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Boundary Value Problems for Nonlinear Fractional Differential Equations With Ψ-Caputo Fractional

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ABSTRACT: In this present paper, we will envisaged the existence and uniqueness of solutions for a boundary value problem for a nonlinear fractional differential equation involving with ψ-Caputo fractional derivative. Our results are proved under Banach contraction principle and Krasnoselkii's fixed point theorem.

Key Words: Nonlinear fractional differential equation, ψ-Caputo fractional derivative, Boundary value problem.

Contents

1. Introduction

Fractional calculus (FC) and fractional differential equations (FDEs) have emerged as the most important and prominent areas of interdisciplinary interest in recent years. FC has a history of more than 300 years, its applicability in different domains has been realized only recently. In the last three decades, the subject witnessed exponential growth and a number of researchers around the globe are actively working on this topic see [7, 8, 9].
Almeida [1] generalized the definition of Caputo fractional derivative by considering the Caputo fractional derivative of a function with respect to another function ψ and studied some useful properties of the fractional calculus. The advantage of this new definition of the fractional derivative is that a higher accuracy of the model could be achieved by choosing a suitable function ψ.
Recently, in [5] Benlabess, Benbachir and Lakrib gave some sifficient conditions for existence of solutions to the linear fractional boundary value problem:

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t, u(t)),\ t \in J := [0,1], 2 < \alpha \leq 3, \\ D^{\alpha-1}_{0^+}u(0) = 0, D^{\alpha-2}_{0^+}u(1) = 0, u(0) = 0. \end{cases}$$

Where $D^{\alpha}_{0^+}$ is the standard Riemann-Liouville fractional differential operator of order α and the non linear function $f : [0,1] \times [0,+\infty) \to \mathbb{R}$ is continuous.
Motivated by the mentioned works, this paper generalize the results obtained in [5] involving $\psi-$Caputo type fractional derivative of order $2 < \alpha \leq 3$ and it deals the existence of solutions for the following nonlinear fractional boundary value problem:

$$\begin{cases} {}^{C}D^{\alpha;\psi}_{0^+}u(t) = f(t, u(t)),\ t \in J := [0,1], \\ u(0) = u'(0) = 0 \text{ and } {}^{C}D^{\alpha-1;\psi}_{0^+}u(1) = 1. \end{cases} \tag{1.1}$$

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Where ${}^C D_{0^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $2 < \alpha \leq 3$ and
$f : J \times [0, \infty) \to \mathbb{R}$ is a given continuous function.
The paper is organized as follows. In section 2, we introduce notations, definitions and preliminary facts which are used .In section 3, we introduce the basic assumptions and the state the main result on the existence and uniqueness of nonlinear fractional boundary value problem.

2. Preliminaries

We start this section by introducing some necessary definition and basic results required for further developments. We denoted by $\mathcal{C}(J, \mathbb{R})$ the Banach space of all continuous functions from $J = [0, 1]$ into $\mathbb{R}$ with the norm $||u||_\infty = \sup\limits_{t \in J} |u(t)|$.

Definition 2.1. *(ψ-Riemann-Liouville fractional integral [4])*
Let $\alpha > 0$, f be an integrable function defined on $[a, b]$ and $\psi : [a, b] \to \mathbb{R}$ that is an increasing differentiable function such that $\psi'(t) \neq 0$, for all $t \in [a, b]$.
The ψ-Riemann-Liouville fractional integral operator of order α of a function f is defined by

$$I_a^{\alpha;\psi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds.$$

Definition 2.2. *(ψ-Riemann-Liouville fractional derivative [4])*
Let $n \in \mathbb{N}$, $f, \psi \in \mathcal{C}^n([a, b])$ be two functions such that ψ is increasing with $\psi'(t) \neq 0$, for all $t \in [a, b]$. ψ-Riemann-Liouville fractional derivative of order α of a function f is defined by

$$\begin{aligned} D_a^{\alpha;\psi} f(t) &= \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n (I_a^{n-\alpha;\psi} f(t)) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.3. *(ψ-Caputo fractional derivative [4])*
Let $n \in \mathbb{N}$, $f, \psi \in \mathcal{C}^n([a, b])$ be two functions such that ψ is increasing with $\psi'(t) \neq 0$, for all $t \in [a, b]$. ψ-Caputo fractional derivative of order α of a function f is defined by

$$\begin{aligned} {}^C D_a^{\alpha;\psi} f(t) &= (I_a^{n-\alpha;\psi} f_\psi^{[n]})(t) \\ &= \frac{1}{\Gamma(n-\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{n-\alpha-1} f_\psi^{[n]}(s) ds, \end{aligned}$$

where $n = [\alpha] + 1$, for $\alpha \notin \mathbb{N}$. And $f_\psi^{[n]}(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n f(t)$ on $[a, b]$.

From the definition, it is clear that when $\alpha = n \in \mathbb{N}$, we have

$${}^C D_a^{\alpha;\psi} f(t) = f_\psi^{[n]}(t).$$

We note that if $f \in \mathcal{C}^n([a, b])$. The ψ-Caputo fractional derivative of order α of f is determined as

$${}^C D_a^{\alpha;\psi} f(t) = D_a^{\alpha;\psi} \left(f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a^+)}{k!} (\psi(t) - \psi(a))^k \right).$$

Theorem 2.4. [4] *Let $f \in \mathcal{C}^n([a, b])$ and $\alpha > 0$. Then we have*

$$I_a^{\alpha;\psi}\ {}^C D_a^{\alpha;\psi} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f_\psi^{[k]}(a^+)}{k!} (\psi(t) - \psi(a))^k.$$

In particular, given $\alpha \in (0, 1)$ we have:

$$I_a^{\alpha;\psi}\ {}^C D_a^{\alpha;\psi} f(t) = f(t) - f(a).$$

Theorem 2.5. *Given a function $f \in \mathcal{C}([a,b])$ and $\alpha > 0$, we have:*

$$ {}^C D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) = \int_a^x f(t)\psi'(t)dt. $$

Proof. By definition,

$$ {}^C D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) = \frac{1}{\Gamma(n-\alpha+1)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha} F_\psi^{[n]}(t)dt, $$

with

$$ F_\psi^{[n]}(x) = \frac{f(a)}{\Gamma(\alpha-n+1)}(\psi(x)-\psi(a))^{\alpha-n} + \frac{1}{\Gamma(\alpha-n+1)} \int_a^x (\psi(x)-\psi(t))^{\alpha-n} f'(t)dt. $$

Then,

$$ \begin{aligned} {}^C D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) &= \frac{f(a)}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_a^x \psi'(t)(\psi(x)-\psi(t))^{n-\alpha}(\psi(t)-\psi(a))^{\alpha-n}dt \\ &+ \frac{1}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_a^x \int_a^t \psi'(t)(\psi(x)-\psi(t))^{n-\alpha}(\psi(t)-\psi(\tau))^{\alpha-n} \\ &\quad f'(\tau)d\tau dt \\ &= \frac{f(a)\times(\psi(x)-\psi(t))^{n-\alpha}}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_a^x \psi'(t)\left(1-\frac{\psi(t)-\psi(a)}{\psi(x)-\psi(a)}\right)^{n-\alpha} (\psi(t)-\psi(a))^{\alpha-n}dt \\ &+ \frac{1}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_a^x \int_a^t \psi'(t)(\psi(x)-\psi(t))^{n-\alpha}(\psi(t)-\psi(\tau))^{\alpha-n} \\ &\quad f'(\tau)d\tau dt. \end{aligned} $$

Using the change of variables $u = \frac{\psi(t)-\psi(a)}{\psi(x)-\psi(a)}$ and the Dirichlet's formula, we deduce:

$$ \begin{aligned} {}^C D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) &= \frac{f(a)\times(\psi(x)-\psi(a))}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_0^1 (1-u)^{n-\alpha}u^{\alpha-n}du \\ &+ \frac{1}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_a^x f'(t)\left\{\int_t^x \psi'(\tau)(\psi(x)-\psi(\tau))^{n-\alpha}\right. \\ &\quad \left.(\psi(\tau)-\psi(t))^{\alpha-n}d\tau\right\} dt \\ &= f(a)\times(\psi(x)-\psi(a)) + \frac{1}{\Gamma(n-\alpha+1)\Gamma(\alpha-n+1)} \int_a^x f'(t) \\ &\int_t^x \psi'(\tau)\left(1-\frac{\psi(\tau)-\psi(t)}{\psi(x)-\psi(t)}\right)^{n-\alpha}\left(\frac{\psi(\tau)-\psi(t)}{\psi(x)-\psi(t)}\right)^{\alpha-n} d\tau dt \\ &= f(a)\times(\psi(x)-\psi(a)) + \int_a^x f'(t)(\psi(x)-\psi(t))dt. \end{aligned} $$

Thus,

$$ {}^C D_{a^+}^{\alpha-1;\psi} I_{a^+}^{\alpha;\psi} f(x) = \int_a^x f(t)\psi'(t)dt. $$

□

Lemma 2.6. *Given $n \le k \in \mathbb{N}$, we have:*

$$ {}^C D_{a+}^{\alpha;\psi}(\psi(t)-\psi(a))^k = \frac{k!}{\Gamma(k+1-\alpha)}(\psi(t)-\psi(a))^{k-\alpha}, $$

and

$$ {}^C D_{b-}^{\alpha;\psi}(\psi(b)-\psi(t))^k = \frac{k!}{\Gamma(k+1-\alpha)}(\psi(b)-\psi(t))^{k-\alpha}. $$

Proof. See [2]. □

Theorem 2.7. *(Krasnselskii's fixed point theorem)*
Let S be a closed convex nom-empty subset of a Banach space X. Suppose that A, B map S into X such that

1. $Au + Bv \in S$, $\forall u, v \in S$,
2. *A is a contraction mapping,*
3. *B is continuous and $B(S)$ is contained in a compact set.*

Then there exists $u \in S$ such that $Au + Bu = u$.

3. Existence of solutions

First of all, we define what we mean by a solution for the boundary value problem (1).

Definition 3.1. *A function $u \in \mathcal{C}(J, \mathbb{R})$ is said to be a solution of* (1) *if, u satisfies the equation*

$$^{C}D_{0^+}^{\alpha;\psi} u(t) = f(t, u(t)), \ t \in J,$$

and the conditions

$$u(0) = u'(0) = 0 \text{ and } {}^{C}D_{0^+}^{\alpha-1;\psi} u(1) = 1.$$

Lemma 3.2. *For a given $h : J \to \mathbb{R}$ continuous, the unique solution of the nonlinear fractional differential equation*

$$\begin{cases} {}^{C}D_{0^+}^{\alpha;\psi} u(t) = h(t), \ t \in J, \\ u(0) = u'(0) = 0 \text{ and } {}^{C}D_{0^+}^{\alpha-1;\psi} u(1) = 1, \end{cases} \tag{3.1}$$

is given by:

$$u(t) = \frac{-\Gamma(4-\alpha)(\psi(1)-\psi(0))^{\alpha-3}}{2}(\psi(t)-\psi(0))^2 \times \left(\int_0^1 h(s)\psi'(s)ds\right) + I_{0^+}^{\alpha;\psi} h(t). \tag{3.2}$$

Proof. Taking the ψ-Riemann-Liouville fractional integral of order α to the first equation of (2), we get:

$$u(t) = c_0 + c_1(\psi(t)-\psi(0)) + c_2(\psi(t)-\psi(0))^2 + I_{0^+}^{\alpha;\psi} h(t)dt.$$

Since $u(0) = 0$ and $u'(0) = 0$, we deduce that $c_0 = c_1 = 0$. Then

$$u(t) = c_2(\psi(t)-\psi(0))^2 + I_{0^+}^{\alpha;\psi} h(t)dt.$$

With the condition ${}^{C}D_{0^+}^{\alpha-1;\psi} u(1) = 1$ and theorem (2.2), we have:

$$^{C}D_{0^+}^{\alpha-1;\psi} u(t) = \frac{2c_2}{\Gamma(4-\alpha)}(\psi(t)-\psi(0))^{3-\alpha} + \int_0^t h(s)\psi'(s)ds,$$

so,

$$c_2 = -\left(\int_0^1 h(s)\psi'(s)ds\right)\frac{\Gamma(4-\alpha)}{2}(\psi(1)-\psi(0))^{\alpha-3}.$$

Thus, we get the integral equation (3) and the converse follows by direct computation which completes the proof. □

Now, we shall present our main result concerning the existence of solutions of problem (1).

Theorem 3.3. *Let $f : J \times [0, \infty) \to \mathbb{R}$ be a continuous function such that:*
(H_f) : *There exists a constant $L > 0$, such that*

$$|f(t, u(t)) - f(t, v(t))| \leq L|u(t) - v(t)|, \ \forall u, v \in \mathbb{R}, \ \forall t \in J.$$

If we have, that

$$L(\psi(1) - \psi(0))^{\alpha} \left\{ \frac{\Gamma(4-\alpha)}{2} + \frac{1}{\Gamma(\alpha+1)} \right\} < 1. \tag{3.3}$$

Then problem (1) *has a unique solution on J.*

Proof. Suppose that:

$$\begin{aligned} Pu(t) = & \frac{-\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha-3}}{2}(\psi(t) - \psi(0))^2 \times \left(\int_0^1 f(s, u(s))\psi'(s)ds \right) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds. \end{aligned}$$

Then,

$$\begin{aligned} |Pu(t) - Pv(t)| \leq & \frac{\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha-3}}{2}(\psi(t) - \psi(0))^2 \times \left(\int_0^1 |f(s, u(s)) - f(s, v(s))|\psi'(s)ds \right) \\ & + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, u(s)) - f(s, v(s))|ds \\ \leq & \frac{L\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha}}{2}||u - v||_\infty + \frac{L}{\Gamma(\alpha+1)}(\psi(1) - \psi(0))^{\alpha}||u - v||_\infty \\ = & L(\psi(1) - \psi(0))^{\alpha} \left\{ \frac{\Gamma(4-\alpha)}{2} + \frac{1}{\Gamma(\alpha+1)} \right\} ||u - v||_\infty. \end{aligned}$$

Since P is a contraction. By Banach fixed point theorem P has a unique fixed point which is a unique solution of problem (1). □

Theorem 3.4. *If f satisfies (H_f) and there exists a constant $\beta > 0$ such that:*

$$|f(t, u| \leq \beta, \ \ \forall t \in J, \forall x \geq 0.$$

And if there exists $\gamma > 0$

$$\beta \left(\frac{\Gamma(4-\alpha)}{2} + \frac{1}{\Gamma(\alpha+1)} \right) (\psi(1) - \psi(0))^{\alpha} \leq \gamma, \tag{3.4}$$

then the problem (1) *has at least one solution on J.*

Proof. We define a subset S of X by:

$$S = \{u \in \mathcal{C}(J, \mathbb{R}), \ ||u||_\infty \leq \gamma\}.$$

Define two operators $A : S \to X$ and $B : S \to X$ by:

$$\begin{aligned} Au(t) == & \frac{-\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha-3}}{2}(\psi(t) - \psi(0))^2 \times \left(\int_0^1 f(s, u(s))\psi'(s)ds \right) \\ & + \frac{1}{2\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds, \end{aligned}$$

and

$$Bu(t) = \frac{1}{2\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds.$$

Then the equation (3) is transformed into the operator equation as

$$u(t) = Au(t) + Bu(t).$$

We show that the operators A and B satisfy all the conditions of theorem (2.3) in several steps.
Step 1. Let $u, v \in S$. Then

$$\begin{aligned} |Au(t) + Bv(t)| &= \Bigg| \frac{-\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha-3}}{2} (\psi(t) - \psi(0))^2 \times \left(\int_0^1 f(s, u(s))\psi'(s)ds \right) \\ &+ \frac{1}{2\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, u(s))ds \\ &+ \frac{1}{2\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s, v(s))ds \Bigg| \\ &\leq \frac{\beta\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha}}{2} + \frac{\beta(\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} \\ &= \beta \left(\frac{\Gamma(4-\alpha)}{2} + \frac{1}{\Gamma(\alpha+1)} \right) (\psi(1) - \psi(0))^{\alpha} \leq \gamma. \end{aligned}$$

Step 2. Let $u, v \in S$. Then

$$\begin{aligned} |Au(t) - Av(t)| &\leq \frac{L\Gamma(4-\alpha)(\psi(1) - \psi(0))^{\alpha}}{2} ||u - v||_\infty + \frac{L(\psi(1) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} ||u - v||_\infty \\ &= L \left(\frac{\Gamma(4-\alpha)}{2} + \frac{1}{\Gamma(\alpha+1)} \right) (\psi(1) - \psi(0))^{\alpha} ||u - v||_\infty \\ &< 1 \quad \text{(by condition (4))}. \end{aligned}$$

Step 3. Let $(u_n)_n$ be a sequence such that $u_n \to u \in \mathcal{C}(J, \mathbb{R})$.
For $t \in J$, we have:

$$\begin{aligned} |Bu_n(t) - Bu(t)| &\leq \frac{1}{2\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} |f(s, u_n(s) - f(s, u(s))| ds \\ &\leq \frac{L(\psi(1) - \psi(0))^{\alpha}}{2\Gamma(\alpha+1)} ||u_n - u||_\infty. \end{aligned}$$

Therefore

$$||Bu_n(t) - Bu(t)||_\infty \to 0 \quad \text{as} \quad ||u_n - u||_\infty \to 0.$$

In order to show that B is compact. Let us take a bounded set $\Omega \subset S$. We are required to show that $B(\Omega)$ is relatively compact in $\mathcal{C}(J, \mathbb{R})$.
For arbitrary $u \in \Omega$ and $t \in J$. We have:

$$||Bu|| \leq \frac{\beta(\psi(1) - \psi(0))^{\alpha}}{2\Gamma(\alpha+1)} = cste.$$

Now, for equi-continuity of B take $t_1, t_2 \in J$ with $t_1 < t_2$, and let $u \in \Omega$. Thus, we get

$$|Bu(t_2) - Bu(t_1)| \leq \frac{\beta}{2\Gamma(\alpha+1)} \{(\psi(t_2) - \psi(0))^{\alpha} + (\psi(t_1) - \psi(0))^{\alpha}\}.$$

From the last estimate, we deduce that $||Bu(t_2) - Bu(t_1)|| \to 0$ when $t_2 \to t_1$. Therefore, B is equicontinuous. Thus, by Ascoli-Arzela theorem, the operator B is compact. Hence the problem (1) has at least one solution on J. □

4. Example

For $t \in [0,1]$ and $x \geq 0$, let define:

$$f(t,x) = \frac{x}{e^t + 1} \text{ and } \psi(x) = x^2.$$

Then

$$|f(t,x) - f(t,y)| \leq \frac{1}{11}|x - y|,$$

which proves that f is a contraction.
To apply theorem (3.1), we should verify that:

$$\frac{1}{11}\left\{\frac{\Gamma(4-\alpha)}{2} + \frac{1}{\Gamma(\alpha+1)}\right\} < 1.$$

Which is the case since, we have:

- For $\alpha = \frac{14}{5}$, then $\frac{1}{11}\left\{\frac{\Gamma(\frac{6}{5})}{2} + \frac{1}{\Gamma(\frac{19}{5})}\right\} = 0.05517 < 1.$
- For $\alpha = \frac{8}{3}$, then $\frac{1}{11}\left\{\frac{\Gamma(\frac{4}{3})}{2} + \frac{1}{\Gamma(\frac{11}{3})}\right\} = 0.114441 < 1.$

By theorem (3.1) we conclude that for f given by $\frac{x}{e^t+1}$ problem (1) admits at least one solution.

5. Conclusion

In this article, we proved the existence and uniqueness of solutions for a boundary value problem for a nonlinear fractional differential equation involving with ψ-Caputo fractional derivative. We use the Banach contraction principle and Krasnoselkii's fixed point theorem in order to prove our results.

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