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Algebraic integers of pure quintic extensions*

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ABSTRACT: Let $\mathbb{Q}$ denote the field of rational numbers and $\mathbb{K}$ be a pure quintic extension, that is, $\mathbb{K} = \mathbb{Q}(\sqrt[5]{d})$, where $d \in \mathbb{Z}$, $d \neq 1$ and square-free. The purpose of this work is to construct an integral basis of $\mathbb{K}$. Furthermore, we present the norm and trace of an element of $\mathbb{K}$ and the discriminant of the field $\mathbb{K}$.

Key Words: Algebraic integer, pure quintic extension, integral basis.

Contents

1. Introduction

In this paper we give a complete answer to an old problem of H. Hasse relative to the existence of a power integral basis of a number field, for the particular case when $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \sqrt[5]{d}$ is the real root of the irreducible polynomial $p(x) = x^5 - d \in \mathbb{Z}[x]$, with $d \neq 1$ and is a square-free integer. Hasse's problem (in fact this problem appeared many years before, even in time of Dedekind) consists in finding necessary and suficient conditions such that the ring of algebraic integers $\mathcal{O}_{\mathbb{K}}$ of a number field $\mathbb{K} = \mathbb{Q}(\alpha)$ is a free $\mathbb{Z}$-module of rank $m = [\mathbb{K} : \mathbb{Q}]$ with a power integral basis, i.e., with an integral basis of the form $\{\beta_1, \beta_2, \ldots, \beta_{m-1}\}$, where α is an algebraic integer in $\mathbb{Q}(\beta)$. In all its generality, up to now, this problem was not completely solved.

2. Basic results

An algebraic number field $\mathbb{K}$ is a field that is a finite degree extension of $\mathbb{Q}$. If $\mathbb{K}$ is a number field of degree n, then $\mathbb{K} = \mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{C}$ is a root of a monic irreducible polynomial $p(x) \in \mathbb{Z}[x]$. The n distinct roots of $p(x)$, namely, $\alpha_1, \alpha_2, \ldots, \alpha_n$, are the conjugates of α. If $\sigma : \mathbb{K} \to \mathbb{C}$ is a $\mathbb{Q}$-embedding then $\sigma(\alpha) = \alpha_i$ for some $i = 1, 2, \ldots, n$. Furthermore, there are exactly n $\mathbb{Q}$-embeddings σ_i, for $i = 1, 2, \ldots, n$, of $\mathbb{K}$ in $\mathbb{C}$. An element $\alpha \in \mathbb{K}$ is called an algebraic integer if there is a monic polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$. The set $\mathcal{O}_{\mathbb{K}} = \{\alpha \in \mathbb{K} : \alpha \text{ is an algebraic integer}\}$ is a ring, called ring of algebraic integers of $\mathbb{K}$. It can be shown that $\mathcal{O}_{\mathbb{K}}$, as a $\mathbb{Z}$-module, has a basis $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ over $\mathbb{Z}$, called integral basis, where n is the degree of $\mathbb{K}$. The trace and the norm of an element $\alpha \in \mathbb{K}$ over $\mathbb{Q}$ are defined as the rational numbers $Tr_{\mathbb{K}}(\alpha) = \displaystyle\sum_{i=1}^{n} \sigma_i(\alpha)$ and $N_{\mathbb{K}}(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha)$. If $\alpha \in \mathcal{O}_{\mathbb{K}}$, then $Tr_{\mathbb{K}}(\alpha)$ and $N_{\mathbb{K}}(\alpha)$ are integers. The characteristic polynomial of α is defined as $f_\alpha(x) = (x - \sigma_1(\alpha))(x - \sigma_2(\alpha_2)) \ldots (x - \sigma_n(\alpha))$ and $\alpha \in \mathcal{O}_{\mathbb{K}}$ if and only if the coefficients of the characteristic polynomial are integers. The discriminant of $\mathbb{K}$ over $\mathbb{Q}$ is defined by $\mathcal{D}_{\mathbb{K}} = \mathcal{D}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \det\limits_{1 \le i,j \le n}(Tr_{\mathbb{K}}(\alpha_i \alpha_j)) = \det\limits_{1 \le i,j \le n}(\sigma_i(\alpha_j))^2$, where $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is an integral basis of $\mathbb{K}$ [1].

An element $\theta \in \mathcal{O}_{\mathbb{K}}$ generates a power integral basis $\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$ for $\mathbb{K}$ if $\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\theta]$. When $\mathbb{K}$ has a power integral basis, the field $\mathbb{K}$ is said to be monogenic. In this work, we show that the field

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$\mathbb{Q}(\sqrt[5]{d})$, where $d \neq 1$ is a square-free integer, if $d \not\equiv \pm 1, \pm 7 (mod\ 25)$. The existence of power integral bases in algebraic number fields is a classical problem in algebraic number theory [2,3,5,4].

3. Pure quintic extension

Let $\mathbb{K}$ be a number field given by $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \sqrt[5]{d}$, $d \in \mathbb{Z}$, $d \neq 1$ and is square-free. The element θ is an algebraic integer, that is, θ is a root of the polynomial $p(x) = x^5 - d \in \mathbb{Z}[x]$. The roots of $p(x)$ are θ, $\theta\zeta_5$, $\theta\zeta_5^2$, $\theta\zeta_5^3$ and $\theta\zeta_5^4$, where ζ_5 is a primitive 5-th root of unity. Let σ_k be the $\mathbb{Q}$-embeddings of $\mathbb{Q}(\theta)$ em $\mathbb{C}$, that is, $\sigma_k(\theta) = \theta\zeta_5^{k-1}$, for $k = 1, 2, 3, 4, 5$. Since $\{1, \theta, \theta^2, \theta^3, \theta^4\}$ is a basis for $\mathbb{K}$ over $\mathbb{Q}$, if $\alpha \in \mathcal{O}_{\mathbb{K}} \subseteq \mathbb{K}$, it follows that $\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 + a_5\theta^5$, where $a_i \in \mathbb{Q}$ for $i = 0, 1, 2, 3, 4$.

Proposition 3.1 *The characteristic polynomial of α is given by*

$$f_\alpha(x) = x^5 - b_4x^4 + b_3x^3 - b_2x^2 + b_1x - b_0,$$

where $b_4 = 5a_0$, $b_3 = 5(2a_0^2 - (a_2a_3 + a_1a_4)d)$, $b_2 = 5(2a_0^3 + (-3a_0a_2a_3 - 3a_0a_1a_4 + a_1a_2^2 + a_1^2a_3)d + (a_3^2a_4 + a_2a_4^2)d^2)$, $b_1 = 5(a_0^4 + (-3a_0^2a_2a_3 - 3a_0^2a_1a_4 + 2a_0a_1a_2^2 + 2a_0a_1^2a_3 - a_1^3a_2)d + (2a_0a_3^2a_4 + 2a_0a_2a_4^2 + a_1^2a_4^2 - a_1a_3^3 - a_1a_2a_3a_4 + a_2^2a_3^2 - a_2^3a_4)d^2 + (-a_3a_4^3)d^3)$ *and* $b_0 = a_0^5 + (-5a_0^3a_2a_3 - 5a_0^3a_1a_4 + 5a_0^2a_1a_2^2 + 5a_0^2a_1^2a_3 - 5a_0a_1^3a_2 + a_1^5)d + (5a_0^2a_3^2a_4 + 5a_0^2a_2a_4^2 + 5a_0a_1^2a_4^2 - 5a_0a_1a_3^3 - 5a_0a_1a_2a_3a_4 + 5a_0a_2^2a_3^2 - 5a_0a_2^3a_4 - 5a_1^3a_3a_4 + 5a_1^2a_2a_3^2 + 5a_1^2a_2^2a_4 - 5a_1a_2^3a_3 + a_2^5)d^2 + (-5a_0a_3a_4^3 + 5a_1a_3^2a_4^2 - 5a_1a_2a_4^3 + a_3^5 - 5a_2a_3^3a_4 + 5a_2^2a_3a_4^2)d^3 + (a_4^5)d^4$.

Proof: Let $\alpha_i = \sigma_i(\alpha)$, for $i = 1, 2, 3, 4, 5$. Thus,

$$\begin{cases} \alpha_1 &=& a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4. \\ \alpha_2 &=& a_0 + a_1\theta\zeta_5 + a_2\theta^2\zeta_5^2 + a_3\theta^3\zeta_5^3 + a_4\theta^4\zeta_5^4. \\ \alpha_3 &=& a_0 + a_1\theta\zeta_5^2 + a_2\theta^2\zeta_5^4 + a_3\theta^3\zeta_5 + a_4\theta^4\zeta_5^3. \\ \alpha_4 &=& a_0 + a_1\theta\zeta_5^3 + a_2\theta^2\zeta_5 + a_3\theta^3\zeta_5^4 + a_4\theta^4\zeta_5^2. \\ \alpha_5 &=& a_0 + a_1\theta\zeta_5^4 + a_2\theta^2\zeta_5^3 + a_3\theta^3\zeta_5^2 + a_4\theta^4\zeta_5. \end{cases}$$

The characteristic polynomial of α is given by

$$\begin{aligned} f_\alpha(x) &= (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)(x-\alpha_4)(x-\alpha_5) = \\ &= x^5 - x^4(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5) + x^3(\alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_2\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_4+ \\ &+ \alpha_3\alpha_4+\alpha_1\alpha_5+\alpha_2\alpha_5+\alpha_3\alpha_5+\alpha_4\alpha_5) - x^2(\alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_2\alpha_5+\alpha_1\alpha_3\alpha_4+ \\ &+ \alpha_1\alpha_3\alpha_5+\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_5+\alpha_2\alpha_4\alpha_5+\alpha_3\alpha_4\alpha_5) + x(\alpha_1\alpha_2\alpha_3\alpha_4+ \\ &+ \alpha_1\alpha_2\alpha_3\alpha_5+\alpha_1\alpha_2\alpha_4\alpha_5+\alpha_1\alpha_3\alpha_4\alpha_5+\alpha_2\alpha_3\alpha_4\alpha_5) - (\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5). \end{aligned}$$

Since $\zeta_5 + \zeta_5^2 + \zeta_5^3 + \zeta_5^4 = -1$, it follows that

1. $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 5a_0$.

2. $\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 + \alpha_3\alpha_4 + \alpha_1\alpha_5 + \alpha_2\alpha_5 + \alpha_3\alpha_5 + \alpha_4\alpha_5 = 5(2a_0^2 - (a_2a_3 + a_1a_4)d)$.

3. $\alpha_1\alpha_2\alpha_3 + \alpha_1\alpha_2\alpha_4 + \alpha_1\alpha_2\alpha_5 + \alpha_1\alpha_3\alpha_4 + \alpha_1\alpha_3\alpha_5 + \alpha_1\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_4 + \alpha_2\alpha_3\alpha_5 + + \alpha_2\alpha_4\alpha_5 + \alpha_3\alpha_4\alpha_5 = 5(2a_0^3 + (-3a_0a_2a_3 - 3a_0a_1a_4 + a_1a_2^2 + a_1^2a_3)d + (a_3^2a_4 + a_2a_4^2)d^2)$.

4. $\alpha_1\alpha_2\alpha_3\alpha_4 + \alpha_1\alpha_2\alpha_3\alpha_5 + \alpha_1\alpha_2\alpha_4\alpha_5 + \alpha_1\alpha_3\alpha_4\alpha_5 + \alpha_2\alpha_3\alpha_4\alpha_5 = 5(a_0^4 + (-3a_0^2a_2a_3 - 3a_0^2a_1a_4 + 2a_0a_1a_2^2 + 2a_0a_1^2a_3 - a_1^3a_2)d + (2a_0a_3^2a_4 + 2a_0a_2a_4^2 + a_1^2a_4^2 - a_1a_3^3 - a_1a_2a_3a_4 + a_2^2a_3^2 - a_2^3a_4)d^2 + (-a_3a_4^3)d^3)$.

5. $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5 = a_0^5 + (-5a_0^3a_2a_3 - 5a_0^3a_1a_4 + 5a_0^2a_1a_2^2 + 5a_0^2a_1^2a_3 - 5a_0a_1^3a_2 + a_1^5)d + (5a_0^2a_3^2a_4 + 5a_0^2a_2a_4^2 + 5a_0a_1^2a_4^2 - 5a_0a_1a_3^3 - 5a_0a_1a_2a_3a_4 + 5a_0a_2^2a_3^2 - 5a_0a_2^3a_4 - 5a_1^3a_3a_4 + 5a_1^2a_2a_3^2 + 5a_1^2a_2^2a_4 - 5a_1a_2^3a_3 + a_2^5)d^2 + (-5a_0a_3a_4^3 + 5a_1a_3^2a_4^2 - 5a_1a_2a_4^3 + a_3^5 - 5a_2a_3^3a_4 + 5a_2^2a_3a_4^2)d^3 + (a_4^5)d^4$.

Thus,

$$\begin{aligned} f_\alpha(x) = {} & x^5 - 5a_0x^4 + (5(2a_0^2 - (a_2a_3 + a_1a_4)d))x^3 - (5(2a_0^3 + (-3a_0a_2a_3 - \\ & - 3a_0a_1a_4 + a_1a_2^2 + a_1^2a_3)d + (a_3^2a_4 + a_2a_4^2)d^2))x^2 + (5(a_0^4 + (-3a_0^2a_2a_3 - \\ & - 3a_0^2a_1a_4 + 2a_0a_1a_2^2 + 2a_0a_1^2a_3 - a_1^3a_2)d + (2a_0a_3^2a_4 + 2a_0a_2a_4^2 + a_1^2a_4^2 - \\ & - a_1a_3^3 - a_1a_2a_3a_4 + a_2^2a_3^2 - a_2^3a_4)d^2 + (-a_3a_4^3)d^3))x - (a_0^5 + (-5a_0^3a_2a_3 - \\ & - 5a_0^3a_1a_4 + 5a_0^2a_1a_2^2 + 5a_0^2a_1^2a_3 - 5a_0a_1^3a_2 + a_1^5)d + (5a_0^2a_3^2a_4 + 5a_0^2a_2a_4^2 + \\ & + 5a_0a_1^2a_4^2 - 5a_0a_1a_3^3 - 5a_0a_1a_2a_3a_4 + 5a_0a_2^2a_3^2 - 5a_0a_2^3a_4 - 5a_1^3a_3a_4 + \\ & + 5a_1^2a_2a_3^2 + 5a_1^2a_2^2a_4 - 5a_1a_2^3a_3 + a_2^5)d^2 + (-5a_0a_3a_4^3 + 5a_1a_3^2a_4^2 - \\ & - 5a_1a_2a_4^3 + a_3^5 - 5a_2a_3^3a_4 + 5a_2^2a_3a_4^2)d^3 + (a_4^5)d^4), \end{aligned}$$

which proves the proposition. □

From [2, Proposition 3], it follows that

$$Tr_{\mathbb{K}}(\theta^k) = \begin{cases} 0, \text{ if } k = 1,2,3,4 \\ 5d^s, \text{ if } k = 5s, \text{ with } s \in \mathbb{N}, \\ 0, \text{ if } k > 5 \text{ and } k \not\equiv 0 (mod\ 5). \end{cases} \tag{3.1}$$

Thus, $Tr_{\mathbb{K}}(1) = 5$, $Tr_{\mathbb{K}}(\theta) = 0$, $Tr_{\mathbb{K}}(\theta^2) = 0$, $Tr_{\mathbb{K}}(\theta^3) = 0$, $Tr_{\mathbb{K}}(\theta^4) = 0$, $Tr_{\mathbb{K}}(d) = 5d$, $Tr_{\mathbb{K}}(\theta^6) = 0$, $Tr_{\mathbb{K}}(\theta^7) = 0$ and $Tr_{\mathbb{K}}(\theta^8) = 0$.

Lemma 3.1 $\mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4) = 3125d^4$.

Proof: The discriminant

$$\begin{aligned} \mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4) &= \det \begin{bmatrix} Tr_{\mathbb{K}}(1) & Tr_{\mathbb{K}}(\theta) & Tr_{\mathbb{K}}(\theta^2) & Tr_{\mathbb{K}}(\theta^3) & Tr_{\mathbb{K}}(\theta^4) \\ Tr_{\mathbb{K}}(\theta) & Tr_{\mathbb{K}}(\theta^2) & Tr_{\mathbb{K}}(\theta^3) & Tr_{\mathbb{K}}(\theta^4) & Tr_{\mathbb{K}}(d) \\ Tr_{\mathbb{K}}(\theta^2) & Tr_{\mathbb{K}}(\theta^3) & Tr_{\mathbb{K}}(\theta^4) & Tr_{\mathbb{K}}(d) & Tr_{\mathbb{K}}(\theta^6) \\ Tr_{\mathbb{K}}(\theta^3) & Tr_{\mathbb{K}}(\theta^4) & Tr_{\mathbb{K}}(d) & Tr_{\mathbb{K}}(\theta^6) & Tr_{\mathbb{K}}(\theta^7) \\ Tr_{\mathbb{K}}(\theta^4) & Tr_{\mathbb{K}}(d) & Tr_{\mathbb{K}}(\theta^6) & Tr_{\mathbb{K}}(\theta^7) & Tr_{\mathbb{K}}(\theta^8) \end{bmatrix} \\ &= \det \begin{bmatrix} 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5d \\ 0 & 0 & 0 & 5d & 0 \\ 0 & 0 & 5d & 0 & 0 \\ 0 & 5d & 0 & 0 & 0 \end{bmatrix} = 5.(-1)^{1+4}.5d \det \begin{bmatrix} 0 & 0 & 5d \\ 0 & 5d & 0 \\ 5d & 0 & 0 \end{bmatrix}, \end{aligned}$$

and therefore, $\mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4) = 3125d^4$. □

Similarly to Lemma 3.1,

1. If $\beta_1 = (1+\theta+\theta^2+\theta^3+\theta^4)/5$, then $\mathcal{D}\left(1,\theta,\theta^2,\theta^3,\beta_1\right) = 125d^4$.
2. If $\beta_2 = (1+4\theta+\theta^2+4\theta^3+\theta^4)/5$, then $\mathcal{D}\left(1,\theta,\theta^2,\theta^3,\beta_2\right) = 125d^4$
3. If $\beta_3 = (1+3\theta+4\theta^2+2\theta^3+\theta^4)/5$, then $\mathcal{D}\left(1,\theta,\theta^2,\theta^3,\beta_3\right) = 125d^4$.
4. If $\beta_4 = (1+2\theta+4\theta^2+3\theta^3+\theta^4)/5$, then $\mathcal{D}\left(1,\theta,\theta^2,\theta^3,\beta_4\right) = 125d^4$.

4. Ring of algebraic integers

In this section, we present the ring of algebraic integers $\mathcal{O}_{\mathbb{K}}$ of $\mathbb{K} = \mathbb{Q}(\theta)$, where $\theta = \sqrt[5]{d}$ with $1 \neq d \in \mathbb{Z}$ and square-free.

Theorem 4.1 *The ring of algebraic integers $\mathcal{O}_{\mathbb{K}}$ of $\mathbb{K}$ is given by*

$$\mathcal{O}_{\mathbb{K}} = \begin{cases} \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2 + \mathbb{Z}\theta^3 + \mathbb{Z}\theta^4, \ \textit{if } d \not\equiv \pm 1, \pm 7 (mod\ 25) \\ \mathbb{Z} + \mathbb{Z}\theta + \mathbb{Z}\theta^2 + \mathbb{Z}\theta^3 + \mathbb{Z}\left(\frac{1+\theta+\theta^2+\theta^3+\theta^4}{5}\right), \ \textit{if } d \equiv \pm 1, \pm 7 (mod\ 25). \end{cases}$$

Proof: If $\alpha \in \mathbb{K}$ is an algebraic integer, then $\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4$, with $a_0, a_1, a_2, a_3, a_4 \in \mathbb{Q}$. From [2, Proposition 4], it follows that $5a_0, 5a_1, 5a_2, 5a_3, 5a_4 \in \mathbb{Z}$. Let $5a_i = p_i$, with $p_i \in \mathbb{Z}$, for all $i = 0, 1, 2, 3, 4$. Thus,

$$a_i = \frac{p_i}{5}, \text{ for } \ i = 0, 1, 2, 3, 4,$$

and p_i can be rewritten as

$$p_i = 5q_i + r_i,$$

where $q_i, r_i \in \mathbb{Z}$ and $r_i \in \{0, 1, 2, 3, 4\}$, for $i = 0, 1, 2, 3, 4$. Therefore,

$$\begin{aligned} \alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 &= \frac{p_0}{5} + \frac{p_1}{5}\theta + \frac{p_2}{5}\theta^2 + \frac{p_3}{5}\theta^3 + \frac{p_4}{5}\theta^4 \\ &= \frac{5q_0 + r_0}{5} + \frac{5q_1 + r_1}{5}\theta + \frac{5q_2 + r_2}{5}\theta^2 + \frac{5q_3 + r_3}{5}\theta^3 + \frac{5q_4 + r_4}{5}\theta^4 \\ &= q_0 + q_1\theta + q_2\theta^2 + q_3\theta^3 + q_4\theta^4 + \frac{r_0}{5} + \frac{r_1}{5}\theta + \frac{r_2}{5}\theta^2 + \frac{r_3}{5}\theta^3 + \frac{r_4}{5}\theta^4. \end{aligned}$$

So,

$$\alpha = \left(q_0 + \frac{r_0}{5}\right) + \left(q_1 + \frac{r_1}{5}\right)\theta + \left(q_2 + \frac{r_2}{5}\right)\theta^2 + \left(q_3 + \frac{r_3}{5}\right)\theta^3 + \left(q_4 + \frac{r_4}{5}\right)\theta^4. \tag{4.1}$$

Since $q = q_0 + q_1\theta + q_2\theta^2 + q_3\theta^3 + q_4\theta^4 \in \mathcal{O}_{\mathbb{K}}$, it follows that

$$\alpha \in \mathcal{O}_{\mathbb{K}} \Leftrightarrow r = \frac{r_0}{5} + \frac{r_1}{5}\theta + \frac{r_2}{5}\theta^2 + \frac{r_3}{5}\theta^3 + \frac{r_4}{5}\theta^4 \in \mathcal{O}_{\mathbb{K}}. \tag{4.2}$$

From Proposition 3.1, it follows that $r \in \mathcal{O}_{\mathbb{K}}$ if and only if

$$\begin{cases} 5[\frac{r_0}{5}] \in \mathbb{Z} \\ 5[\frac{2r_0^2}{25} - (\frac{r_2r_3}{25} + \frac{r_1r_4}{25})d] \in \mathbb{Z}, \\ 5[\frac{2r_0^3}{125} + (-\frac{3r_0r_2r_3}{125} - \frac{3r_0r_1r_4}{125} + \frac{r_1r_2^2}{125} + \frac{r_1^2r_3}{125})d + (\frac{r_3^2r_4}{125} + \frac{r_2r_4^2}{125})d^2] \in \mathbb{Z}, \\ 5[\frac{r_0^4}{625} + (-\frac{3r_0^2r_2r_3}{625} - \frac{3r_0^2r_1r_4}{625} + \frac{2r_0r_1r_2^2}{625} + \frac{2r_0r_1^2r_3}{625} - \frac{r_1^3r_2}{625})d + (\frac{2r_0r_3^2r_4}{625} + \\ + \frac{2r_0r_2r_4^2}{625} + \frac{r_1^2r_4^2}{625} - \frac{r_1r_3^3}{625} - \frac{r_1r_2r_3r_4}{625} + \frac{r_2^2r_3^2}{625} - \frac{r_2^3r_4}{625})d^2 + (-\frac{r_3r_4^3}{625})d^3] \in \mathbb{Z} \text{ and} \\ \frac{r_0^5}{3125} + (-\frac{5r_0^3r_2r_3}{3125} - \frac{5r_0^3r_1r_4}{3125} + \frac{5r_0^2r_1r_2^2}{3125} + \frac{5r_0^2r_1^2r_3}{3125} - \frac{5r_0r_1^3r_2}{3125} + \frac{r_1^5}{3125})d + \\ +(\frac{5r_0^2r_3^2r_4}{3125} + \frac{5r_0^2r_2r_4^2}{3125} + \frac{5r_0r_1^2r_4^2}{3125} - \frac{5r_0r_1r_3^3}{3125} - \frac{5r_0r_1r_2r_3r_4}{3125} + \frac{5r_0r_2^2r_3^2}{3125} - \frac{5r_0r_2^3r_4}{3125} - \\ - \frac{5r_1^3r_3r_4}{3125} + \frac{5r_1^2r_2r_3^2}{3125} + \frac{5r_1^2r_2^2r_4}{3125} - \frac{5r_1r_2^3r_3}{3125} + \frac{r_2^5}{3125})d^2 + (-\frac{5r_0r_3r_4^3}{3125} + \frac{5r_1r_3^2r_4^2}{3125} - \\ - \frac{5r_1r_2r_4^3}{3125} + \frac{r_3^5}{3125} - \frac{5r_2r_3^3r_4}{3125} + \frac{5r_2^2r_3r_4^2}{3125})d^3 + (\frac{r_4^5}{3125})d^4 \in \mathbb{Z}. \end{cases} \tag{4.3}$$

Let

$$\omega_1 = \frac{2r_0^2 - (r_2r_3 + r_1r_4)d}{5}. \tag{4.4}$$

$$\omega_2 = \frac{2r_0^3 + (-3r_0r_2r_3 - 3r_0r_1r_4 + r_1r_2^2 + r_1^2r_3)d + (r_3^2r_4 + r_2r_4^2)d^2}{25}. \tag{4.5}$$

$$\begin{aligned} \omega_3 = &\frac{r_0^4 + (-3r_0^2r_2r_3 - 3r_0^2r_1r_4 + 2r_0r_1r_2^2 + 2r_0r_1^2r_3 - r_1^3r_2)d + (2r_0r_3^2r_4)d^2}{125} + \\ &+ \frac{(2r_0r_2r_4^2 + r_1^2r_4^2 - r_1r_3^3 - r_1r_2r_3r_4 + r_2^2r_3^2 - r_2^3r_4)d^2 + (-r_3r_4^3)d^3}{125}. \end{aligned} \tag{4.6}$$

$$\begin{aligned}\omega_4 = &\frac{r_0^5+(-5r_0^3r_2r_3-5r_0^3r_1r_4+5r_0^2r_1r_2^2+5r_0^2r_1^2r_3-5r_0r_1^3r_2+r_1^5)d}{3125}+\\&+\frac{(5r_0^2r_3^2r_4+5r_0^2r_2r_4^2+5r_0r_1^2r_4^2-5r_0r_1r_3^3-5r_0r_1r_2r_3r_4+5r_0r_2^2r_3^2)d^2}{3125}+\\&+\frac{(-5r_0r_2^3r_4-5r_1^3r_3r_4+5r_1^2r_2r_3^2+5r_1^2r_2^2r_4-5r_1r_2^3r_3+r_2^5)d^2}{3125}+\\&+\frac{(-5r_0r_3r_4^3+5r_1r_3^2r_4^2-5r_1r_2r_4^3+r_3^5-5r_2r_3^3r_4+5r_2^2r_3r_4^2)d^3+(r_4^5)d^4}{3125}.\end{aligned}\tag{4.7}$$

From Equations (4.2) and (4.3), it folllows that

$$\alpha\in\mathcal{O}_{\mathbb{K}}\Leftrightarrow\omega_1,\omega_2,\omega_3,\omega_4\in\mathbb{Z}.\tag{4.8}$$

Since $r_0,r_1,r_2,r_3,r_4\in\{0,1,2,3,4\}$, it follows that there exist 3125 possibilities for $s_j=(r_0,r_1,r_2,r_3,r_4)$. Let $\alpha=\left(q_0+\frac{r_0}{5}\right)+\left(q_1+\frac{r_1}{5}\right)\theta+\left(q_2+\frac{r_2}{5}\right)\theta^2+\left(q_3+\frac{r_3}{5}\right)\theta^3+\left(q_4+\frac{r_4}{5}\right)\theta^4$, where $q_i\in\mathbb{Z}$ and $r_i\in\{0,1,2,3,4\}$, for $i=0,1,2,3,4$.

1. If

   $$\alpha=z_0+z_1\theta+z_2\theta^2+z_3\theta^3+z_4\theta^4\in\mathcal{O}_{\mathbb{K}},$$

   with $z_0,z_1,z_2,z_3,z_4\in\mathbb{Q}$, then

   $$z_0=q_0+\frac{r_0}{5},\quad z_1=q_1+\frac{r_1}{5},\quad z_2=q_2+\frac{r_2}{5},\quad z_3=q_3+\frac{r_3}{5}\quad\text{and}\quad z_4=q_4+\frac{r_4}{5}.$$

   Since $\{1,\theta,\theta^2,\theta^3,\theta^4\}$ is an integral basis for $\mathbb{K}$ if and only if $z_0,z_1,z_2,z_3,z_4\in\mathbb{Z}$ and $r_i\in\{0,1,2,3,4\}$, for $i=0,1,2,3,4$, it follows that

   $$r_0=r_1=r_2=r_3=r_4=0.$$

   Therefore, the only solution is $s_1=(0,0,0,0,0)$.

2. If

   $$\begin{aligned}\alpha&=z_0+z_1\theta+z_2\theta^2+z_3\theta^3+z_4\left(\frac{1+\theta+\theta^2+\theta^3+\theta^4}{5}\right)\\&=\left(z_0+\frac{z_4}{5}\right)+\left(z_1+\frac{z_4}{5}\right)\theta+\left(z_2+\frac{z_4}{5}\right)\theta^2+\left(z_3+\frac{z_4}{5}\right)\theta^3+\left(\frac{z_4}{5}\right)\theta^4,\end{aligned}$$

   with $z_0,z_1,z_2,z_3,z_4\in\mathbb{Q}$, then

   $$\begin{cases} z_0+\frac{z_4}{5}=q_0+\frac{r_0}{5} & z_1+\frac{z_4}{5}=q_1+\frac{r_1}{5} \qquad z_2+\frac{z_4}{5}=q_2+\frac{r_2}{5}\\ z_3+\frac{z_4}{5}=q_3+\frac{r_3}{5} & \frac{z_4}{5}=q_4+\frac{r_4}{5}.\end{cases}$$

   Since $\left\{1,\theta,\theta^2,\theta^3,\frac{1+\theta+\theta^2+\theta^3+\theta^4}{5}\right\}$ is an integral basis if and only if $z_0,z_1,z_2,z_3,z_4\in\mathbb{Z}$, and $r_i\in\{0,1,2,3,4\}$, for $i=0,1,2,3,4$, it follows that

   $$r_4=0,1,2,3\text{ or }4\text{ and }r_0=r_1=r_2=r_3=r_4.$$

   Therefore, the solutions are $s_1=(0,0,0,0,0)$, $s_2=(1,1,1,1,1)$, $s_3=(2,2,2,2,2)$, $s_4=(3,3,3,3,3)$ and $s_5=(4,4,4,4,4)$.

3. If

$$\begin{aligned}\alpha &= z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + z_4\left(\frac{1+4\theta+\theta^2+4\theta^3+\theta^4}{5}\right)\\ &= \left(z_0+\frac{z_4}{5}\right)+\left(z_1+\frac{4z_4}{5}\right)\theta+\left(z_2+\frac{z_4}{5}\right)\theta^2+\left(z_3+\frac{4z_4}{5}\right)\theta^3+\left(\frac{z_4}{5}\right)\theta^4,\end{aligned}$$

with $z_0, z_1, z_2, z_3, z_4 \in \mathbb{Q}$, then

$$\begin{cases} z_0+\frac{z_4}{5}=q_0+\frac{r_0}{5} & z_1+\frac{4z_4}{5}=q_1+\frac{r_1}{5} & z_2+\frac{z_4}{5}=q_2+\frac{r_2}{5}\\ z_3+\frac{4z_4}{5}=q_3+\frac{r_3}{5} & \frac{z_4}{5}=q_4+\frac{r_4}{5}. \end{cases}$$

Since $\left\{1,\theta,\theta^2,\theta^3,\frac{1+4\theta+\theta^2+4\theta^3+\theta^4}{5}\right\}$ is an integral basis if and only if $z_0, z_1, z_2, z_3, z_4 \in \mathbb{Z}$, and $r_i \in \{0,1,2,3,4\}$, for $i = 0,1,2,3,4$, it follows that

$$r_4 = 0,1,2,3 \text{ or } 4, \quad r_1 \equiv r_3 \equiv 4r_4 (mod\ 5) \quad \text{and} \quad r_0 = r_2 = r_4.$$

Therefore, the solutions are $s_1 = (0,0,0,0,0)$, $s_6 = (1,4,1,4,1)$, $s_7 = (2,3,2,3,2)$, $s_8 = (3,2,3,2,3)$ and $s_9 = (4,1,4,1,4)$.

4. If

$$\begin{aligned}\alpha &= z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + z_4\left(\frac{1+3\theta+4\theta^2+2\theta^3+\theta^4}{5}\right)\\ &= \left(z_0+\frac{z_4}{5}\right)+\left(z_1+\frac{3z_4}{5}\right)\theta+\left(z_2+\frac{4z_4}{5}\right)\theta^2+\left(z_3+\frac{2z_4}{5}\right)\theta^3+\left(\frac{z_4}{5}\right)\theta^4,\end{aligned}$$

with $z_0, z_1, z_2, z_3, z_4 \in \mathbb{Q}$, then

$$\begin{cases} z_0+\frac{z_4}{5}=q_0+\frac{r_0}{5} & z_1+\frac{3z_4}{5}=q_1+\frac{r_1}{5} & z_2+\frac{4z_4}{5}=q_2+\frac{r_2}{5}\\ z_3+\frac{2z_4}{5}=q_3+\frac{r_3}{5} & \frac{z_4}{5}=q_4+\frac{r_4}{5} \end{cases}$$

Since $\left\{1,\theta,\theta^2,\theta^3,\frac{1+3\theta+4\theta^2+2\theta^3+\theta^4}{5}\right\}$ is an integral basis if and only if $z_0, z_1, z_2, z_3, z_4 \in \mathbb{Z}$, and $r_i \in \{0,1,2,3,4\}$, for $i = 0,1,2,3,4$, it follows that

$$r_0 = r_4 = 0,1,2,3 \text{ or } 4, \quad r_3 \equiv 2r_4 (mod\ 5), \quad r_2 \equiv 4r_4 (mod\ 5) \text{ and } r_1 \equiv 3r_4 (mod\ 5).$$

Therefore, the solutions are $s_1 = (0,0,0,0,0)$, $s_{10} = (1,3,4,2,1)$, $s_{11} = (2,1,3,4,2)$, $s_{12} = (3,4,2,1,3)$ and $s_{13} = (4,2,1,3,4)$.

5. If

$$\begin{aligned}\alpha &= z_0 + z_1\theta + z_2\theta^2 + z_3\theta^3 + z_4\left(\frac{1+2\theta+4\theta^2+3\theta^3+\theta^4}{5}\right)\\ &= \left(z_0+\frac{z_4}{5}\right)+\left(z_1+\frac{2z_4}{5}\right)\theta+\left(z_2+\frac{4z_4}{5}\right)\theta^2+\left(z_3+\frac{3z_4}{5}\right)\theta^3+\left(\frac{z_4}{5}\right)\theta^4,\end{aligned}$$

with $z_0, z_1, z_2, z_3, z_4 \in \mathbb{Q}$, then

$$\begin{cases} z_0+\frac{z_4}{5}=q_0+\frac{r_0}{5} & z_1+\frac{2z_4}{5}=q_1+\frac{r_1}{5} & z_2+\frac{4z_4}{5}=q_2+\frac{r_2}{5}\\ z_3+\frac{3z_4}{5}=q_3+\frac{r_3}{5} & \frac{z_4}{5}=q_4+\frac{r_4}{5} & . \end{cases}$$

Since $\left\{1, \theta, \theta^2, \theta^3, \dfrac{1+2\theta+4\theta^2+3\theta^3+\theta^4}{5}\right\}$ is an integral basis for $\mathbb{K}$ if and only if $z_0, z_1, z_2, z_3, z_4 \in \mathbb{Z}$, and $r_i \in \{0,1,2,3,4\}$, for $i = 0,1,2,3,4$, it follows that

$$r_0 = r_4 = 0,1,2,3 \text{ or } 4, \quad r_3 \equiv 3r_4 (mod\ 5), \quad r_2 \equiv 4r_4 (mod\ 5) \ \text{ and } \ r_1 \equiv 2r_4 (mod\ 5).$$

Therefore, the solutions are $s_1 = (0,0,0,0,0)$, $s_{14} = (1,2,4,3,1)$, $s_{15} = (2,4,3,1,2)$, $s_{16} = (3,1,2,4,3)$ and $s_{17} = (4,3,1,2,4)$.

Thus, there are 17 solutions $s_j = (r_0, r_1, r_2, r_3, r_4)$, for $j = 1,2,\ldots,17$. Now, replacing these solutions in the Equations (4.4), (4.5), (4.6) and (4.7), we found the equivalences of d modulo 25 such that $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{Z}$.

Table 1: $\mathbb{K} = \mathbb{Q}(\sqrt[5]{d})$, $d \in \mathbb{Z}$, $d \neq 1$ and square-free

s_j	ω_1	ω_2	ω_3	ω_4	$d \equiv (?)$
s_1	0	0	0	0	$\forall d$
s_2	$\frac{2(1-d)}{5}$	$\frac{2(1-d)^2}{25}$	$\frac{(1-d)^3}{125}$	$\frac{(1-d)^4}{3125}$	$d \equiv 1$
s_3	$\frac{8(1-d)}{5}$	$\frac{16(1-d)^2}{25}$	$\frac{16(1-d)^3}{125}$	$\frac{32(1-d)^4}{3215}$	$d \equiv 1$
s_4	$\frac{18(1-d)}{5}$	$\frac{54(1-d)^2}{25}$	$\frac{81(1-d)^3}{125}$	$\frac{243(1-d)^4}{3125}$	$d \equiv 1$
s_5	$\frac{32(1-d)}{5}$	$\frac{128(1-d)^2}{25}$	$\frac{256(1-d)^3}{125}$	$\frac{1024(1-d)^4}{3125}$	$d \equiv 1$
s_6	$\frac{2(1-4d)}{5}$	$\frac{2+44d+17d^2}{25}$	$\frac{1+48d-207d^2-4d^3}{125}$	$\frac{1+1004d-1119d^2+1004d^3+d^4}{3125}$	$d \equiv -1$
s_7	$\frac{4(2-3d)}{5}$	$\frac{16-33d+26d^2}{25}$	$\frac{16-42d+43d^2-24d^3}{125}$	$\frac{32+3d-58d^2+3d^3+32d^4}{3125}$	$d \equiv -1$
s_8	$\frac{6(3-2d)}{5}$	$\frac{54-82d+39d^2}{25}$	$\frac{81-192d+173d^2-54d^3}{125}$	$\frac{243-778d+1083d^2-778d^3+243d^4}{3125}$	$d \equiv -1$
s_9	$\frac{8(4-d)}{5}$	$\frac{128-79d+68d^2}{25}$	$\frac{256-252d+303d^2-64d^3}{125}$	$\frac{1024-1279d+1644d^2-1279d^3+1024d^4}{3125}$	$d \equiv -1$
s_{10}	$\frac{2-11d}{5}$	$\frac{2+33d+8d^2}{25}$	$\frac{1-9d-23d^2-2d^3}{125}$	$\frac{1-22d+119d^2+22d^3+d^4}{3125}$	$d \equiv 7$
s_{11}	$\frac{2(4-7d)}{5}$	$\frac{16-71d+44d^2}{25}$	$\frac{16-119d+182d^2-32d^3}{125}$	$\frac{32-329d+933d^2-296d^3+32d^4}{3125}$	$d \equiv 7$
s_{12}	$\frac{2(9-7d)}{5}$	$\frac{54-94d+21d^2}{25}$	$\frac{81-314d+222d^2-27d^3}{125}$	$\frac{243-1346d+2417d^2-1154d^3+243d^4}{3125}$	$d \equiv 7$
s_{13}	$\frac{32-11d}{5}$	$\frac{2(64-59d+26d^2)}{25}$	$\frac{256-424d+407d^2-192d^3}{125}$	$\frac{1024-2528d+3731d^2-3097d^3+1024d^4}{3125}$	$d \equiv 7$
s_{14}	$\frac{2(1-7d)}{5}$	$\frac{2+2d+13d^2}{25}$	$\frac{1+14d+32d^2-3d^3}{125}$	$\frac{1+22d+119d^2-22d^3+d^4}{3125}$	$d \equiv -7$
s_{15}	$\frac{8-11d}{5}$	$\frac{2(8-7d+7d^2)}{25}$	$\frac{16-116d+47d^2-8d^3}{125}$	$\frac{32-296d+933d^2-329d^3+32d^4}{3125}$	$d \equiv -7$
s_{16}	$\frac{18-11d}{5}$	$\frac{54-91d+66d^2}{25}$	$\frac{81-251d+357d^2-108d^3}{125}$	$\frac{243-1154d+2417d^2-1346d^3+243d^4}{3125})$	$d \equiv -7$
s_{17}	$\frac{2(16-7d)}{5}$	$\frac{128-147d+32d^2}{25}$	$\frac{256-531d+352d^2-128d^3}{125}$	$\frac{1024-3097d+3731d^2-2528d^3+1024d^4}{3125}$	$d \equiv -7$

1. For $d \not\equiv \pm 1, \pm 7 (mod\ 25)$ exclusively, solution is s_1. In this case, the integral basis is given by
$$\left\{1, \theta, \theta^2, \theta^3, \theta^4\right\}.$$

2. For $d \equiv 1 (mod\ 25)$ exclusively, solutions are s_2, s_3, s_4 and s_5. In this case, the integral basis is
$$\left\{1, \theta, \theta^2, \theta^3, \frac{1+\theta+\theta^2+\theta^3+\theta^4}{5}\right\}.$$

3. For $d \equiv -1 (mod\ 25)$ exclusively, solutions are s_6, s_7, s_8 and s_9. In this case, the integral basis is
$$\left\{1, \theta, \theta^2, \theta^3, \frac{1+4\theta+\theta^2+4\theta^3+\theta^4}{5}\right\}.$$

4. For $d \equiv 7 (mod\ 25)$ exclusively, solutions are s_{10}, s_{11}, s_{12} and s_{13}. In this case, the integral basis is
$$\left\{1, \theta, \theta^2, \theta^3, \frac{1+3\theta+4\theta^2+2\theta^3+\theta^4}{5}\right\}.$$

5. For $d \equiv -7(mod\ 25)$ exclusively, solutions are s_{14}, s_{15}, s_{16} and s_{17}. In this case, the integral basis is
$$\left\{1, \theta, \theta^2, \theta^3, \frac{1+2\theta+4\theta^2+3\theta^3+\theta^4}{5}\right\}.$$

Since $\mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4) = 3125d^4$ and $\mathcal{D}(1,\theta,\theta^2,\theta^3,\beta_1) = \mathcal{D}(1,\theta,\theta^2,\theta^3,\beta_2) = \mathcal{D}(1,\theta,\theta^2,\theta^3,\beta_3) = \mathcal{D}(1,\theta,\theta^2,\theta^3,\beta_4) = 125d^4$, where $\beta_1 = \left(\frac{1+\theta+\theta^2+\theta^3+\theta^4}{5}\right)$, $\beta_2 = \left(\frac{1+4\theta+\theta^2+4\theta^3+\theta^4}{5}\right)$, $\beta_3 = \left(\frac{1+3\theta+4\theta^2+2\theta^3+\theta^4}{5}\right)$, $\beta_4 = \left(\frac{1+2\theta+4\theta^2+3\theta^3+\theta^4}{5}\right)$, it follows that the ring of algebraic integers of $\mathbb{K}$ is given by

$$\mathcal{O}_{\mathbb{K}} = \begin{cases} \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\theta^4, \text{ if } d \not\equiv \pm1, \pm7(mod\ 25) \\ \mathbb{Z}+\mathbb{Z}\theta+\mathbb{Z}\theta^2+\mathbb{Z}\theta^3+\mathbb{Z}\left(\frac{1+\theta+\theta^2+\theta^3+\theta^4}{5}\right), \text{ if } d \equiv \pm1, \pm7(mod\ 25), \end{cases}$$

which proves the theorem. □

From Theorem 4.1, it follows that the discriminant of $\mathbb{K}$ is given by

$$\mathcal{D}(\mathbb{K}) = \begin{cases} 3125d^4, \text{ if } d \not\equiv \pm1, \pm7(mod\ 25) \\ 125d^4, \text{ if } d \equiv \pm1, \pm7(mod\ 25). \end{cases}$$

Furthermore, if $\alpha = a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 \in \mathbb{K}$, with $a_0, a_1, a_2, a_3, a_4 \in \mathbb{Q}$, then the trace α is $Tr_{\mathbb{K}}(\alpha) = 5a_0$ and the norm of α is given by

$$\begin{aligned}\mathcal{N}(\alpha) = a_0^5 &+ (-5a_0^3a_2a_3 - 5a_0^3a_1a_4 + 5a_0^2a_1a_2^2 + 5a_0^2a_1^2a_3 - 5a_0a_1^3a_2 + a_1^5)d + \\ &+ (5a_0^2a_3^2a_4 + 5a_0^2a_2a_4^2 + 5a_0a_1^2a_4^2 - 5a_0a_1a_3^3 - 5a_0a_1a_2a_3a_4 + 5a_0a_2^2a_3^2 - 5a_0a_2^3a_4 - \\ &- 5a_1^3a_3a_4 + 5a_1^2a_2a_3^2 + 5a_1^2a_2^2a_4 - 5a_1a_2^3a_3 + a_2^5)d^2 + (-5a_0a_3a_4^3 + 5a_1a_3^2a_4^2 - \\ &- 5a_1a_2a_4^3 + a_3^5 - 5a_2a_3^3a_4 + 5a_2^2a_3a_4^2)d^3 + (a_4^5)d^4.\end{aligned}$$

Finally, if $\theta \in \mathbb{C}$ is such that $\mathbb{K} = \mathbb{Q}(\theta)$, where $p(x) = x^5 + ax + b \in \mathbb{Z}[x]$, with $a \neq 0$, is the minimal polynomial of θ, then $\{1, \theta, \theta^2, \theta^3, \theta^4\}$ is a basis for $\mathbb{K}$ over $\mathbb{Q}$ contained in $\mathcal{O}_{\mathbb{K}}$ and

$$\begin{aligned}\mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4) &= (-1)^{\frac{5(5-1)}{2}}[5^5b^{5-1} + (-1)^{5+1}(5-1)^{5-1}a^5] = 256a^5 + 3125b^4 \\ &= 256a^5 + 3125b^4.\end{aligned}$$

If $\mathcal{D}(1,\theta,\theta^2,\theta^3,\theta^4)$ is square-free, then the ring of algebraic integers of $\mathbb{K}$ is given by

$$\mathcal{O}_{\mathbb{K}} = \mathbb{Z}[\theta] = \{a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + a_4\theta^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{Z}\}.$$

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