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Multiplicity of an Implicit Differential Equation of Degree 3

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ABSTRACT: In this paper we introduce the concept of the multiplicity of an implicit differential equation of degree 3

$$F(x,y,p)=a(x,y)p^3+b(x,y)p^2+c(x,y)p+d(x,y)=0,$$

where $p=\frac{dy}{dx}$ and a, b, c, d are real analytic functions at an isolated singular point. We also show that the multiplicity introduced here is an invariant under smooth equivalence (Theorem 3.5) and an algebraic formula can be obtained to calculate it under certain reasonable hypothesis.

Key Words: Multiplicity, index, implicit differential equation.

Contents

1. Introduction

Let $F(x,y,p)=0$ be an implicit differential equation (IDE), where F is an analytic function and $p=\frac{dy}{dx}$. If $F_p(q_0)\neq 0$ at $q_0\in\mathbb{R}^3$, the IDE can be written locally in the form $p=g(x,y)$ and studied using methods from the theory of ordinary differential equations. When $F_p(q_0)=0$, the equation may define locally more than one direction in the plane. The cases that have been most intensively studied are the IDE's that define at most two directions in the plane. This is the case, for example, when:

$$F(x,y,p)=0,\quad F(q_0)=F_p(q_0)=0,\quad F_{pp}(q_0)\neq 0. \tag{1.1}$$

A natural way to study these equations is to lift the multi-valued direction field determined by the IDE to a single field ξ on the surface $M=F^{-1}(0)$. (This field is determined by the restriction to the manifold of the contact planes associated with the standard contact form $dy-pdx$ in $\mathbb{R}^3$). If 0 is a regular value of F, then M is smooth and the singularity of the projection to the plane is, generically, a fold or cusp. The critical set of this projection is called the criminant and its image is the discriminant of the equation.

In [9], Davydov classified (following the work of Dara [8]) generic bi-valued fields when the discriminant is smooth and showed that the topological normal form of the IDE acquires moduli when the discriminant is a cusp.

Implicit differential equations have extensive applications to differential geometry of surfaces, partial differential equations, control theory and singularity theory. For example, lines of curvature, asymptotic and characteristic lines on a smooth surface in $\mathbb{R}^3$ are given by IDE's ([4]) and the characteristic lines of a general linear second-order differential equation are also given by an IDE ([15]).

Bruce and Tari introduced in [3] the multiplicity of an IDE of degree 2, at a singular point, as the maximum number of singular points of the implicit differential equation which emerge when perturbing the equation F. In [5] and [6] the author defined the index of an IDE of degree 2 in terms of generic perturbations of the IDE and showed that this index is independent of the choice of a generic perturbation. One of the main results in [5] and [6] is the invariance of the index by smooth equivalences.

In this work we define the multiplicity of an IDE of degree 3 at an isolated singular point.

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2. Implicit Differential Equations

As mentioned in the introduction, an implicit differential equation is of the form

$$F(x, y, p) = 0, \tag{2.1}$$

where $p = \frac{dy}{dx}$ and F is an analytic function in $\mathbb{R}^3$. An integral curve of the IDE (2.1) is a smooth curve $\alpha = (\alpha_1, \alpha_2) : (-1, 1) \longrightarrow \mathbb{R}^2$ such that $\alpha_1'(t) \neq 0$ and $F(\alpha(t), \frac{\alpha_2'(t)}{\alpha_1'(t)}) = 0$.

Consider the surface $M = F^{-1}(0)$, and the projection $\pi : M \longrightarrow \mathbb{R}^2$, given by $\pi(x, y, p) = (x, y)$. Generically M is a smooth surface and the projection π is generically a submersion or has a singularity of type fold, cusp or two transverse folds ([2]). The critical set $F = F_p = 0$ of this projection is called the criminant and its image is the discriminant of the IDE.

The multi-valued direction field in the plane determined by the IDE lifts to a single vector field tangent to M given by

$$\xi = F_p \frac{\partial}{\partial x} + pF_p \frac{\partial}{\partial y} - (F_x + pF_y) \frac{\partial}{\partial p}.$$

Equivalently, this vector field is determined by the restriction of the canonical 1-form $dy - pdx$ to the surface M. Note that the vector field ξ may generically have an elementary zero, it is of type saddle, node or focus. One of the properties of this vector field is that the image by the projection π, of the integral curves of ξ on M, corresponds to integral curves of the IDE.

We denote by C_F the criminant set of the IDE. We also denote by $\omega_{|C_F}$ the restriction of the 1-form $\omega = dy - pdx$ on C_F.

Definition 2.1 *We say that $z_0 \in \mathbb{R}^2$ is a singular point of the IDE (2.1) if there exists $p_0 \in \mathbb{R}$ such that $q_0 = (z_0, p_0)$ is a zero of the 1-form $\omega_{|C_F}$.*

It is easy to verify that this definition reduces to Definition 2.3 given in [5] when the IDE defines at most two directions in the plane.

Proposition 2.1 *([3]) Let $q_0 \in \mathbb{R}^3$ be a point on the criminant. Then q_0 is a zero of the 1-form $\omega_{|C_F}$ if and only if q_0 is a zero of the vector field ξ or $F_{pp}(q_0) = 0$.*

From Proposition 2.1 it follows that the singular points of the IDE correspond to zeros of the vector field ξ or cusps of the natural projection π. We denote by (F, z_0) the germ of the IDE (2.1) at an isolated singular point z_0.

Definition 2.2 *We say that (F, z_1) and (G, z_2) are equivalent (resp. topologically equivalent) if there exists a germ of diffeomorphism (resp. homeomorphism) $h : (\mathbb{R}^2, z_2) \longrightarrow (\mathbb{R}^2, z_1)$ that sends integral curves of (G, z_2) to integral curves of (F, z_1).*

If $q_0 = (z_0, p_0)$ is a fold singularity of the projection π and is an elementary zero of ξ, then (F, z_0) is topologically equivalent to a well folded singularity $(p^2 - y + \lambda x^2, 0)$, $\lambda \neq 0, \frac{1}{4}$. We have a well folded saddle if $\lambda < 0$, well folded node if $0 < \lambda < \frac{1}{4}$ and a well folded focus if $\lambda > \frac{1}{4}$ (see [9]). When π has a cusp singularity at q_0, the equation has functional moduli with respect to topological equivalence [9].

Definition 2.3 *([8]) Let $q_0 \in \mathbb{R}^3$ be a point on the criminant, such that $F_{pp}(q_0) = 0$ and $F_{ppp}(q_0)F_x(q_0) \neq 0$.*

(i) We say that q_0 is an elliptic cusp if $(F_xF_{py} - F_yF_{px})(q_0) > 0$.

(ii) We say that q_0 is a hyperbolic cusp if $(F_xF_{py} - F_yF_{px})(q_0) < 0$.

In [8], Dara studied singularities of the IDE's, describing the singularities appearing in an open and dense set in the space of all functions F with the Whitney C^3-topology. The generic singularities described by Dara are the well folded saddle, well folded node, well folded focus and the elliptical and hyperbolic cusp.

A particular class of implicit differential equations are the binary differential equations (BDE's) of degree n, that is differential equations of the form

$$a_0(x,y)dy^n + a_1(x,y)dy^{n-1}dx + \ldots + a_n(x,y)dx^n = 0, \tag{2.2}$$

where a_i are analytic function defined on $U \subset \mathbb{R}^2$. If $dx = 0$ is not a solution of the Equation (2.2), we can set $p = \frac{dy}{dx}$ and reduce (2.2) to the IDE

$$F(x,y,p) = a_0(x,y)p^n + a_1(x,y)p^{n-1} + \ldots + a_n(x,y) = 0. \tag{2.3}$$

We say that the IDE of degree n given by (2.3) is totally real if $a_i(0,0) = 0$ (for any $i = 0,1,\ldots,n$) and for all $(x,y) \in U$, $(x,y) \neq 0$, the Equation (2.2) has exactly n different integral curves. An IDE of degree 1 is always totally real. In the case $n = 2$, an IDE is totally real if it is positive in the sense of [14]. In [12], Fukui and Nuño-Ballesteros introduce the concept of index for totally real IDE and produced a classification of generic singularities of this type of equations. Also, a generalization of the Poincaré-Hopf theorem and the Bendixon formula is obtained in [12].

In [6] the author defined the index of an IDE of degree 2 not necessarily totally real. One of the main results in [6] is the invariance of the index by smooth equivalences. We set $\delta = a_1^2 - 4a_0a_2$.

Theorem 2.1 *([6]) Let $(F,0)$ be the germ of an IDE of degree 2. If 0 is an isolated zero of the map $(\delta, a_0\delta_x - a_1\delta_y)$ and (δ,δ_y), then the index of $(F,0)$ at 0 is given by*

$$I(F,0) = \frac{1}{2}Ind_0(\delta,(a_0\delta_x - a_1\delta_y)a_0\delta_y) - \frac{1}{2}Ind_0(a_0,a_1) - \frac{1}{2}Ind_0(\delta\delta_x,\delta_y) + \frac{1}{2}Ind_0\nabla\delta.$$

When $F_{pp}(0) \neq 0$, the formula of the index simplifies. We denote by $\mathrm{Ind}_{q_0}\xi$ the index of the vector field ξ at $q_0 \in M$, introduced by W. Ebeling and S. M. Gusein-Zade in [10].

Theorem 2.2 *([6]) Let $(F,0)$ be the germ of an IDE of degree 2, and let 0 be a zero of the vector field ξ. If $F_{pp}(0) \neq 0$, then*

$$I(F,0) = \frac{1}{2}[Ind_0\xi + Ind_0(F_{pp}F_y, F_p, F_x + pF_y)].$$

3. Multiplicity of an Implicit Differential Equation of Degree 3

Definition 3.1 *We say that z_0 is a non-degenerate singular point of the IDE (2.1) if (F,z_0) is topologically equivalent to a well folded singularity, an elliptic cusp or a hyperbolic cusp.*

If z_0 is a non-degenerate singular point of the IDE (2.1), then there exists $p_0 \in \mathbb{R}$ such that (z_0,p_0) is a saddle, node or focus of the vector field ξ or elliptic cusp or hyperbolic cusp. So we can associate a number to each non-degenerate singular point z_0 of the IDE (2.1), as follows:

(i) $L_F(z_0) = 1$ if (z_0,p_0) is a saddle, node or focus of the vector field ξ.

(iii) $K_F(z_0) = 1$ if (z_0,p_0) is an elliptic or hyperbolic cusp.

A 1-parameter perturbation F^t of the IDE (2.3) is determined by the 1-parameter smooth perturbations $a_t^i(x,y) = \tilde{a}_i(x,y,t)$ (for any $i = 0,1,\ldots,n$) of its coefficients.

Definition 3.2 *We say that F^t is a good perturbation of $(F,0)$ if all the singular points of F^t are well folded singularities, or elliptical or hyperbolic cusps, for $t \neq 0$ sufficiently close to zero.*

Bruce and Tari introduced in [3] the multiplicity of an IDE of degree 2

$$F(x,y,p) = a(x,y)p^2 + 2b(x,y)p + c(x,y) = 0, \tag{3.1}$$

at a singular point, as the maximum number of singularities of well folded saddle, node or focus type appearing in a good perturbation of the IDE. We denote by $BT(F,0)$ the multiplicity of an IDE introduce by Bruce and Tari.

Definition 3.3 *([3]) Let F^t be a good perturbation of the germ $(F,0)$ given by (3.1). Then the multiplicity of $(F,0)$ at 0 is defined by*

$$BT(F,0) = \sum_i L_{F^t}(z_i),$$

where z_i are non-degenerate singular points of F^t of well folded saddle, node or focus type.

We set $\delta = b^2 - ac$.

Theorem 3.1 *([3]) Let $(F,0)$ be the germ of an IDE of degree 2. Then*

$$BT(F,0) = \dim_{\mathbb{R}} \mathcal{E}_2/(\delta, a\delta_x - b\delta_y) - \dim_{\mathbb{R}} \mathcal{E}_2/(a,b).$$

One of the main results in [3] is the invariance of the multiplicity by smooth equivalences.

Theorem 3.2 *([3]) The multiplicity $BT(F,0)$ is invariant under a smooth change of coordinates in the plane.*

Let $(F,0)$ be the germ of an IDE of degree 3 given by

$$F(x,y,p) = a(x,y)p^3 + b(x,y)p^2 + c(x,y)p + d(x,y) = 0, \tag{3.2}$$

such that $F(0) = F_p(0) = F_{pp}(0) = 0$. Using Equation (3.2) we deduce that

$$\begin{aligned} 27a^2F &= [3ap+b]^3 + 9ap[3ac-b^2] + 27a^2d - b^3 \\ 3aF_p &= [3ap+b]^2 + 3ac - b^2 \\ F_{pp} &= 2[3ap+b]. \end{aligned}$$

From the above equation, we obtain that the discriminant of the IDE (3.2) is given by

$$\delta = (27a^2d - 9abc + 2b^3)^2 + 4(3ac-b^2)^3.$$

We denote by $\mathcal{E}_n$ the ring of function germs in $\mathbb{R}^n$ at 0

Remark 3.1 *If 0 is an elliptic or a hyperbolic cusp of the germ $(F,0)$ (3.2), then*

$$\dim_{\mathbb{R}} \mathcal{E}_3/(F_{ppp}F, F_p, F_{pp})) = \dim_{\mathbb{R}} \mathcal{E}_2/(9ad - bc, 3ac - b^2) = K_F(0)$$

Let F^t be a 1-parameter perturbation of the IDE (3.2) given by

$$F^t(x,y,p) = a_t(x,y)p^3 + b_t(x,y)p^2 + c_t(x,y)p + d_t(x,y) = 0. \tag{3.3}$$

To show that F^t is a good perturbation of the IDE (3.2), it is sufficient to prove that 0 is a regular value of the map $(9a_td_t - b_tc_t, 3a_tc_t - b_t^2)$, for all $t \neq 0$. We denote by $\mathcal{P}_k(\mathbb{R}^2)$ the set of all polynomials of 2 variables and degree less than or equal to k. Let $\Phi : \mathbb{R}^2 \times \mathcal{P}_k^4(\mathbb{R}^2) \longrightarrow \mathbb{R}^2$ be a smooth map defined by

$$\Phi(x,y,r) = [9(a+\tilde{a})(d+\tilde{d}) - (b+\tilde{b})(c+\tilde{c}), 3(a+\tilde{a})(c+\tilde{c}) - (b+\tilde{b})^2](x,y),$$

where $r = (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) \in \mathcal{P}_k^4(\mathbb{R}^2)$. We set $\Phi_r(x,y) = \Phi(x,y,r)$.

Lemma 3.1 *There exists an open and dense set Δ of $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in \Delta$, 0 is a regular value of Φ_r.*

Proof: By Thom transversality lemma, there exists a dense set Δ of $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in \Delta$, 0 is a regular value of Φ_r, that is Φ_r intersect 0 transversally. It is not difficult to show that $H : \mathcal{P}_k^4(\mathbb{R}^2) \longrightarrow C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ given by

$$H(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = [9(a+\tilde{a})(d+\tilde{d}) - (b+\tilde{b})(c+\tilde{c}), 3(a+\tilde{a})(c+\tilde{c}) - (b+\tilde{b})^2]$$

is continuous. Let $r_0 = (a_0, b_0, c_0) \in \Delta$. As the set of maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ which intersect 0 transversally is open, we have that there exists a neighborhood U of r_0 in $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in U$, $H(r) = \Phi_r$ intersect 0 transversally. The result now follows. □

Theorem 3.3 *If a, b, c, d are analytic functions, then there exists a good perturbation F^t of $(F, 0)$.*

Proof: By Lemma 3.1, there exists an open and dense set Δ of $\mathcal{P}_k^4(\mathbb{R}^2)$ such that for all $r \in \Delta$, 0 is a regular value of Φ_r. It is not difficult to show that there exists a smooth curve $\alpha : (-1, 1) \to \mathcal{P}_k^4(\mathbb{R}^2)$ such that $\alpha[(-1, 1) - \{0\}] \subseteq \Delta$ and $\alpha(0) = 0$. Let $(a_t, b_t, c_t, d_t) = (a, b, c, d) + \alpha(t)$. Then, $F^t = a_t p^3 + b_t p^2 + c_t p + d_t = 0$ is a good perturbation of the IDE (3.2). □

Definition 3.4 *Let F^t be a good perturbation of the germ $(F, 0)$ given by (3.2). Then the multiplicity of $(F, 0)$ at 0 is defined by*

$$M(F, 0) = \sum_i K_{F^t}(z_i),$$

where z_i are non-degenerate singular points of F^t of elliptic or hyperbolic cusp type.

The next theorem shows that the multiplicity $M(F, 0)$ can be expressed in terms of the coefficients a, b, c and d. We denote by $\mathcal{E}_n^p$ the ring of function germs on $\mathbb{R}^n$ at p.

Theorem 3.4 *Let $(F, 0)$ be the germ of an IDE given by (3.2). If $F_{pp}(0) = 0$, then*

$$M(F, 0) = \dim_{\mathbb{R}} \mathcal{E}_2/(9ad - bc, 3ac - b^2) - \dim_{\mathbb{R}} \mathcal{E}_2/(a, b).$$

Proof: It follows from Theorem 3.3 that there exists a good perturbation F^t of the IDE (3.2) such that 0 is a regular value of $(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2)$, $t \neq 0$. Then using Proposition 2.2 in [7] we obtain

$$\dim_{\mathbb{R}} \mathcal{E}_2/(9ad - bc, 3ac - b^2) = \sum \dim_{\mathbb{R}} \mathcal{E}_2^{z_i}/(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2). \tag{3.4}$$

By Remark 3.1,

$$\sum_{a_t(z_i) \neq 0} \dim_{\mathbb{R}} \mathcal{E}_2^{z_i}/(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2) = K_{F^t}(z_i). \tag{3.5}$$

As 0 is a regular value of $(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2)$, we get

$$\begin{aligned}\sum_{a_t(z_i) = 0} \dim_{\mathbb{R}} \mathcal{E}_2^{z_i}/(9a_t d_t - b_t c_t, 3a_t c_t - b_t^2) &= \dim_{\mathbb{R}} \mathcal{E}_2^{z_i}/(a_t, b_t) \\ &= \dim_{\mathbb{R}} \mathcal{E}_2/(a, b).\end{aligned}$$

The result now follows. □

Theorem 3.5 *Let $(F, 0)$ and $(G, 0)$ be the germs of IDE's of degree 3. If $(F, 0)$ and $(G, 0)$ are equivalent, then $M(F, 0) = M(G, 0)$.*

Proof: It follows from the hypothesis that there exist a germ of diffeomorphim $h = (h_1, h_2) : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^2, 0)$ and a germ of function $\rho : (\mathbb{R}^2, 0) \longrightarrow \mathbb{R}$, $\rho(0) \neq 0$ such that $G = \rho \cdot (F \circ H)$, where

$$H(x, y, p) = (h(x, y), \frac{h_{2x}(x, y) + h_{2y}(x, y)p}{h_{1x}(x, y) + h_{1y}(x, y)p}).$$

We set $T(x, y, p) = \frac{h_{2x}(x,y)+h_{2y}(x,y)p}{h_{1x}(x,y)+h_{1y}(x,y)p}$. Then,

$$\begin{pmatrix} \rho & 0 & 0 \\ 0 & \rho T_p & 0 \\ 0 & T_{pp} & \rho T_p^2 \end{pmatrix} \begin{pmatrix} F \circ H \\ F_p \circ H \\ F_{pp} \circ H \end{pmatrix} = \begin{pmatrix} G \\ G_p \\ G_{pp} \end{pmatrix}, \tag{3.6}$$

$G_{ppp} = \rho[(F_{ppp} \circ H)T_p^3 + 3(F_{pp} \circ H)T_p T_{pp} + (F_p \circ H)T_{ppp}]$ and $T_p = \frac{det[dh]}{[h_{1x}+h_{1y}p]^2}$.

By Theorem 3.3, there exists a good perturbation F^t of $(F, 0)$. Using Equation (3.6), it is not difficult to show that $G^t = \rho \cdot (F^t \circ H)$ is a good perturbation of $(G, 0)$ and

$$\dim_{\mathbb{R}} \mathcal{E}_3^{r_i}/(F^t_{ppp}F^t, F^t_p, F^t_{pp}) = \dim_{\mathbb{R}} \mathcal{E}_3^{q_i}/(G^t_{ppp}G^t, G^t_p, G^t_{pp}),$$

where $H(q_i) = r_i$. The result follows by Remark 3.1 and Definition 3.4.

□

Corollary 3.1 *Let $(F, 0)$ be the germ of an IDE given by (3.2). If $F_{ppp}(0) \neq 0$ and $F_{pp}(0) = 0$, then*

$$M(F, 0) = \dim_{\mathbb{R}} \mathcal{E}_3/(F, F_p, F_{pp}).$$

Proof: The proof follows by using Theorem 3.4 and Remark 3.1. □

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