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# Existence and Uniqueness of the Weak Solution for Keller-Segel model coupled with the heat equation

Ali Slimani* and Amar Guesmia

ABSTRACT: Keller-Segel chemotaxis model is described by a system of nonlinear PDE : a convection diffusion equation for the cell density coupled with a reaction-diffusion equation for chemoattractant concentration. In this work, we study the phenomenon of Keller Segel model coupled with a heat equation, because The heat has an effect the density of the cells as well as the signal of chemical concentration, since the heat is a factor affecting the spread and attraction of cells as well in relation to the signal of chemical concentration, The main objectives of this work is the study of the global existence and uniqueness and boundedness of the weak solution for the problem defined in (2.4) for this we use the technical of Galerkin method.

Key Words: Chemotaxis, Keller-Segel, global existence, boundedness, Galerkin method.

## Contents



## 1. Introduction

Biological pattern formation is a topic of growing interest in the field of applied mathematics, both due to the possibility of developing new mathematics as well as for the broad range of important applications it might have [16]. At the cellular level, it is known that chemotaxis plays a fundamental role in the self-organization of many biological systems. Chemotaxis is the directed movement of an organism in response to ambient chemical gradients, that are oftenly segregated by the cells themselves. In those cases where the chemical products are attractive (and they are therefore called chemoattractants), they lead to the phenomenon known as chemotactic aggregation. One of the most important partial differential systems for understanding chemotactic aggregation is Keller-Segel model [14,20,21,22], which in a simplified form reads:

$$\partial_t \rho = D_\rho \triangle \rho - k\nabla(\rho\nabla c), \tag{1.1}$$
$$\delta c_t = D_c \triangle c - \tau c + \beta\rho. \tag{1.2}$$

Here $D_\rho$ is the cellular diffusion constant, $k$ the chemotactic coefficient, $\beta$ the rate of attractant production $\tau$ the rate of attractant depletion, $D_c$ the chemical diffusion constant, $\rho$ is the cell density, and $c$ is the chemical density. The terms in $Eq_1$ include the diffusion of the cells and chemotactic and $Eq_2$ expresses the diffusion and production of attractant. This system is known to have finite time blowing up solutions for large enough initial conditions in dimensions $d \geq 2$, but all the solutions are regular for $d = 1$,

* Corresponding author

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there is a large literature on the analysis of this problem for the Keller-Segel model as well as for some simplifications [5], [13], [17], [8], [9], [10], [18], [11], [12] the biological meaning of this mathematical fact was examined in Refs [4], [1]. So assuming the phenomenology, the one dimensional result is not just a model problem, but has relevance in predicting the behaviour of the biological system. Interestingly, the three-dimensional modalities of The fractional Keller-Segel model chemotactic collapse allowed by the Keller-Segel system have already been observed in experiments performed with Escherichia coli [2], [3], this would constitute an achievement of fundamental importance in mathematical biology. In this papers, we consider the regularity problem as

$$\begin{cases} u_t - \nabla(m\nabla u) + \nabla(\zeta u\nabla c + \mu u\nabla v) = 0 \quad , (x,t) \in \Omega\times\mathbb{R}^+, \\ \delta c_t - \triangle c + \tau c + \rho u + Kcv = 0 \quad , (x,t)\in\Omega\times\mathbb{R}^+, \\ v_t - \alpha\triangle v + \lambda(u+c) = 0 \quad , (x,t)\in\Omega\times\mathbb{R}^+. \end{cases} \tag{1.3}$$

Where $u = u(x,t)$ denots the density of the cells in position $x \in \mathbb{R}^d$ at time $t$, $c = c(x,t)$ is the concentration of chemical signal substance, $\delta \geq 0$ represents the relaxation time, the parameter $\zeta$ is the chemotactic coefficient and $m, \tau, \mu$ and $\rho, k, \lambda$ are given smooth functions, the term $v_t$ is heat distrubition over time and the term $\triangle v$ corresponds to a variation of $v$ compared to its average and $\alpha$ is the heat coefficient and the terms $\nabla(\mu u\nabla v), Kcv$ are directed cell movement by a heat and chemical degradation by a heat factor (Respectively). This problem deals with the extent of the influence of heat on the attraction and graduation of cell density and concentration of the chemical solution signal.
The main objectives of this work is to study of the problem Keller-Segel coupled with a heat equation on the form: $\delta = 1$ and $m, \tau$ and $\rho, k, \lambda$ are the positive constants and $\mu = \zeta$ are the negatif constant and $\alpha = 1$. In this paper we demonstrate the global existence and uniqueness of a weak solution for parabolic-parabolic-parabolic problem with the Dirichlet conditions and initials conditions defined as:

$$\begin{cases} P1\begin{cases} u_t - \nabla(m\nabla u) - \nabla(\zeta(u\nabla c + u\nabla v)) = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ u = 0, \quad \text{in } \Gamma, \\ u(0,x) = u_0, \quad x\in\Omega, \end{cases} \\ \\ P2\begin{cases} c_t - \triangle c + \tau c + \rho u + Kcv = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ c = 0, \quad \text{in } \Gamma, \\ c(0,x) = c_0, \quad x\in\Omega, \end{cases} \\ \\ P3\begin{cases} v_t - \triangle v + \lambda(u+c) = 0, \quad (x,t)\in\Omega\times\mathbb{R}^+, \\ v = 0, \quad \text{in } \Gamma, \\ v(0,x) = v_0, \ \ x\in\Omega. \end{cases} \end{cases} \tag{1.4}$$

## 2. Existence and uniqueness of weak solution of the problem

To simplify the weak solution of the problem (1.4) a decomposition into three subproblems $(P1)$ and $(P2)$ and $(P3)$ are adopted. We use the Galerkin method we can demonstrate the existence and uniquness of a weak solution of subproblems $(P1)$ and $(P2)$ and $(P3)$ therefore we have the existence and uniqueness of a weak solution of the problem (1.4). The following initial-boundary conditions assumption is used to prove the proposed solution of (1.4)

$$u_0 \in L^2(\Omega), \tag{2.1}$$

$$c_0 \in L^2(\Omega), \tag{2.2}$$

$$v_0 \in L^2(\Omega). \tag{2.3}$$

### 2.1. Existence and uniqueness of weak solution of the problem (P1)

In subsection, we state and prove the existence and uniqueness of weak solution result of the problem $(P1)$ .

**Definition 2.1** *We say* $u \in L^2(0,T;H_0^1(\Omega)) \times H_0^1(\Omega)$ *with* $u_t \in L^2(0,T;H^{-1}(\Omega))$ *is a weak solution of the problem* $(P1)$ *if and only if*

$$\langle u_t, \Phi\rangle + B(u,\Phi,t) = 0, \tag{2.4}$$

*where*

$$B(u,\Phi,t) = \int_\Omega [m(\nabla u \nabla \Phi) + \zeta(u\nabla c\nabla\Phi + u\nabla v)\nabla\Phi]dx, \tag{2.5}$$

*for all* $\Phi \in H_0^1(\Omega)$ *,*$0 \le t \le T$*,and*

$$u(0,x) = u_0 \in L^2(\Omega). \tag{2.6}$$

**Remark 2.1** *Note that* $u \in C([0,T];L^2(\Omega))$ *as* $u \in L^2(0,T;H_0^1(\Omega))$ *and* $u_t \in L^2(0,T;H^{-1}(\Omega))$ *Then equality* (2.6) *makes sense.*

Before proving the existence and uniqueness of weak solution of the problem $(P1)$, we need the following lemma:

**Lemma 2.1** *i) For all* $\Phi \in H_0^1(\Omega)$ *the* $B(u,\Phi,t)$*is continuous in* $H_0^1(\Omega)\times H_0^1(\Omega)$*, there exists a constant positive* $C$ *such that*

$$\mid B(u,\Phi,t) \mid \le C \parallel u \parallel_{H_0^1(\Omega)} \parallel \Phi \parallel_{H_0^1(\Omega)} . \tag{2.7}$$

*ii)For any* $u \in H_0^1(\Omega)$ *Then there exists a constant positive* $\beta$ *such that*

$$\beta \parallel u \parallel_{H_0^1(\Omega)} \le B(u,u,t), \quad \forall u \in H_0^1(\Omega). \tag{2.8}$$

**Proof:** i) We use the Cauchy-Schwarz inequality on (2.5) we obtain

$$\begin{aligned} \mid B(u,\Phi,t) \mid \quad &\le \mid m \mid \parallel \nabla u \parallel_{L^2(\Omega)} \parallel \nabla\Phi \parallel_{L^2(\Omega)} \\ &+ \mid \zeta \mid \parallel u \parallel_{L^2(\Omega)} \parallel \nabla c \parallel_{L^4(\Omega)} \parallel \nabla \Phi \parallel_{L^4(\Omega)} \\ &\times \mid \zeta \mid \parallel u \parallel_{L^2(\Omega)} \parallel \nabla v \parallel_{L^4(\Omega)} \parallel \nabla \Phi \parallel_{L^4(\Omega)}, \end{aligned}$$

and we have

$$\mid B(u,\Phi,t) \mid \le C \parallel u \parallel_{H^1(\Omega)} \parallel \Phi \parallel_{H^1(\Omega)} .$$

ii) The expression of $B(u,u,t)$ we obtain

$$B(u,u,t) = \textstyle\int_\Omega (m(\nabla u)^2 + \zeta(u\nabla c \nabla u + u\nabla v\nabla u)dx,$$

and we have

$$B(u,u,t) \ge \textstyle\int_\Omega (m(\nabla u)^2 dx = \grave{m} \parallel \nabla u \parallel^2_{L^2(\Omega)},$$

finally, inequality Poincares, gives $B(u,u,t) \ge \beta \parallel u \parallel^2_{H_0^1(\Omega)}$ . □

To demonstrate the existence of weak solution of problem $(P1)$ we use the method of Galerkin we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\left\{ \begin{array}{l} w_i \in H_0^1(\Omega), \\ \forall m; w_t....w_m, \text{ its linearly independent,} \\ \text{ the finite linear combinations of } w_i \text{ are dense in } H_0^1(\Omega). \end{array} \right. \tag{2.9}$$

We are looking for $u_m = u_m(t)$ solution ≺≺approached≻≻ of the problem in the form

$$u_m(t) = \sum_{i=1}^{m} g_{im}(t) w_i, \tag{2.10}$$

and $g_{im}$ to be determined by the conditions:

$$\left\{ \begin{array}{c} \langle u'_m, w_j\rangle + B(u_m, w_j, t) = 0, \\ 1 \le j \le m. \end{array} \right. \tag{2.11}$$

The nonlinear differential equation system is to be completed by the conditional:

$$u_m(0) = u_{0m}, \; u_{0m} = \textstyle\sum_{i=1}^m \alpha_{im} w_j \to u_0 \text{ in } H_0^1(\Omega), \text{ when } m \to \infty.$$

*2.1.1. Energy estimates.*
We propose now to send $m$ to infinity and show a subsequence of our solutions $u_m$ of the approximation problems (2.11) and (2.12) converges to a weak solution of $(P1)$. For this we will need some uniform estimates.

**Theorem 2.1** *(Energy estimates.) There exists a constant $C$ depending only on $\Omega$, $T$ and $c$, such that*

$$\max_{0\leq t\leq T} \| u_m \|_{L^2(\Omega)} + \| u_m \|_{L^2(0,T;H_0^1(\Omega))} + \| u'_m \|_{L^2(0,T;H^{-1}(\Omega))} \leq C \| u_0 \|_{L^2(\Omega)} . \tag{2.12}$$

**Proof:** In order to prove the estimation (2.12) we will estime each terms in the left side of (2.10) one by one as follows:

1. Multiplying equation (2.11) by $g_{jm}(t)$ and summing for $k = 1...m$, and then recalling (2.10) we find

$$\langle u_m^{'}, u_m\rangle + B(u_m, u_m, t) = 0, \tag{2.13}$$

and we have

$$\frac{1}{2}\frac{d}{dt}[\| u_m \|^2_{L^2(\Omega)}] + B(u_m, u_m, t) = 0, \tag{2.14}$$

From Lemma (2.1) there exists constant $\beta > 0$ such that

$$\beta \| u_m \|^2_{H_0^1(\Omega)} \leq B(u_m, u_m, t), \forall 0 \leq t \leq T, \tag{2.15}$$

and we have

$$\frac{d}{dt}(\| u_m \|^2_{L^2(\Omega)}) + \beta \| u_m \|^2_{H_0^1(\Omega)} \leq 0, \tag{2.16}$$

this implies that

$$\| u_m \|^2_{L^2(\Omega)} \leq \| u_m(0) \|^2_{L^2(\Omega)} \leq \| u_0 \|^2_{L^2(\Omega)}, \tag{2.17}$$

so we have

$$\max_{0\leq t\leq T} \| u_m \|_{L^2(\Omega)} \leq \| u_0 \|_{L^2(\Omega)} . \tag{2.18}$$

2. Integrate inequality (2.16) from 0 to $T$ and we employ the inequality (2.18) to find

$$\| u_m \|^2_{L^2(0,T;H_0^1(\Omega))} = \int_0^T \| u_m \|^2_{H_0^1(\Omega)} dt. \tag{2.19}$$

3. Fix any $v \in H_0^1(\Omega)$, with $\| v \|^2_{H_0^1(\Omega)} \leq 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$, $(k = 1, ..., m)$, we use (2.11) we deduce for all $0 \leq t \leq T$ that

$$(u_m^{'}, v^1) + B(u_m, v^1, t) = 0,$$

then (2.10) implies

$$\langle u_m^{'}, v\rangle = (u_m^{'}, v) = (u_m^{'}, v^1) = -B(u_m, v^1, t),$$

consequently

$$| \langle u_m^{'}, v\rangle | \leq C \| u_m \|^2_{H_0^1(\Omega)},$$

since

$$\| v^1 \|^2_{H_0^1(\Omega)} \leq \| v \|^2_{H_0^1(\Omega)} \leq 1,$$

we have

$$\| u_m^{'} \|_{H^{-1}(\Omega)} \leq C \| u_m \|_{H_0^1(\Omega)}$$

and therefore

$$\| u_m^{'} \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| u_m^{'} \|^2_{H^{-1}(\Omega)} dt \leq C \int_0^T \| u_m \|^2_{H_0^1(\Omega)} dt \leq C \| u_0 \|^2_{L^2(\Omega)} .$$

□

*2.1.2. Existence and uniqueness of weak solution.*
Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P1)$.

**Theorem 2.2** *(Existence of weak solution.) Under hypothesis (2.1), there exists a weak solution of problem $(P1)$.*

**Proof:** According to the energy estimates (2.12), we see that the sequence $\{u_m\}_{m=1}^{\infty}$is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{u_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ Consequently there exists a subsequence which is also noted by $\{u_m\}_{m=1}^{\infty}$ and a function $u \in L^2(0,T;H_0^1(\Omega))$, with $u^{'} \in L^2(0,T;H^{-1}(\Omega))$, such that

$$u_m \to u \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \tag{2.20}$$
$$u_m^{'} \to u^{'} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \tag{2.21}$$

2. Next fix an integer $N$ and choose a function $v \in C^1(0,T;H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} g^{(k)}(t) w_k, \tag{2.22}$$

where $\{g^{(k)}\}_{k=1}^{N}$ are given smooth functions, we choose $m \geq N$ and multipling equation (2.11) by $g^{(k)}(t)$ $\forall K = 1 ... N$, and then integrate with respect to $t$ to find

$$\int_0^t \langle u_m^{'}, v\rangle + B(u_m, v, t)dt = 0, \tag{2.23}$$

we recall (2.20) and to find upon passing to weak limits that

$$\int_0^t \langle u^{'}, v\rangle + B(u, v, t)dt = 0, \quad \forall v \in L^2(0,T;H_0^1(\Omega)), \tag{2.24}$$

as functions of the form (2.22) are dense in $L^2(0,T;H_0^1(\Omega))$. Hence in particular

$$\langle u', v\rangle + B(u, v, t) = 0, \quad \forall v \in H_0^1(\Omega) \text{ and } \forall t \in [0,T], \tag{2.25}$$

and from Remark (2.1) we have $u \in C(0,T;L^2(\Omega))$.

3. In order to prove $u(0) = u_0$, we first note from (2.6) that

$$\int_0^t -\langle u, v'\rangle + B(u, v, t) = (u(0), v(0)), \tag{2.26}$$

for each $v \in C^1(0,T;H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (2.23) we obtain

$$\int_0^t -\langle u_m, v'\rangle + B(u_m, v, t)dt = (u_0, v(0)), \tag{2.27}$$

we use again (2.26), we obtain

$$\int_0^t -\langle u, v'\rangle + B(u, v, t)dt = (u_0, v(0)), \tag{2.28}$$

since $u_m(0) \to u_0$ in $L^2(\Omega)$. Comparing (2.26) and (2.28), we conclude $u(0) = u_0$. □

**Theorem 2.3** *(Uniqueness of weak solutions. ) A weak solution of problem $(P1)$ is unique.*

**Proof:** We suppose there exists two weak solution $u_1$ and $u_2$and we put $U = u_2 - u_1$ then $U$ is also a solution of problem $(P1)$ with $U_0 = (u_2 - u_1)(0) \equiv 0$. Setting $v = U$ in identity (2.14) we are

$$\frac{d}{dt}(\frac{1}{2} \parallel U \parallel^2_{L^2(\Omega)}) + B(U,U,t) = 0,$$

from Lemma (2.1), we have $B(U,U,t) \geq \beta \parallel U \parallel^2_{H^1_0(\Omega)} \geq 0$, so $\frac{d}{dt}(\frac{1}{2} \parallel U \parallel^2_{L^2(\Omega)}) \leq 0$, then integrate with respect to $t$ to find

$$\parallel U \parallel^2_{L^2(\Omega)} \leq \parallel U_0 \parallel^2_{L^2(\Omega)} = 0,$$

Thus $U \equiv 0$. □

### 2.2. Existence and uniqueness of weak solution of problem (P2)

In subsection, we state and prove the existence and uniqueness of weak solution result of the problem $(P2)$

**Definition 2.2** *We say $c \in L^2(0,T;H^1_0(\Omega))$ with $c_t \in L^2(0,T;H^{-1}(\Omega))$ is a weak solution of the problem $(P2)$ if and only if*

$$\langle c_t, q\rangle + L(c,q,t) = 0, \tag{2.29}$$

*where*

$$L(c,q,t) = \int_\Omega [(\nabla c \nabla q) + \tau cq + \rho uq + Kvcq]dx, \tag{2.30}$$

*for all $q \in H^1_0(\Omega)$ ,$0 \leq t \leq T$,and*

$$c(0,x) = c_0 \in L^2(\Omega). \tag{2.31}$$

**Remark 2.2** *Note that $c \in C([0,T];L^2(\Omega))$ as $c \in L^2(0,T;H^1_0(\Omega))$ and $c_t \in L^2(0,T;H^{-1}(\Omega))$ Then equality (2.31) makes sense.*

To demonstrate existence of weak solution of problem (P1) we use the Galerkin method, we assume $w_k = w_k(x)$ are smooth functions verifying:

$$\begin{cases} w_i \in H^1_0(\Omega), \\ \forall m; w_t....w_m \text{ its linearly independent }, \\ \text{ the finite linear combination of } w_i \text{ are dense in } H^1_0(\Omega). \end{cases} \tag{2.32}$$

We are looking for $c_m = c_m(t)$ solution ≺≺approached≻≻ of the problem in the form

$$c_m(t) = \sum_{i=1}^m d_{im}(t) w_i, \tag{2.33}$$

the $d_{im}$ to be determined by the conditions:

$$\begin{cases} \langle c'_m, w_j\rangle + L(c_m, w_j, t) = 0, \\ 1 \leq j \leq m. \end{cases} \tag{2.34}$$

The system of nonlinear differential equations is to be completed by the initial conditions:

$$c_m(0) = c_{0m},\ c_{0m} = \sum_{i=1}^m \beta_{im} w_j \to c_0 \text{ in } H^1_0(\Omega),\ \text{when } m \to \infty. \tag{2.35}$$

We now propose to send $m$ to infinity and to show a subsequence of our solutions $c_m$ approximation problems (2.34) and (2.35) converges towards a weak solution of the problem $(P2)$. For this we need uniform estimates.

*2.2.1. Energy estimates.*
We propose now to send m to infinity and show a subsequence of our solutions $c_m$ of the approximation problems (2.34) and (2.35) converges to a weak solution of problem $(P2)$. For this we will need some uniform estimates.

**Theorem 2.4** *(Energy estimates.) They exists a constant $C$ depending only on $\Omega, T$ such that*

$$\max_{0\le t\le T} \| c_m \|_{L^2(\Omega)} + \| c_m \|_{L^2(0,T;H_0^1(\Omega))} + \| c'_m \|_{L^2(0,T;H^{-1}(\Omega))} \le C \| c_0 \|_{L^2(\Omega)} . \tag{2.36}$$

**Proof:** In order to prove the estimation (2.36) we will estime each terms in the left side of (2.34) one by one as follows:

1. Multiplying equation (2.34) by $d_{jm}(t)$ and summing for $j$ we find

$$\langle c'_m, w_j\rangle + B(c_m, c_m, t) = 0, \tag{2.37}$$

and we have

$$\frac{1}{2}\frac{d}{dt}[\| c_m \|^2_{L^2(\Omega)}] + L(c_m(t), c_m(t)) = 0, \tag{2.38}$$

and we put $\| v \| = \sqrt{L(v,v)}$ ($=$ is norm in $H_0^1(\Omega)$), so

$$\frac{1}{2}\frac{d}{dt}(\| c_m \|^2_{L^2(\Omega)}) + \| c_m \|^2_{H_0^1(\Omega)} = 0, \tag{2.39}$$

we have

$$\frac{d}{dt}(\| c_m \|^2_{L^2(\Omega)}) \le 0,$$

and we have

$$\| c_m \|^2_{L^2(\Omega)} \le \| c_m(0) \|^2_{L^2(\Omega)} \le \| c_0 \|^2_{L^2(\Omega)}, \tag{2.40}$$

so we are

$$\max_{0\le t\le T} \| c_m \|_{L^2(\Omega)} \le \| c_0 \|_{L^2(\Omega)} . \tag{2.41}$$

2. Integrate inequality (2.39) from 0 to $T$ and we use (2.41) to find

$$\| c_m \|^2_{L^2(0,T;H_0^1(\Omega))} = \int_0^T \| c_m \|^2_{H_0^1(\Omega)} dt. \tag{2.42}$$

3. Fix any $v \in H_0^1(\Omega)$, with $\| v \|^2_{H_0^1(\Omega)} \le 1$, and write $v = v^1 + v^2$, where $v^1 \in (w_k)_{k=1}^{k=m}$ and $(v^2, w_k) = 0$ for all $(k = 1, ..., m)$. we use (2.34) from all $0 \le t \le T$ that

$$(c'_m, v^1) + L(c_m, v^1, t) = 0.$$

Then (2.33) implies

$$(c'_m, v) = \langle c'_m, v\rangle = \langle c'_m, v^1\rangle = -L(c_m, v^1, t),$$

consequently

$$| \langle c'_m, v\rangle | \le C \| c_m \|^2_{H_0^1(\Omega)},$$

and as

$$\| v^1 \|^2_{H_0^1(\Omega)} \le \| v \|^2_{H_0^1(\Omega)} \le 1,$$

thus

$$\| c'_m \|_{H^{-1}(\Omega)} \le C \| c_m \|_{H_0^1(\Omega)},$$

and therefore

$$\| c'_m \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| c'_m \|^2_{H^{-1}(\Omega)} dt \le C \int_0^T \| c_m \|^2_{H_0^1(\Omega)} dt \le C \| c_0 \|^2_{L^2(\Omega)} .$$

□

*2.2.2. Existence and uniqueness of weak solution.*
Next, we pass to limits as $m \to \infty$, to build a weak solution of our initial boundary-value problem $(P2)$.

**Theorem 2.5** *(Existence of weak solution.) Under hypothesis* (2.2)*, there exists a weak solution of* $(P2)$*.*

**Proof:** According to the energy estimates (2.36), we see that the sequence $\{c_m\}_{m=1}^{\infty}$is bounded in $L^2(0,T;H_0^1(\Omega))$ and $\{c_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0,T;H^{-1}(\Omega))$ Consequently there exists a subsequence which is also noted by $\{c_m\}_{m=1}^{\infty}$ and a function $c \in L^2(0,T;H_0^1(\Omega))$ with $c^{'} \in L^2(0,T;H^{-1}(\Omega))$, such that

$$\begin{aligned} c_m &\to c \text{ weakly in } L^2(0,T;H_0^1(\Omega)), \\ c_m^{'} &\to c^{'} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)). \end{aligned} \tag{2.43}$$

2. Next fix an integer $N$ and choose a function $v \in C^1(0,T;H_0^1(\Omega))$ having the form

$$v(t) = \sum_{k=1}^{N} d^{(k)}(t) w_k, \tag{2.44}$$

where $\{d^{(k)}\}_{k=1}^{N}$ are given smooth functions. We choose $m \geq N$, multiply equation (2.34) by $d^{(k)}(t)$ $\forall K = 1...N$, and then integrate with respect to $t$ to find

$$\int_0^t \langle c_m^{'}, v\rangle + L(c_m, v, t)dt = 0, \tag{2.45}$$

we recall (2.43) to find upon passing to weak limits that

$$\int_0^t \langle c^{'}, v\rangle + L(c, v, t)dt = 0, \ \forall v \in L^2(0,T;H_0^1(\Omega)). \tag{2.46}$$

As functions of the form (2.44) are dense in $L^2(0,T;H_0^1(\Omega))$. Hence in particular

$$\langle c', v\rangle + L(c, v, t) = 0, \ \forall v \in H_0^1(\Omega) \text{ et } \forall t \in [0,T] \tag{2.47}$$

and as (2.31) and from Remark (2.2) we have $c \in C(0,T;L^2(\Omega))$.
3. In order to prove for prouver $c(0) = c_0$, we first note from (2.31) that

$$\int_0^t -\langle c, v'\rangle + L(c, v, t)dt = (c(0), v(0)), \tag{2.48}$$

for each $v \in C^1(0,T;H_0^1(\Omega))$ with $v(T) = 0$. Similary, from (2.45) we obtain

$$\int_0^t -\langle c_m, v'\rangle + B(c_m, v, t)dt = (c_0, v(0)), \tag{2.49}$$

we use again (2.48), we obtain

$$\int_0^t -\langle c, v'\rangle + B(c, v, t)dt = (c_0, v(0)), \tag{2.50}$$

since $c_m(0) \to c_0$ in $L^2(\Omega)$. Comparing (2.48) and (2.50), we conclude $c(0) = c_0$. □

**Theorem 2.6** (*Uniqueness of weak solutions.* ) *A weak solution of problem* $(P2)$ *is unique.*

**Proof:** We suppose there exists two weak solution $c_1$ et $c_2$ and we put that $C = c_2 - c_1$ then $C$ is also a solution of $(P2)$ with $C_0 = (c_2 - c_1)(0) \equiv 0$. Setting $v = C$ in identity (2.47) we have

$$\frac{d}{dt}(\frac{1}{2} \parallel C \parallel^2_{L^2(\Omega)}) + L(C, C, t) = 0,$$

and as $\parallel C \parallel = \sqrt{L(C,C)}$ (= norm in $H^1_0(\Omega)$), there $L(C,C,t) = \parallel C \parallel^2_{H^1_0(\Omega)} \geq 0$, then we have

$$\tfrac{d}{dt}(\tfrac{1}{2} \parallel C \parallel^2_{L^2(\Omega)}) \leq 0,$$

then integrate with respect to t to find

$$\parallel C \parallel^2_{L^2(\Omega)} \leq \parallel C_0 \parallel^2_{L^2(\Omega)} = 0,$$

then $C \equiv 0$. □

### 2.3. Existence and uniqueness of weak solution of the problem (P3)

In subsection, we state and prove existence and uniqueness of weak solution result of the problem $(P3)$

**Definition 2.3** *We say $v \in L^2(0,T; H^1_0(\Omega))$ with $v_t \in L^2(0,T; H^{-1}(\Omega))$ is a weak solution to the problem $(P3)$ if and only if*

$$\langle v_t, \Psi \rangle + A(v, \Psi, t) = 0, \tag{2.51}$$

*when*

$$A(v, \Psi, t) = \int_\Omega \nabla v \nabla \Psi + \lambda(u + c)\Psi dx, \tag{2.52}$$

*for all* $\Psi \in H^1_0(\Omega)$ *,*$0 \leq t \leq T$*,and*

$$v(0, x) = v_0 \in L^2(\Omega). \tag{2.53}$$

**Remark 2.3** *Note that $v \in C([0,T]; L^2(\Omega))$ as $v \in L^2(0,T; H^1_0(\Omega))$ and $v_t \in L^2(0,T; H^{-1}(\Omega))$ Then equality (2.53) makes sense.*

Before proving the existence and uniqueness of weak solution of problem $(P3)$, we need the following lemma:

**Lemma 2.2** *i)For all $\Psi \in H^1_0(\Omega)$ and $A(v, \Psi, t)$is continuous in $H^1_0(\Omega) \times H^1_0(\Omega)$, there exists a constant positive $C$such that*

$$| A(v, \Psi, t) | \leq C \parallel v \parallel_{H^1_0(\Omega)} \parallel \Psi \parallel_{H^1_0(\Omega)} . \tag{2.54}$$

*ii)For any $v \in H^1_0(\Omega)$ Then there exists a constant positive $\beta$ such that*

$$\beta \parallel v \parallel_{H^1_0(\Omega)} \leq A(v, v, t), \ \forall v \in H^1_0(\Omega). \tag{2.55}$$

**Proof:** i) We use the Cauchy-Schwarz inequality on (2.52) we obtain

$$| \quad A(u, v, t) | \leq \parallel \nabla v \parallel_{L^2(\Omega)} \parallel \nabla \Psi \parallel_{L^2(\Omega)} \lambda \parallel (u + c) \parallel_{L^2(\Omega)},$$

and

$$| \quad A(u, v, t) | \leq C \parallel v \parallel_{H^1(\Omega)} \parallel \Psi \parallel_{H^1(\Omega)} .$$

ii) The expression of $A(v, v, t)$ becomes

$$A(v, v, t) = \int_\Omega [(\nabla v)^2 + \lambda(u + c)]dx \geqslant \int_\Omega (\nabla v)^2 dx = \parallel \nabla v \parallel^2_{L^2(\Omega)},$$

finally, Poincares inequality, gives $A(v,v,t) \geq \beta \parallel v \parallel^2_{H^1_0(\Omega)}$ . □

To demonstrate existence of a weak solution of $(P3)$ we use the Galerkin method we suppose that $w_k = w_k(x)$ are smooth functions checking:

$$\begin{cases} w_i \in H^1_0(\Omega), \\ \forall m; w_t....w_m \text{ its linearly independent}, \\ \text{the finite linear combination of } w_i \text{ are dense in } H^1_0(\Omega). \end{cases} \tag{2.56}$$

we are looking for $v_m = v_m(t)$ solution ≺≺approached≻≻ of the problem in the form

$$v_m(t) = \sum_{i=1}^{m} l_{im}(t) w_i, \tag{2.57}$$

the $l_{im}$ to be determined by the conditions:

$$\begin{cases} \langle v_m^{'}, w_j \rangle + A(v_m, w_j, t) = 0, \\ \qquad 1 \leq j \leq m. \end{cases} \tag{2.58}$$

The system of nonlinear differential equations is to be completed by the initial condition:

$$v_m(0) = v_{0m}, \ v_{0m} = \sum_{i=1}^{m} \alpha_{im} w_j \to v_0 \text{ in } H^1_0(\Omega), \text{ when } m \to \infty. \tag{2.59}$$

We now propose to send $m$ to infinity and to show a subsequence of our solutions $v_m$ of approximation problems (2.58) and (2.59) converges to a weak solution of problem $(P3)$. For this we will need uniform estimates.

*2.3.1. Energy estimates.*

**Theorem 2.7** *(Energy estimates.) There is a constant $C$ depending only on $\Omega, T$ such that*

$$\max_{0 \leq t \leq T} \parallel v_m \parallel_{L^2(\Omega)} + \parallel v_m \parallel_{L^2(0,T;H^1_0(\Omega))} + \parallel v_m' \parallel_{L^2(0,T;H^{-1}(\Omega))} \leq C \parallel v_0 \parallel_{L^2(\Omega)} . \tag{2.60}$$

**Proof:** Multiplying equation (2.58) index $j$ by $l_{jm}(t)$ and we are in $j$ he comes :

$$\langle v_m^{'}, w_j \rangle + A(v_m, v_m, t) = 0, \tag{2.61}$$

where

$$\frac{1}{2}\frac{d}{dt}[\parallel v_m \parallel^2_{L^2(\Omega)}] + A(v_m, v_m, t) = 0, \tag{2.62}$$

we have according to the lemma (2.2) there is a constant $\beta > 0$ such That

$$\beta \parallel v_m \parallel^2_{H^1_0(\Omega)} \leq A(v_m, v_m, t), \forall 0 \leq t \leq T, \tag{2.63}$$

so

$$\frac{d}{dt}(\parallel v_m \parallel^2_{L^2(\Omega)}) + \beta \parallel v_m \parallel^2_{H^1_0(\Omega)} \leq 0, \tag{2.64}$$

therefore

$$\parallel v_m \parallel^2_{L^2(\Omega)} \leq \parallel v_m(0) \parallel^2_{L^2(\Omega)} \leq \parallel v_0 \parallel^2_{L^2(\Omega)}, \tag{2.65}$$

then we have

$$\max_{0 \leq t \leq T} \parallel v_m \parallel_{L^2(\Omega)} \leq \parallel v_0 \parallel_{L^2(\Omega)} . \tag{2.66}$$

Integrated inequality (2.64) from 0 to $T$ and we use (2.66) find

$$\parallel v_m \parallel^2_{L^2(0,T;H^1_0(\Omega))} = \int_0^T \parallel v_m \parallel^2_{H^1_0(\Omega)} dt, \tag{2.67}$$

fixed everything $u \in H_0^1(\Omega)$, with $\| u \|^2_{H_0^1(\Omega)} \leq 1$, and write $u = u^1 + u^2$
where $u^1 \in (w_k)_{k=1}^{k=m}$ and $(u^2, w_k) = 0,\ (k = 1, ..., m)$. we use (2.58) from all $0 \leq t \leq T$ that

$$(v_m^{'}, u^1) + A(v_m, u^1, t) = 0,$$

when (2.57) we find

$$(v_m^{'}, u) = (v_m^{'}, u^1) = -A(v_m, u^1, t),$$

therefore

$$| (v_m^{'}, u) | \leq C \| v_m \|^2_{H_0^1(\Omega)}$$

and as

$$\| u^1 \|^2_{H_0^1(\Omega)} \leq \| u \|^2_{H_0^1(\Omega)} \leq 1,$$

therefore

$$\| v_m^{'} \|_{H^{-1}(\Omega)} \leq C \| v_m \|_{H_0^1(\Omega)},$$

and when

$$\| v_m^{'} \|^2_{L^2(0,T;H^{-1}(\Omega))} = \int_0^T \| v_m^{'} \|^2_{H^{-1}(\Omega)} dt \leq C \int_0^T \| v_m \|^2_{H_0^1(\Omega)} dt \leq C \| v_0 \|^2_{L^2(\Omega)} .$$

□

*2.3.2. Existence and uniqueness of weak solution.*
Then we pass the limit as $m \to \infty$, to build a weak solution for the initial problem condition $(P3)$.

**Theorem 2.8** *Under hypothesis (2.3), There is a weak solution of problem $(P3)$.*

**Proof:** According to energy estimates (2.60), we see that the sequence $\{v_m\}_{m=1}^{\infty}$is bounded in $L^2(0, T; H_0^1(\Omega))$ and $\{v_m^{'}\}_{m=1}^{\infty}$ is bounded in $L^2(0; T; H^{-1}(\Omega))$ Therefore, there is a subsequence which is also noted by$\{v_m\}_{m=1}^{\infty}$ and a function $v \in L^2(0, T; H_0^1(\Omega))$, with $v^{'} \in L^2(0; T; H^{-1}(\Omega))$, such that

$$v_m \rightarrow v \text{ weakly in } L^2(0, T; H_0^1(\Omega)), \tag{2.68}$$
$$v_m^{'} \rightarrow v^{'} \text{ weakly in } L^2(0; T; H^{-1}(\Omega)). \tag{2.69}$$

Next fix an integer $N$ and takes a function $u \in C^1(0, T; H_0^1(\Omega))$ under the form

$$u(t) = \sum_{k=1}^{N} g^{(k)}(t) w_k, \tag{2.70}$$

where $\{g^{(k)}\}_{k=1}^{N}$ are given smooth functions and we choose $m \geq N$, multipling the equation (2.58) by $g^{(k)}(t)$ and $\forall K = 1...N$, then integrate with respect to $t$ to find

$$\int_0^t \langle v_m^{'}, u \rangle + A(v_m, u, t) dt = 0, \tag{2.71}$$

we recall and (2.69) to find passing low limits that

$$\int_0^t \langle v^{'}, u \rangle + A(v, u, t) dt = 0,\ \forall u \in L^2(0, T; H_0^1(\Omega)), \tag{2.72}$$

as functions of the form (2.70) are dense in $L^2(0, T; H_0^1(\Omega))$. In particular

$$\langle v', u \rangle + A(v, u, t) = 0,\ \forall u \in H_0^1(\Omega) \text{ and } \forall t \in [0, T] \tag{2.73}$$

and as (2.53) and from Remark(2.3) we have $v \in C(0, T; L^2(\Omega))$.
To prove $v(0) = v_0$, we first note of (2.6) that

$$\int_0^t -\langle v, u' \rangle + A(v, u, t) = (v(0), u(0)), \tag{2.74}$$

for each $u \in C^1(0, T; H_0^1(\Omega))$ with $u(T) = 0$. Similary, from (2.71) we obtian

$$\int_0^t -\langle v_m, u' \rangle + A(v_m, u, t)dt = (v_0, u(0)), \tag{2.75}$$

we use again (2.68), we obtian

$$\int_0^t -\langle v, u' \rangle + A(v, u, t)dt = (v_0, u(0)), \tag{2.76}$$

since $v_m(0) \to v_0$ in $L^2(\Omega)$. Comparing (2.75) and (2.76), we conclude $v(0) = v_0$. □

**Theorem 2.9** *(Uniqueness of weak solutions.) A weak solution of problem* $(P3)$ *is unique.*

**Proof:** We suppose there exists two weak solution $v_1$ and $v_2$ and we put that $V = v_2 - v_1$ then $V$ is also a solution of $(P3)$ with $V_0 = (v_2 - v_1)(0) \equiv 0$. Setting $u = V$ in identity (2.73), we have

$$\frac{d}{dt}\left(\frac{1}{2} \parallel V \parallel^2_{L^2(\Omega)}\right) + A(V, V, t) = 0,$$

for lemma (2.2), we have $A(V, V, t) \geq \beta \parallel V \parallel^2_{H_0^1(\Omega)} \geq 0$, so $\frac{d}{dt}(\frac{1}{2} \parallel V \parallel^2_{L^2(\Omega)}) \leq 0$,
then integrate with respect to $t$ to find

$$\parallel V \parallel^2_{L^2(\Omega)} \leq \parallel V_0 \parallel^2_{L^2(\Omega)} = 0,$$

thus $V \equiv 0$. □

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*Ali Slimani,*
*Department of Software and Information Systems Technologies,*
*Faculty of New Information and Communication Technologies,*
*University of Abdelhamid Mehri Constantine 2,*
*Algeria.*
*E-mail address:* ali.slimani@univ-constantine2.dz, ali.slimani@univ-skikda.dz

*and*

*Amar Guesmia,*
*Department of Mathematics,*
*University 20 august 1955,*
*Algeria.*
*E-mail address:* a.guesmia@univ-skikda.dz.