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New bounds for spectral radius and the geometric-arithmetic energy of graphs

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ABSTRACT: In this paper, new bounds on the GA-energy of graphs are established. Moreover, we show the our bounds are stronger than some previously known lower and upper bounds in the literature.

Key Words: Spectral radius, Geometric-arithmetic energy.

Contents

1. Introduction

Throughout this paper, $G = (V(G), E(G))$ denotes an undirected finite simple graph without isolated vertices. By n and m we denote the cardinality of the set of vertices of G and the cardinality of the set of edges of G, respectively. We denote by $N(v)$ the set of all vertices adjacent to $v \in V(G)$. The *degree* of vertex $v \in V(G)$ is $d_i = dv_i = |N(v)|$. As usual C_n and K_n denotes the cycle and complete graphs on n vertices, respectively.

The geometric-arithmetic index or GA-index is defined in [14] by

$$GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d_u d_v}}{d_u + d_v}.$$

Suppose $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of adjacency matrix $A(G)$. We know that

$$\det A = \prod_{i=1}^{n} \lambda_i.$$

If $\det(A) = 0$, we call G singular, otherwise we call it non-singular. The *energy* of a graph G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} \mid \lambda_i \mid .$$

This concept was introduced by Gutman and is intensively studied in chemistry, since it can be used to approximate the total π-*electron* energy of a *molecule* (see [4,5,15]).

The geometric-arithmetic matrix (GA-matrix) of a graph G, $Aga(G)$, is defined in [11] as following

$$g_{ij} = \begin{cases} \frac{2\sqrt{d_i d_j}}{d_i + d_j} & \text{if } v_i v_j \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

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We denote the eigenvalues of $A_{ga}(G)$ by $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$. For any odd integer $k \geq 1$, let

$$N_k(G) = \sum_{i=1}^{n} |\kappa_i|^k \tag{1.1}$$

where k may be an odd integer, but also any real-valued number. The special case $k = 1$, is the geometric-arithmetic energy (GA-energy), denoted by $\mathcal{E}_{ga}(G)$. Rodriguez and Sigaretta [11] studied the properties the geometric-arithmetic energy.

Then, in this paper, we establish new bounds for the spectral radius GA-adjacency matrix and the GA- energy. Some of these bounds improve previous results.

2. Preliminaries

In this section, we recall some results that will be used in the sequel.

Lemma 2.1 *([2]) For positive real numbers y_i such that $0 < y_1 \leq \cdots \leq y_i \leq \cdots \leq y_s \leq \cdots \leq y_n$, we have*

$$\sum_{j=1}^{n} y_j - n\sqrt{y_1 y_2 \dots y_n} \geq Q\left(\sqrt{y_s} - \sqrt{y_i}\right)^2 \tag{2.1}$$

where

$$Q = \begin{cases} \frac{2i(n-s+1)}{n+i-s+1} & \text{if } i+s \leq n+1, \\ n-s+1, & \text{if } i+s \geq n+1. \end{cases}$$

Lemma 2.2 *([16]) If $a_1, a_2, \dots, a_n$ are non-negative numbers, then*

$$n\left(\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right) \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n}\sqrt{a_i}\right)^2 \leq n(n-1)\left(\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{1/n}\right). \tag{2.2}$$

Lemma 2.3 *[7] Let $x_1, \dots, x_n$ be non-negative numbers and let $X = \frac{1}{n}\sum_{i=1}^{n} x_i$ and $Y = \left(\prod_{i=1}^{n} x_i\right)^{1/n}$. Then*

$$\frac{1}{n(n-1)}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2 \leq X - Y \leq \frac{1}{n}\sum_{i<j}\left(\sqrt{x_i} - \sqrt{x_j}\right)^2.$$

Lemma 2.4 *[13] If $0 < a < A$, and $a_1, \dots, a_n \in [a, A]$, then*

$$\left(\frac{1}{n}\sum_{i=1}^{n} a_i\right)\left(\frac{1}{n}\sum_{i=1}^{n}\frac{1}{a_i}\right) \leq \frac{(a+A)^2}{4Aa}. \tag{2.3}$$

The proof of the following theorem can be found in [8].

Theorem 2.1 *Suppose $\phi_1 \geqslant \phi_2 \geqslant \cdots \geqslant \phi_n$ be roots of an arbitrary polynomial $\varphi_n(\phi)$ and*

$$\bar{\phi} = \frac{1}{n}\sum_{i=1}^{n}\phi_i,$$

$$\Lambda = n\sum_{i=1}^{n}\phi_i^2 - \left(\sum_{i=1}^{n}\phi_i\right)^2.$$

Then, we have

$$\begin{aligned}
&\bar{\phi} + \frac{1}{n}\sqrt{\frac{\Lambda}{n-1}} \leqslant \phi_1 \leqslant \bar{\phi} + \frac{1}{n}\sqrt{(n-1)\Lambda}, \\
&\bar{\phi} - \frac{1}{n}\sqrt{\frac{i-1}{n-i+1}\Lambda} \leqslant \phi_i \leqslant \bar{\phi} + \frac{1}{n}\sqrt{\frac{n-i}{i}\Lambda}, \quad 2 \leqslant i \leqslant n-1, \\
&\bar{\phi} - \frac{1}{n}\sqrt{(n-1)\Lambda} \leqslant \phi_n \leqslant \bar{\phi} - \frac{1}{n}\sqrt{\frac{\Lambda}{n-1}}.
\end{aligned} \tag{2.4}$$

The next lemma plays a vital role in obtaining the results of this paper.

Lemma 2.5 *[11] For $A_{ga}(G)$ matrix with eigenvalues $\kappa_1 \geqslant \kappa_2 \geqslant \cdots \geqslant \kappa_n$, we have*

$$\sum_{i=1}^{n} \kappa_i = 0, \quad \sum_{i=1}^{n} \kappa_i^2 = tr(A_{ga}^2), \quad \sum_{i=1}^{n} \kappa_i^2 \leq 2m, \quad \textit{and} \quad \sum_{i=1}^{n} \kappa_i^4 = tr(A_{ga}^4).$$

3. Spectral properties of the geometric-arithmetic matrix

In what follows, we give some lower and upper bounds on spectral of the geometric-arithmetic matrix. We first present a relation between κ_1 and $tr(A_{ga}^2)$ in a graph G.

Theorem 3.1 *If G be a graph, then*

$$\kappa_1(G) \leq \sqrt{\frac{(n-1)(tr(A_{ga}^2)}{n}}. \tag{3.1}$$

Proof. Note that $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_n$ are eigenvalues of $A_{ga}(G)$. Using Cauchy-Schwarz inequality we obtain

$$\left(\sum_{i=2}^{n} \kappa_i\right)^2 \leq (n-1)\sum_{i=2}^{n} \kappa_i^2. \tag{3.2}$$

By Lemma 2.5, we have $\sum_{i=2}^{n} \kappa_i = -\kappa_1$ and $\sum_{i=2}^{n} \mu_i^2 = tr(A_{ga}^2) - \kappa_1^2$. Then from (3.2), we have

$$(-\kappa_1)^2 \leq (n-1)(tr(A_{ga}^2) - \kappa_1^2)$$

that is, $\kappa_1 \leq \sqrt{\frac{(n-1)(tr(A_{ga}^2)}{n}}$. □

Corollary 3.1 *([11]) If G be a graph, then $\kappa_1 \leq n-1$.*

Proof. Using $2m = \sum_{i=1}^{n} d_i \leq n\Delta \leq n(n-1)$ and the upper bound $tr(A_{ga}^2) \leq 2m$ [11], we obtain $tr(A_{ga}^2) \leq n(n-1)$ and this leads to $\sqrt{\frac{(n-1)(tr(A_{ga}^2)}{n}} \leq n-1$. It follows from Theorem 3.1 that $\kappa_1 \leq n-1$.
□

Next result is an immediate consequence of Theorem 2.1 and Lemma 2.5.

Lemma 3.1 *If G be a graph of order $n \geq 2$, then*

$$\kappa_1 \geq \sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}}, \tag{3.3}$$

$$\kappa_{n-1} \leq \sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}} \tag{3.4}$$

and

$$-\sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}} \leq \kappa_2 \leq \sqrt{\frac{(n-2)tr(A_{ga}^2)}{2n}},$$

$$-\sqrt{\frac{(n-1)tr(A_{ga}^2)}{n}} \leq \kappa_n \leqslant -\sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}}.$$

Corollary 3.2 *([12]) For any graph G, $GA(G) \geq \frac{n\kappa_1^2}{2(n-1)}$.*

Proof. Using the inequality $tr(A_{ga}^2) \leq 2GA(G)$ [12], we get $\sqrt{\frac{(n-1)(tr(A_{ga}^2)}{n}} \leq \sqrt{\frac{2(n-1)GA(G)}{n}}$, and Theorem 3.1 leads to the desired bound. □

4. Bounds for the geometric-arithmetic energy

In this section, we establish new bounds for the GA-energy. The first result gives a relation between the geometric-arithmetic energy and $tr(A_{ga}^2)$.

Theorem 4.1 *If G be a connected graph of order $n \geq 2$, then*

$$\mathcal{E}_{ga}(G) \geq 2\sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}}. \tag{4.1}$$

Proof: Let $\kappa_1, \ldots, \kappa_p$ be the positive eigenvalues of A_{ga}. By Lemma 2.5, we have

$$\mathcal{E}_{ga}(G) = 2\sum_{i=1}^{p} \kappa_i.$$

We deduce from Lemma 3.1 that

$$\mathcal{E}_{ga}(G) = 2\sum_{i=1}^{p} \kappa_i \geq 2\kappa_1 \geq 2\sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}},$$

as desired. □

Corollary 4.1 *([11]) For any graph connected graph G of order $n \geq 2$, $\mathcal{E}_{ga}(G) \geq \frac{tr(A_{ga}^2)}{n(n-1)}$.*

Proof: As in the proof of Corollary 3.1, we have $tr(A_{ga}^2) \leq n(n-1)$ and hence $\sqrt{tr(A_{ga}^2)} \leq \sqrt{n(n-1)} < 2\sqrt{n(n-1)}$. Hence $2 > \sqrt{\frac{tr(A_{ga}^2)}{n(n-1)}}$. Now Theorem 4.1 implies that $\mathcal{E}_{ga}(G) > \frac{tr(A_{ga}^2)}{n(n-1)}$. □

The following result relates GA-energy and the $tr(A_{ga}^2)$.

Theorem 4.2 *Let G be a connected graph of order $n \geq 2$ and let $\kappa'_1, \ldots, \kappa'_n$ be the eigenvalues of GA-matrix such that $|\kappa'_1| \geq |\kappa'_2| \geq \cdots \geq |\kappa'_n|$. Then*

$$\mathcal{E}_{ga}(G) \leq \sqrt{ntr(A_{ga}^2) - \frac{n}{2}\left(|\kappa'_1| - |\kappa'_n|\right)^2}. \tag{4.2}$$

The bound is sharp for $G \cong \bar{K}_n$, $G \cong \frac{n}{2}K_2$ and $G \cong C_4$.

Proof: Lagrange's inequality [10] implies that

$$0 \leq ntr(A_{ga}^2) - \mathcal{E}_{ga}(G)^2 = n\sum_{i=1}^{n} |\kappa'_i|^2 - \left(\sum_{i=1}^{n} |\kappa'_i|\right)^2 = \sum_{1 \leq i < j \leq n} \left(|\kappa'_i| - |\kappa'_j|\right)^2.$$

It follows that

$$ntr(A_{ga}^2) - \mathcal{E}_{ga}(G)^2 \geq \sum_{i=2}^{n-1} \left(\left(|\kappa'_1| - |\kappa'_i|\right)^2 + \left(|\kappa'_i| - |\kappa'_n|\right)^2\right) + \left(|\kappa'_1| - |\kappa'_n|\right)^2.$$

Jennsen's inequality [9] gives

$$ntr(A_{ga}^2) - \mathcal{E}_{ga}(G)^2 \geq \frac{n-2}{2}\left(|\kappa'_1| - |\kappa'_n|\right)^2 + \left(|\kappa'_1| - |\kappa'_n|\right)^2 = \frac{n}{2}\left(|\kappa'_1| - |\kappa'_n|\right)^2, \tag{4.3}$$

and this leads to the desired result. □

The next result is an immediate consequence of Theorem 4.2.

Corollary 4.2 *([11]) For any graph G of order $n \geq 2$, $\mathcal{E}_{ga}(G) \leq \sqrt{ntr(A_{ga}^2)}$.*

In the following, we give a result relating the geometric-arithmetic energy and $tr(A_{ga}^2)$.

Theorem 4.3 *If G is a connected graph of order $n \geq 2$ and $|\kappa_1| \geq |\kappa_2| \geq \cdots \geq |\kappa_n|$ are the absolute eigenvalues of $A_{ga}(G)$, then*

$$\mathcal{E}_{ga}(G) \geq \sqrt{ntr(A_{ga}^2) - \omega(n)(|\kappa_1| - |\kappa_n|)^2}. \tag{4.4}$$

The equality holds if and only if $|\kappa_1| = |\kappa_2| = \cdots = |\kappa_n|$. Moreover, the equality holds if $G \in \{\overline{K_n}, C_4, \frac{n}{2}K_2\}$.

Proof: For real numbers x_i and y_i and constants x, y, X and Y, such that $1 \leq i \leq n$, $x \leq x_i \leq X$ and $y \leq y_i \leq Y$, it is proved in [1] that

$$\left|n\sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i \sum_{i=1}^{n} y_i\right| \leq \omega(n)(X - x)(Y - y), \tag{4.5}$$

where $\omega(n) = n[\frac{n}{2}]\left(1 - \frac{1}{n}[\frac{n}{2}]\right)$. Equality in (4.5) holds if and only if $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$. Takeing $x_i = y_i = |\kappa_i|$ for each $1 \leq i \leq n$, $x = y = |\kappa_n|$ and $X = Y = |\kappa_1|$ in Inequality (4.5), we get

$$\left|n\sum_{i=1}^{n} (\kappa_i)^2 - \left(\sum_{i=1}^{n} |\kappa_i|\right)^2\right| \leq \omega(n)(|\kappa_1| - |\kappa_n|)^2.$$

Since $\mathcal{E}_{ga}(G) \leq \sqrt{ntr(A_{ga}^2)}$ (see [11]), we get

$$ntr(A_{ga}^2) - \mathcal{E}_{ga}(G)^2 \leq \omega(n)(|\kappa_1| - |\kappa_n|)^2,$$

and this leads to the desired bound.

Since equality in (4.5) holds if and only $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$. Hence, equality in (4.4) holds if and only if $|\kappa_1| = |\kappa_2| = \cdots = |\kappa_n|$. □

Since, $n[\frac{n}{2}]\left(1 - \frac{1}{n}[\frac{n}{2}]\right) \leq \frac{n^2}{4}$, the next result is an immediate consequence of Theorem 4.3.

Corollary 4.3 *[11] For any connected graph G of order $n \geq 2$, $\mathcal{E}_{ga}(G) \geq \sqrt{ntr(A_{ga}^2) - \frac{n^2}{4}(|\kappa_1| - |\kappa_n|)^2}$.*

The following results relate the geometric-arithmetic energy to $tr(A_{ga}^2)$ and $|\det A_{ga}|$.

Theorem 4.4 *If G is a connected graph of order $n \geq 2$, then*

$$\mathcal{E}_{ga}(G) \geq \frac{tr(A_{ga}^2)}{n-1} + (n-1)\left(\frac{(n-1)|\det A_{ga}|}{tr(A_{ga}^2)}\right)^{1/n-1}. \tag{4.6}$$

Proof: Applying the arithmetic-geometric mean inequality, we have

$$E_{ga}(G) = \kappa_1 + \sum_{i=2}^{n} |\kappa_i| \geq \kappa_1 + (n-1)\left(\prod_{i=2}^{n} |\kappa_i|\right)^{1/n-1} = \kappa_1 + (n-1)\left(\frac{|\det A_{ga}|}{\kappa_1}\right)^{1/n-1}.$$

Let us consider the function $g(y)$, as

$$g(y) = y + (n-1)\left(\frac{|\det A_{ga}|}{y}\right)^{1/n-1}.$$

It is easy to observe that for $y \geq |\det A_{ga}|)^{1/n}$, $g(y)$ is increasing. From the above and the fact that $\kappa_1 \geq \frac{tr(A_{ga}^2)}{n-1}$ (see [11]), we arrive at

$$\mathcal{E}_{ga}(G) \geq \frac{tr(A_{ga}^2)}{n-1} + (n-1)\left(\frac{(n-1)|\det A_{ga}|}{tr(A_{ga}^2)}\right)^{1/n-1}.$$

This completes the proof. □

It is obvious that the bound in (4.6) is better than the bound in Corollary 4.1.

In the next theorem, a relationship between the geometric-arithmetic energy and $|\det A_{ga}|$ is provided.

Theorem 4.5 *If G is a connected graph of order $n \geq 2$, then*

$$\mathcal{E}_{ga}(G) \leq \frac{n|\det A_{ga}|^{1/n}\left(|\kappa_1| + |\kappa_n|\right)^2}{4|\kappa_1||\kappa_n|}.$$

Proof: Seeting $a_i = |\kappa_i|$ for $1 \leq i \leq n$, the inequality (2.3) transforms into

$$\frac{1}{n^2}\left(|\kappa_1| + |\kappa_2| + \cdots + |\kappa_n|\right)\left(\frac{1}{|\kappa_1|} + \frac{1}{|\kappa_2|} + \cdots + \frac{1}{|\kappa_n|}\right) \leq \frac{\left(|\kappa_1| + |\kappa_n|\right)^2}{4|\kappa_1||\kappa_n|}. \tag{4.7}$$

By applying the arithmetic-geometric mean inequality to the positive numbers $\frac{1}{|\kappa_1|}, \frac{1}{|\kappa_2|}, \ldots, \frac{1}{|\kappa_n|}$ we get

$$\frac{1}{|\kappa_1|} + \frac{1}{|\kappa_2|} + \cdots + \frac{1}{|\kappa_n|} \geq \frac{n}{|\kappa_1 + \kappa_2 + \cdots + \kappa_n|^{1/n}} = \frac{n}{|\det A_{ga}|^{1/n}}. \tag{4.8}$$

Based on (4.8) and (4.7), we obtain

$$\frac{1}{n}\frac{\mathcal{E}_{ga}(G)}{|\det A_{ga}|^{1/n}} \leq \frac{\left(|\kappa_1| + |\kappa_n|\right)^2}{4|\kappa_1||\kappa_n|}$$

that is, $\mathcal{E}_{ga}(G) \leq \frac{n|\det A_{ga}|^{1/n}(|\kappa_1|+|\kappa_n|)^2}{4|\kappa_1||\kappa_n|}$, as desired. □

The next theorem reveals a connection among the geometric-arithmetic energy, $tr(A_{ga})^2$ and $|\det A_{ga}|$.

Theorem 4.6 *If G is a connected graph of order $n \geq 2$, then*

$$\mathcal{E}_{ga}(G) \leq \sqrt{(n-1)tr(A_{ga})^2 + n\left(|\det A_{ga}|\right)^{2/n}}.$$

Proof: Setting $a_i = \kappa_i^2, i = 1, \ldots, n$, in inequality (2.2), we have

$$nS \leq n\sum_{i=1}^{n}\kappa_i^2 - \left(\sum_{i=1}^{n}|\kappa_i|\right)^2 \leq n(n-1)S \tag{4.9}$$

that is,

$$nS \leq ntr(A_{ga})^2 - \left(\mathcal{E}_{ga}(G)\right)^2 \leq n(n-1)S$$

where $S = \left(\frac{1}{n}\sum_{i=1}^{n}\kappa_i^2 - \left(\prod_{i=1}^{n}\kappa_i^2\right)^{1/n}\right) = \frac{tr(A_{ga})^2}{n} - \left(|\det A_{ga}|\right)^{2/n}$.

□

By the same argument as before and by Inequality (4.9), we can prove the next result.

Corollary 4.4 *([11]) For any graph G of order $n \geq 2$,*

$$\mathcal{E}_{ga}(G) \geq \sqrt{tr(A_{ga})^2 + n(n-1)\left(|\det A_{ga}|\right)^{2/n}}. \tag{4.10}$$

In the next result, we determine a lower bound on the geometric-arithmetic energy in terms of order, $tr(A_{ga}^2)$ and $|\det A_{ga}|$.

Theorem 4.7 *If G is a connected graph of order $n \geq 3$, then*

$$\mathcal{E}_{ga}(G) \geq \sqrt{tr(A_{ga}^2) + n(n-1)\left(|\det A_{ga}|\right)^{2/n} + \frac{4}{(n+1)(n-2)} \sum_{i<j\leq k<l} \left(\sqrt{|\kappa_i||\kappa_j|} - \sqrt{|\kappa_k||\kappa_l|}\right)^2}. \tag{4.11}$$

The equality holds if $G \cong \overline{K_n}$.

Proof: By definition, we have

$$\mathcal{E}_{ga}(G) = \sum_{i=1}^{n} |\kappa_i|^2 + 2\sum_{i<j} |\kappa_i||\kappa_j|. \tag{4.12}$$

Setting $N = \frac{n(n-1)}{2}$ and

$$(x_1, x_2, \ldots, x_N) = (|\kappa_1||\kappa_2|, |\kappa_1||\kappa_3|, \ldots, |\kappa_1||\kappa_n|, \ldots, |\kappa_2||\kappa_n|, \ldots, |\kappa_{n-1}||\kappa_n|)$$

in Lemma 2.3, we obtain

$$\sum_{1\leq i<j\leq n} |\kappa_i||\kappa_j| \geq \frac{n(n-1)}{2}\left(\prod_{i=1}^{N} |\kappa_i|\right)^{2/n} + \frac{2}{n^2-n-2} \sum_{i<j\leq k<l} \left(\sqrt{|\kappa_i||\kappa_j|} - \sqrt{|\kappa_k||\kappa_l|}\right)^2$$

yielding

$$2\sum_{1\leq i<j\leq n} |\kappa_i||\kappa_j| \geq n(n-1)\left(\det A_{ga}\right)^{2/n} + \frac{4}{(n+1)(n-2)} \sum_{i<j\leq k<l} \left(\sqrt{|\kappa_i||\kappa_j|} - \sqrt{|\kappa_k||\kappa_l|}\right)^2.$$

Combining the above inequality with (4.12) leads to the desired inequality. □

Since $\left(\sqrt{|\kappa_i||\kappa_j|} - \sqrt{|\kappa_k||\kappa_l|}\right)^2 \geq 0$, we have

$$\begin{aligned}\mathcal{E}_{ga}(G) &\geq \sqrt{tr(A_{ga}^2) + n(n-1)\left(|\det A_{ga}|\right)^{2/n} + \frac{4}{(n+1)(n-2)} \sum_{i<j\leq k<l} \left(\sqrt{|\kappa_i||\kappa_j|} - \sqrt{|\kappa_k||\kappa_l|}\right)^2} \\ &\geq \sqrt{tr(A_{ga})^2 + n(n-1)\left(|\det A_{ga}|\right)^{2/n}}.\end{aligned}$$

Thus, the bound in (4.11) is better than the bound in (4.10).

The proof of the next lower bound can be found in [11].

$$\mathcal{E}_{ga}(G) \geq n\left(|\det A_{ga}|\right)^{1/n}. \tag{4.13}$$

The following results relate the geometric-arithmetic energy and $|\det A_{ga}|$.

Theorem 4.8 *Let G be a connected graph of order $n \geq 3$ and let $|\kappa_1'| \geq |\kappa_2'| \geq \cdots \geq |\kappa_n'|$ be the absolute eigenvalue of $A_{ga}(G)$. If $A_{ga}(G)$ is a non-singular graph, then*

$$\mathcal{E}_{ga}(G) \geq n\left(|\det A_{ga}|\right)^{1/n} + \left(\sqrt{|\kappa_n'|} - \sqrt{|\kappa_1'|}\right)^2. \tag{4.14}$$

Proof: Setting $a_i = |\kappa_i|$, $s = n$ and $i = 1$, in Inequality (2.1), we get that $Q = 1$ and hence

$$\sum_{i=1}^{n} |\kappa'_i| \geq n \sqrt[n]{|\kappa'_1||\kappa'_2| \dots |\kappa'_n|} + \left(\sqrt{|\kappa'_n|} - \sqrt{|\kappa'_1|}\right)^2$$
$$= n \left(|\det A_{ga}|\right)^{1/n} + \left(\sqrt{|\kappa'_n|} - \sqrt{|\kappa'_1|}\right)^2.$$

□

Since for non-singular matrix $A_{ga}(G)$, we have $\left(\sqrt{|\kappa'_n|} - \sqrt{|\kappa'_1|}\right)^2 > 0$, the bound (4.14) is better than the bound in (4.13) for non-singular graphs.

Theorem 4.9 *Let G be a connected graph of order $n \geq 2$. Then*

$$\mathcal{E}_{ga}(G) \geqslant \frac{(tr(A_{ga}^2))^2}{\sum_{i=1}^{n} (|\kappa_i|)^3}. \tag{4.15}$$

Proof: For $1 \leq i \leq n$, let h_i and k_i be non-negative real numbers. By Hölder's inequality we have

$$\sum_{i=1}^{n} h_i k_i \leqslant \left(\sum_{i=1}^{n} h_i^r\right)^{\frac{1}{r}} \left(\sum_{i=1}^{n} k_i^s\right)^{\frac{1}{r}}. \tag{4.16}$$

If we take $h_i = |\kappa_i|^{\frac{1}{2}}$, $k_i = |\kappa_i|^{\frac{3}{2}}$, $r = 2$ and $s = 2$, in Inequality (4.16), we obtain

$$\sum_{i=1}^{n} |\kappa_i|^2 = \sum_{i=1}^{n} |\kappa_i|^{\frac{1}{2}} \left(|\kappa_i|^3\right)^{\frac{1}{2}} \leqslant \left(\sum_{i=1}^{n} |\kappa_i|\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |\kappa_i|^3\right)^{\frac{1}{2}}. \tag{4.17}$$

Since G is a connected graph of order $n \geq 2$, we have $\sum_{i=1}^{n} |\kappa_i|^3 \neq 0$, and Inequality (4.17) gives

$$\sum_{i=1}^{n} |\kappa_i| \geqslant \frac{\left(\sum_{i=1}^{n} |\kappa_i^2|\right)^2}{\sum_{i=1}^{n} (|\kappa_i|)^3}.$$

This inequality leads to the desired bound. □

Since for any connected graph G of order $n \geq 2$ we have $\sqrt{(tr(A_{ga}^2))(tr(A_{ga}^4))} \geq \sum_{i=1}^{n} (|\kappa_i|)^3$, the next result is an immediate consequence of Theorem 4.9.

Corollary 4.5 *([11]) For any nontrivial connected graph G, $\geq \sqrt{\frac{(tr(A_{ga}^2))^3}{(tr(A_{ga}^4))}}$.*

Theorem 4.10 *Let G be a connected graph of order $n \geq 2$ and let a, b, c be non-negative real numbers such that $4a = b + c + 2$. Then*

$$\mathcal{E}_{ga}(G) \geq \frac{(N_a(G))^2}{\sqrt{N_b(G) N_c(G)}}. \tag{4.18}$$

Proof: We use the following inequality published in [17]. For positive real numbers z_j $j = 1, 2 \dots, n$, and non-negative real numbers a, b, c, such that $4a = b + c + 2$,

$$\left(\sum_{j=1}^{n} (z_j)^a\right)^4 \leq \left(\sum_{j=1}^{n} z_j\right)^2 \sum_{j=1}^{n} (z_j)^b \sum_{j=1}^{n} (z_j)^c. \tag{4.19}$$

Moreover, if $(b,c) \neq (1,1)$, then the equality in (4.19) holds if and only if $z_1 = z_2 = \cdots = z_n$.

Let $\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_\ell$ be the non-zero GA- eigenvalues of the graph G. Since G is a connected graph of order at least two, $\kappa_1 > 0$ and $\kappa_2 < 0$. For $z_j = |\kappa_j|$, $j = 1, 2, \ldots, \ell$, the inequality (4.19) transforms into

$$\left(\sum_{j=1}^{\ell} (|\kappa_j|)^a\right)^4 \leq \left(\sum_{j=1}^{\ell} |\kappa_j|\right)^2 \sum_{j=1}^{\ell} (|\kappa_j|)^b \sum_{j=1}^{\ell} (|\kappa_j|)^c \tag{4.20}$$

that is

$$(N_a)^4 \leq (\mathcal{E}_{ga}(G))^2 N_b N_c$$

and this leads to the desired bound. □

Theorem 4.10 has the following consequence for $b = 0$, $c = 2$ which implies $a = 1$.

Corollary 4.6 *Let G be a connected graph of order $n \geq 2$ and let $A_{ga}(G)$ has τ zero eigenvalues. Then*

$$\mathcal{E}_{ga}(G) \leq \sqrt{2(n-\tau) tr(A_{ga}^2)}. \tag{4.21}$$

Note that if we take, $b = 2$, $c = 4$ in Theorem 4.10, we obtain

$$\mathcal{E}_{ga}(G) \geq \sqrt{\frac{\left(tr(A_{ga}^2)\right)^3}{tr(A_{ga}^4)}}.$$

Conclusion

In this paper, we studied the eigenvalues of the geometric-arithmetic matrix and established bounds for the spectral radius of this matrix. Finally, we obtained new bounds for the geometric-arithmetic energy and shown that some of our bounds improved the previously published bounds.

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