+7.5) \\ &\in (7\varphi+4\varsigma, \varphi+2\varsigma+7.5). \end{aligned}$$

Thus,

$$d(A\varphi, B\varsigma) \preceq \vartheta^* m(\varphi,\varsigma)\vartheta \quad \forall \quad \varphi,\varsigma \in (1,2] \quad with \quad \|\vartheta\| = .9, a=b=\frac{9}{20}.$$

The pair (A, f) and (B, g) are weakly compatible. Hence, by the Theorem (2.3), the mappings A, B, f and g have unique common fixed point. Indeed, 0 is a common unique fixed point.

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3. Declaration

Competing interests

The author declares that they do not have any competing interests.

Conflict of interest

All authors declare no conflict of interest.

Ethical approval

This article contains no studies with human participants or animals performed by authors.

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Rishi Dhariwal,
Department of Mathematics,
Lovely Professional University,
Phagwara, India, 144411.
E-mail address: rishidhariwal51@gmail.com

and

Deepak Kumar,
Department of Mathematics,
Lovely Professional University,
Phagwara, India, 144411.
E-mail address: deepakanand@live.in