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Improvement of the piecewise polynomial collocation method for Fredholm integro-differential equations

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ABSTRACT: In this work, in the first step, the error estimation by using defect correction principle is studied for the numerical approximation of Fredholm integro-differential equations. Based on theoretical study, it is shown that for m degree piecewise polynomial collocation method, the deviation of the error is $\mathcal{O}(h^{m+1})$. In the next step by using the deviation of the error the collocation solution has been improved. Also in the last step of this paper, simulated results to investigate the theory results are given.

Key Words: Fredholm integro-differential equations, collocation, finite difference, error analysis.

Contents

1. Introduction

Many phenomena in nature are expressed by different types of equations. One of the most important equations used in various branches of science and engineering is the Fredholm integro-differential equation (FIDE) equations [1]. In recent years, various methods have been studied for numerical solution of these equations, such as the iterated Galerkin approximation and finite element solutions are studied in [2,3]. Also, other methods can be found in [4,5]. One of the efficient methods to solve this type of equations is the piecewise polynomial collocation method that can be found in [6,7,8]. In addition to this category of equations, fractional, delay and systems of equations have also received attention in the last few years. Readers can find more information about these topics in [9,10,11,12,13,14,15]. Furthermore, other collocation methods with different basic functions have been employed to solve integral equations. For instance, the Bessel collocation method is utilized in [16,17], and the Pell-Lucas collocation method is employed in [18]. Also, based on the behavior of the kernel of integral equations, various methods have been used to approximate the solution, which can be referred to the methods described [19,20]. In this paper, this method is improved by using Error estimation based on defect correction principle. Now, we continue to introduce and state the conditions for this equation. This type of equation is defined as

$$y'(t) = F\big(t, y(t), z[y](t)\big), \quad t \in I := [a,b], \tag{1.1}$$
$$\alpha y(b) + \beta y(a) = r, \tag{1.2}$$

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where

$$z[y](t) := \int_a^b K\big(t, s, y(s)\big)ds. \tag{1.3}$$

and $a, b, \beta, \alpha, r \in R = (-\infty, \infty)$, $\alpha + \beta \neq 0$ and $b > a$. We define W and S as follows

$$W := \{(t, y, z) \,|\, t \in I, y, z \in (-\infty, \infty)\},$$
$$S := \{(t, s, u) \,|\, t, s \in I, u \in (-\infty, \infty)\}.$$

In this paper, we shall assume that F and K are uniformly continuous in W and S, respectively. We say that $z[y](t)$ is linear if we can write $z[y](t)$ as

$$z[y](t) = \int_a^b \Lambda(t, s)y(s)ds, \tag{1.4}$$

where $\Lambda(t, s)$ is sufficiently smooth in $J := \{(t, s) \,|t, s \in I\}$. Also, we say that F is linear if we can write $F(t, y(t), z[y](t))$ as

$$F(t, y(t), z[y](t)) = a_1(t)y(t) + a_2(t) + z[y](t).$$

In the nonlinear case, we assume that $F(t, y, z)$, $F_t(t, y, z)$, $F_y(t, y, z)$ and $F_z(t, y, z)$ are Lipschitz-continuous. Also, when $z[y](t)$ is nonlinear we assume that $K(t, s, u)$ and $K_u(t, s, u)$ are Lipschitz-continuous. We say FIDE with boundary condition (1.2) is linear if we can write (1.1) as follows

$$y'(t) = a_1(t)y(t) + a_2(t) + z[y](t), \quad t \in [a, b], \tag{1.5}$$

with linear $z[y](t)$. Also in the linear case, we assume that $a_1(t), a_2(t) \in \mathcal{C}(I)$. In this paper we study the deviation of the error for the linear and nonlinear Fredholm integro-differential equations. We show that for m degree piecewise polynomial collocation method the order of the deviation of the error is at least $\mathcal{O}(h^{m+1})$. In this paper we find the deviation of the error by using first-order finite difference scheme although we can use second-order finite difference scheme and a similar argument can be studied for this finite difference scheme. The general studies on the structure of defect correction principle can be found in [21,22]. The deviation of the error estimation based on piecewise polynomial collocation method for linear and nonlinear first order and second order boundary value problems can be found in [23,24]. Also, in the present paper, the error correction method based on defect correction principle is used to improve the collocation method. The error correction method based on other methods as spectral deferred is used in [25]. Although the error correction method in the present paper and in [25,26] has been used to improve the collocation method, the details of the structures of the two methods are very different. The reader can refer to the articles [21] and [26] to see the details of the difference between the two methods. Also, the discussion of the error estimation for other integro-differential equations have been studied by the authors in [27,28,29]

The outline of the paper is as follows. In Section 2, piecewise polynomial collocation method, finite differences method and exact difference scheme are introduced. In Section 3, we give a complete analysis of the deviation of the error for linear and nonlinear cases. The main results of the paper are formulate in Theorems 3.1, 3.2, 3.3 and 3.4. In Section 4, we present the results of numerical experiments that demonstrate our theoretical results. Also conclusions are included in Section 5.

2. Description of the method

In this section, we introduce some details about the deviation of the error estimation.

2.1. Collocation method

In this subsection, we introduce the piecewise polynomial collocation method for solution of the FID problem (1.1)-(1.2). Let

$$a = \tau_0 < \tau_1 < \ldots < \tau_n = b,\ (n \geq 1),$$
$$0 = \rho_0 < \rho_1 < \ldots < \rho_m < \rho_{m+1} = 1,$$

and $h_i := \tau_{i+1} - \tau_i$, we define X_n, Z_n and $S_m^{(0)}(Z_n)$ as follows

$$X_i := \{t_{i,j} := \tau_i + \rho_j h_i; j = 1, \ldots, m\},$$

$$Z_n := \{t_{i,0} := \tau_i;\ i = 0, \ldots, n\},$$

$$S_m^{(0)}(Z_n) := \{p \in \mathcal{C}(I);\ p \restriction [\tau_i, \tau_{i+1}] \in \Pi_m([\tau_i, \tau_{i+1}])(i = 0, \ldots, n-1)\}.$$

In the above definition, $\Pi_m([\tau_i, \tau_{i+1}])$ is space of real polynomial functions on $[\tau_i, \tau_{i+1}]$ of degree$\leqslant m$. We define h (the diameter of grid Z_n) and h' as

$$h := \max\{h_i;\ i = 0, \ldots, n-1\},\ h' := \min\{h_i;\ i = 0, \ldots, n-1\}.$$

In this paper, the set $X(n) := \bigcup_{i=0}^{n-1} X_i$ is called the set of collocation points. According to the piecewise polynomial collocation method, we are looking to find a $p \in S_m^{(0)}(Z_n)$ so that (1.1)-(1.2) holds for all $t \in X(n)$. In the collocation method, since we can not determine exact value for $z[p](t)$, therefore we use the following method to determine $z[p](t)$. In the first step, we define

$$L_j(\rho) := \prod_{\substack{i=1\\ i\neq j}}^{m+1} \frac{\rho - \rho_i}{\rho_j - \rho_i}, \quad L_j^{[a',b']}(\rho) := L_j(\frac{\rho - a'}{b' - a'}), \quad a \leq a' < b' \leq b,$$

then by using quadrature method, we find

$$z[p](t_{i,j}) \approx \sum_{k=0}^{n-1} \sum_{z=1}^{m+1} \alpha_{z,k} K\big(t_{i,j}, t_{k,z}, p(t_{k,z})\big) =: \widetilde{z}[p](t_{i,j}), \tag{2.1}$$

where the quadrature weights are given by

$$\alpha_{z,k} := \int_{\tau_k}^{\tau_{k+1}} L_z^{[\tau_k, \tau_{k+1}]}(s) ds.$$

By using Interpolation error theorem [30], the following lemma is easily proved.

Lemma 2.1 *For sufficiently smooth f, the following estimate holds*

$$|z[f](t_{i,j}) - \widetilde{z}[f](t_{i,j})| = \mathcal{O}(h^{m+1}), \tag{2.2}$$

where $\widetilde{z}[\cdot](t_{i,j})$ is defined in (2.1).

In a similar way to [6,7], for the piecewise polynomial collocation method, we can find the following theorem.

Theorem 2.1 *Assume that the FID problem (1.1)-(1.2) has a unique and sufficiently smooth solution $y(t)$. Also assume that $p(t)$ is a piecewise polynomial collocation solution of degree$\leq m$. Then for sufficiently small h, the collocation solution $p(t)$ is well-defined and the following uniform estimate at least holds:*

$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^m),\ j = 0, 1.$$

Remark 2.1 In special case, we can see that for equidistant collocation grid points with odd m the following uniform estimate holds

$$\|y^{(j)}(t) - p^{(j)}(t)\|_\infty = \mathcal{O}(h^{m+1}),\ j = 0, 1.$$

Lemma 2.2 *For linear and nonlinear $z[\cdot](t)$ we have*

$$|\widetilde{z}\,[p](t_{i,j}) - \widetilde{z}\,[y](t_{i,j})| = \mathcal{O}(h^m).$$

Proof: By using Lemma 2.1, Theorem 2.1 and the Integral mean value theorem, for linear case we get

$$\begin{aligned}
\widetilde{z}\,[p](t_{i,j}) - \widetilde{z}\,[y](t_{i,j}) &= \underbrace{\widetilde{z}\,[e](t_{i,j}) - z[e](t_{i,j})}_{\mathcal{O}(h^{m+1})} + z[e](t_{i,j}) = \\
&= \int_a^b \Lambda(t_{i,j}, s) e(s) ds + \mathcal{O}(h^{m+1}) \\
&= (b-a)\,\Lambda(t_{i,j}, \zeta_{i,j})\,\underbrace{e(\zeta_{i,j})}_{\mathcal{O}(h^m)} + \mathcal{O}(h^{m+1}) \\
&= \mathcal{O}(h^m),
\end{aligned}$$

where $\zeta_{i,j} \in [a,b]$ and $e(s) = y(s) - p(s)$. For nonlinear case by using Lemma 2.1 we get

$$\begin{aligned}
&\widetilde{z}\,[y](t_{i,j}) - \widetilde{z}\,[p](t_{i,j}) = \widetilde{z}\,[y](t_{i,j}) - z[y](t_{i,j}) - \widetilde{z}\,[p](t_{i,j}) + z[p](t_{i,j}) \\
&+ z[y](t_{i,j}) - z[p](t_{i,j}) = \int_a^b \Big(K\big(t_{i,j}, s, y(s)\big) - K\big(t_{i,j}, s, p(s)\big)\Big) ds \\
&+ \mathcal{O}(h^{m+1}) = \mathcal{O}(h^m),
\end{aligned} \tag{2.3}$$

We note that in (2.3) by using the Lipschitz condition for K we find

$$|K\big(t_{i,j}, s, y(s)\big) - K\big(t_{i,j}, s, p(s)\big)| \le C|y(s) - p(s)| = \mathcal{O}(h^m),$$

which completes the proof. □

2.2. Finite difference scheme

We define $\mathcal{A}$ and $\mathcal{B}$ as follows

$$\mathcal{A} := \{(i,j);\ t_{i,j} \in X(n) \cup Z_n\}, \qquad \mathcal{B} := \mathcal{A} - \{(n,0)\}.$$

We write a general one-step finite difference scheme as

$$(L^{(1)}_{\mathcal{A}}\eta)_{i,j} := \frac{\eta_{i,j+1} - \eta_{i,j}}{\delta_{i,j}} = F(t_{i,j}, \eta_{i,j}, \chi[\eta]_{i,j}),\ (i,j) \in \mathcal{B}, \tag{2.4}$$

$$\alpha\,\eta_{n,0} + \beta\,\eta_{0,0} = r, \tag{2.5}$$

where $\delta_{i,j} := t_{i,j+1} - t_{i,j}$ and

$$\chi[\eta]_{i,j} := \sum_{(l,v)\in\mathcal{B}} \delta_{l,v} K(t_{i,j}, t_{l,v}, \eta_{l,v}).$$

Definition 2.1 For any function u, we define

$$\mathcal{R}(u) := \{u(t_{i,j})\,;\ (i,j) \in \mathcal{A}\},$$

also we define

$$\eta := \{\eta_{i,j}\,;\ (i,j) \in \mathcal{A}\}, \quad L^{(1)}_{\mathcal{A}}\eta := \{(L^{(1)}_{\mathcal{A}}\eta)_{i,j}\,; (i,j) \in \mathcal{A}\}.$$

By using Taylor expansions, the following lemma is obtained easily.

Lemma 2.3 *If the function f has a continuous first derivative, then there exists a number $\xi_{i,j} \in [t_{i,j}, t_{i,j+1}]$, such that*

$$\int_{t_{i,j}}^{t_{i,j+1}} f(x)dx - \delta_{i,j} f(t_{i,j}) = \frac{\delta_{i,j}^2}{2} f'(\xi_{i,j}).$$

By using the above lemma, we can find the following lemma.

Lemma 2.4 *For sufficiently smooth f, the following estimate holds*

$$|\chi[f]_{i,j} - z[f](t_{i,j})| = \mathcal{O}(h).$$

Then by using Taylor expansion and Lemma 2.4, we have the following estimate

$$\|\eta - \mathcal{R}(y)\|_\infty = \mathcal{O}(h), \tag{2.6}$$

$$\|L_{\mathcal{A}}^{(1)}\eta - \mathcal{R}(y')\|_\infty = \mathcal{O}(h), \tag{2.7}$$

where η and $L_{\mathcal{A}}^{(1)}\eta$ is defined in the Definition 2.1.

2.3. Deviation of the error estimation for FID equations

Now we find the deviation of the error estimation for (1.1)-(1.2) by using defect correction principle. We consider $y'(t) = f(t)$, $a \le t \le b$, where $f(t)$ is permitted to have jump discontinuities in the points belonging to Z_n. Using Taylor expansion we can find

$$(L_{\mathcal{A}}^{(1)} y)_{i,j} := \frac{y(t_{i,j+1}) - y(t_{i,j})}{\delta_{i,j}} = \int_0^1 f(t_{i,j} + \xi\delta_{i,j})d\xi =: \mathcal{I}_{\mathcal{A}}(f, t_{i,j}).$$

Therefore, we find "exact finite difference scheme" for $y'(t) = f(t)$, which is satisfied by the exact solution. Then we can say that a solution of problem (1.1)-(1.2) satisfies in the following exact finite difference scheme

$$(L_{\mathcal{A}}^{(1)} y)_{i,j} = \mathcal{I}_{\mathcal{A}}\big(F(\cdot, y, z[y]), t_{i,j}\big).$$

Also according to the collocation method, we have the following relation.

$$p'(t_{i,j}) - F\big(t_{i,j}, p(t_{i,j}), z[p](t_{i,j})\big) \equiv 0,\ (i,j) \in X(n),$$

therefore we define defect at $t_{i,j}$ as follows

$$D_{i,j} := (L_{\mathcal{A}}^{(1)} p)_{i,j} - \mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, z[p]), t_{i,j}\big),\ (i,j) \in \mathcal{B}. \tag{2.8}$$

In order to compute integral in (2.8), we use quadrature formula:

$$\begin{aligned}\mathcal{I}_{\mathcal{A}}\big(F(\cdot, p, z[p]), t_{i,j}\big) &\approx Q_{\mathcal{A}}\big(F(\cdot, p, \widetilde{z}[p]), t_{i,j}\big)\\ &:= \sum_{k=1}^{m+1} \gamma_{i,j}^k F\big(t_{i,k}, p(t_{i,k}), \widetilde{z}[p](t_{i,k})\big),\end{aligned}$$

where

$$\gamma_{i,j}^k := \int_0^1 L_k(\rho_j + \frac{\xi\delta_{i,j}}{h_i})d\xi.$$

With standard arguments we can show that for sufficiently smooth f the following error holds

$$\mathcal{I}_{\mathcal{A}}\big(f, t_{i,j}\big) - Q_{\mathcal{A}}\big(f, t_{i,j}\big) = \mathcal{O}(h^{m+1}).$$

Also when m is odd and the nodes ρ_i are symmetrically, we can see the following relation.

$$\mathcal{I}_{\mathcal{A}}\big(f,t_{i,j}\big)-Q_{\mathcal{A}}\big(f,t_{i,j}\big)=\mathcal{O}(h^{m+2}).$$

In this step we define $\pi=\{\pi_{i,j}\,;\,(i,j)\in\mathcal{A}\}$ as the solution of the following finite difference

$$(L_{\mathcal{A}}^{(1)}\pi)_{i,j}:=\frac{\pi_{i,j+1}-\pi_{i,j}}{\delta_{i,j}}=F(t_{i,j},\pi_{i,j},\chi[\pi]_{i,j})+D_{i,j},\ (i,j)\in\mathcal{B}, \tag{2.9}$$

$$\alpha\,\pi_{n,0}+\beta\,\pi_{0,0}=r. \tag{2.10}$$

We define $\mathbf{D}:=\{D_{i,j}\,;\,(i,j)\in\mathcal{B}\}$. For small value $\mathbf{D}$, we have

$$\pi-\mathcal{R}(p)\approx\eta-\mathcal{R}(y),$$

where η can be found in (2.4)-(2.5). We define ε and e as

$$\varepsilon:=\pi-\eta\approx\mathcal{R}(p)-\mathcal{R}(y):=e.$$

An estimate for the error e can be found in Theorem 2.1. The deviation of the error can be written in the following form

$$\theta:=e-\varepsilon.$$

In the next section, we study the order of the deviation of the error estimate for FID problems.

Lemma 2.5 *The defined defect in (2.8) has order* $\mathcal{O}(h^m)$.

Proof: We can write

$$\begin{aligned}D_{i,j}&=(L_{\mathcal{A}}^{(1)}p)_{i,j}-Q_{\mathcal{A}}\big(F(.,p,\widetilde{z}[p]),t_{i,j}\big)+\mathcal{O}(h^{m+1})\\&=\mathcal{I}_{\mathcal{A}}(p',t_{i,j})-Q_{\mathcal{A}}\big(F(.,p,\widetilde{z}[p]),t_{i,j}\big)+\mathcal{O}(h^{m+1})\\&=\underbrace{\mathcal{I}_{\mathcal{A}}(p',t_{i,j})-Q_{\mathcal{A}}(p',t_{i,j})}_{S_1}-\underbrace{Q_{\mathcal{A}}\big(p'-F(.,p,\widetilde{z}[p]),t_{i,j}\big)}_{S_2}+\mathcal{O}(h^{m+1}).\end{aligned}$$

Since p' is a polynomial of degree $m-1$, therefore $S_1=0$. Also according to the definition of collocation solution, we can say that $S_2=0$ at all collocation grid points $t_{i,j}$. For grid point τ_i, we have

$$\begin{aligned}p'(\tau_i)-F\big(\tau_i,p(\tau_i),\widetilde{z}[p](\tau_i)\big)&=\underbrace{y'(\tau_i)-F\big(\tau_i,y(\tau_i),z[y](\tau_i)\big)}_{=0}\\&+\underbrace{e'(\tau_i)}_{\mathcal{O}(h^m)}+\underbrace{F\big(\tau_i,y(\tau_i),z[y](\tau_i)\big)-F\big(\tau_i,p(\tau_i),\widetilde{z}[p](\tau_i)\big)}_{S_3}.\end{aligned}$$

For S_3 by using Lipschitz condition and Lemma 2.1 we can obtain

$$|S_3|\le C_1|\underbrace{y(\tau_i)-p(\tau_i)}_{\mathcal{O}(h^m)}|+C_2|\underbrace{z[y](\tau_i)-\widetilde{z}[p](\tau_i)}_{\mathcal{O}(h^m)}|=\mathcal{O}(h^m).$$

which completes the proof. □

The following lemma is a consequence of the above lemma.

Lemma 2.6 *The* $\pi-\eta$ *has order* $\mathcal{O}(h^m)$.

By using the deviation of the error estimate for FID problems, we can improve the collocation method. The structure of the improved collocation method is given in the following algorithm. Also in the next section, it can see that the order of convergence of the improved method is one unit higher than the order of convergence the collocation method.

Algorithm 2.1
1- By using collocation method in section 2, find collocation solution as $\mathcal{R}(p)$;
2- Find the defect values as D by (2.8);
3- Find π and η by solving (2.4)-(2.5) and (2.9)-(2.10);
4- Define $\varepsilon = \pi - \eta$;
5- Improve collocation solution by $\mathcal{R}(p) - \varepsilon$;

3. Analysis of the deviation of the error

Definition 3.1 In this section, we define $\overline{\varepsilon}$ and $\widehat{\varepsilon}$ as follows

$$\overline{\varepsilon} := \pi - \mathcal{R}(p),$$
$$\widehat{\varepsilon} := \eta - \mathcal{R}(y).$$

By using Theorem 2.1, (2.6), (2.7) and Lemma 2.6, we can find the following lemma.

Lemma 3.1 *The $\overline{\varepsilon}$ and the $\widehat{\varepsilon}$ have order $\mathcal{O}(h)$.*

Lemma 3.2 *We have*

$$||\overline{\varepsilon} - \widehat{\varepsilon}||_\infty = \mathcal{O}(h^m).$$

Proof: By using Lemma 3.1 and Theorem 2.1, we can write

$$||\overline{\varepsilon} - \widehat{\varepsilon}||_\infty \leq \underbrace{||\pi - \eta||_\infty}_{\mathcal{O}(h^m)} + \underbrace{||\mathcal{R}(p) - \mathcal{R}(y)||_\infty}_{\mathcal{O}(h^m)} = \mathcal{O}(h^m).$$

□

Lemma 3.3 *For linear and nonlinear $z[\cdot](t)$, we have*

$$|\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})| = \mathcal{O}(h), \tag{3.1}$$
$$|\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})| = \mathcal{O}(h). \tag{3.2}$$
$$|\chi[\eta]_{i,j} - \chi[\pi]_{i,j}| = \mathcal{O}(h^m), \tag{3.3}$$
$$|\chi[y]_{i,j} - \chi[p]_{i,j}| = \mathcal{O}(h^m), \tag{3.4}$$
$$|\chi[\widehat{\varepsilon}]_{i,j} - \chi[\overline{\varepsilon}]_{i,j}| = \mathcal{O}(h^m), \tag{3.5}$$
$$|\chi[\widehat{\varepsilon}]_{i,j}| = \mathcal{O}(h). \tag{3.6}$$

Proof: In the first step, we prove (3.1). By using Lemma 2.1 and Lemma 2.4, we have

$$|\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})| = |\underbrace{\chi[y]_{i,j} - z[y](t_{i,j})}_{\mathcal{O}(h)} + \underbrace{z[y](t_{i,j}) - \widetilde{z}[y](t_{i,j})}_{\mathcal{O}(h^{m+1})}| = \mathcal{O}(h).$$

Similarly we can prove (3.2). Now we prove (3.3). When $z[\cdot](t)$ is linear we find

$$\begin{aligned} \chi[\eta]_{i,j} - \chi[\pi]_{i,j} = \chi[\eta - \pi]_{i,j} &= \sum_{l,v \in \mathcal{B}} \delta_{l,v} \Lambda(t_{i,j}, t_{l,v}) \varepsilon_{l,v} \\ &\leq (b-a)(m+1)\frac{h}{h'} \max_{l,v \in \mathcal{B}} \Lambda(t_{i,j}, t_{l,v}) \underbrace{\varepsilon_{l,v}}_{\mathcal{O}(h^m)} \\ &= \mathcal{O}(h^m). \end{aligned}$$

From the Lipschitz condition for nonlinear K and Lemma 2.6, we get

$$|K\big(t_{i,j},t_{l,v},\pi_{l,v}\big)-K\big(t_{i,j},t_{l,v},\eta_{l,v}\big)|\le C|\pi_{l,v}-\eta_{l,v}|=\mathcal{O}(h^m).$$

Then for nonlinear case, we can get

$$\begin{aligned}\chi[\pi]_{i,j}-\chi[\eta]_{i,j}&=\sum_{l,v\in\mathcal{B}}\delta_{l,v}\Big(K\big(t_{i,j},t_{l,v},\pi_{l,v}\big)-K\big(t_{i,j},t_{l,v},\eta_{l,v}\big)\Big)\\&\le(b-a)(m+1)\frac{h}{h'}\max_{l,v\in\mathcal{B}}\Big(K\big(t_{i,j},t_{l,v},\pi_{l,v}\big)-K\big(t_{i,j},t_{l,v},\eta_{l,v}\big)\Big)=\mathcal{O}(h^m).\end{aligned}$$

Similarly, we can prove (3.4),(3.5) and (3.6). □

3.1. Linear case

Theorem 3.1 *Consider the FID problem (1.5) with boundary condition (1.2). Assume that the FID problem has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty=||e-\varepsilon||_\infty=\mathcal{O}(h^{m+1}),$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

Proof: Since F is linear then by using (2.4) and (2.9) we get

$$(L^{(1)}_{\mathcal{A}}\varepsilon)_{i,j}=a_1(t_{i,j})\varepsilon_{i,j}+\chi[\varepsilon]_{i,j}+D_{i,j}. \tag{3.7}$$

Therefore we can write

$$\begin{aligned}(L^{(1)}_{\mathcal{A}}e)_{i,j}&=(L^{(1)}_{\mathcal{A}}p)_{i,j}-(L^{(1)}_{\mathcal{A}}y)_{i,j}\\&=(L^{(1)}_{\mathcal{A}}p)_{i,j}-\mathcal{I}_{\mathcal{A}}(a_1y+a_2+z[y],t_{i,j})\\&=\underbrace{(L^{(1)}_{\mathcal{A}}p)_{i,j}-Q_{\mathcal{A}}(a_1p+a_2+\widetilde{z}[p],t_{i,j})}_{D_{i,j}}\\&+Q_{\mathcal{A}}(a_1p+a_2+\widetilde{z}[p],t_{i,j})-\mathcal{I}_{\mathcal{A}}(a_1y+a_2+z[y],t_{i,j})\\&=D_{i,j}+Q_{\mathcal{A}}(a_1p+a_2+\widetilde{z}[p],t_{i,j})-\mathcal{I}_{\mathcal{A}}(a_1y+a_2+z[y],t_{i,j})\\&+Q_{\mathcal{A}}(a_1y+a_2+\widetilde{z}[y],t_{i,j})-Q_{\mathcal{A}}(a_1y+a_2+\widetilde{z}[y],t_{i,j})\\&=D_{i,j}+Q_{\mathcal{A}}(a_1e+\widetilde{z}[e],t_{i,j})\\&+\underbrace{\big(Q_{\mathcal{A}}(a_1y+a_2+\widetilde{z}[y],t_{i,j})-\mathcal{I}_{\mathcal{A}}(a_1y+a_2+z[y],t_{i,j})\big)}_{\mathcal{O}(h^{m+1})}\\&=D_{i,j}+Q_{\mathcal{A}}(a_1e+\widetilde{z}[e],t_{i,j})+\mathcal{O}(h^{m+1}).\end{aligned} \tag{3.8}$$

Then from (3.7) and (3.8), we find

$$\begin{aligned}(L^{(1)}_{\mathcal{A}}\theta)_{i,j}&=(L^{(1)}_{\mathcal{A}}e)_{i,j}-(L^{(1)}_{\mathcal{A}}\varepsilon)_{i,j}\\&=D_{i,j}+Q_{\mathcal{A}}(a_1e+\widetilde{z}[e],t_{i,j})-a_1(t_{i,j})\varepsilon_{i,j}-\chi[\varepsilon]_{i,j}\\&-D_{i,j}+a_1(t_{i,j})e_{i,j}+\chi[e]_{i,j}\\&-a_1(t_{i,j})e_{i,j}-\chi[e]_{i,j}+\mathcal{O}(h^{m+1})\\&=a_1(t_{i,j})\theta_{i,j}+\chi[\theta]_{i,j}\\&+\underbrace{Q_{\mathcal{A}}(a_1e,t_{i,j})-a_1(t_{i,j})e_{i,j}}_{S_1}+\underbrace{Q_{\mathcal{A}}(\widetilde{z}[e],t_{i,j})-\chi[e]_{i,j}}_{S_2}+\mathcal{O}(h^{m+1}).\end{aligned} \tag{3.9}$$

Since $\sum_{k=1}^{m+1} \gamma_{i,j}^k = 1$, then we rewrite S_1 as

$$S_1 = \sum_{k=1}^{m+1} \gamma_{i,j}^k \big(a_1(t_{i,k})e(t_{i,k}) - a_1(t_{i,j})e(t_{i,j})\big),$$

by using Taylor expansion, we have

$$\begin{aligned} a_1(t_{i,k})e(t_{i,k}) - a_1(t_{i,j})e(t_{i,j}) &= \underbrace{(t_{i,k} - t_{i,j})}_{\mathcal{O}(h)}\big(a_1'(\xi_i)\underbrace{e(\xi_i)}_{\mathcal{O}(h^m)} + a_1(\xi_i)\underbrace{e'(\xi_i)}_{\mathcal{O}(h^m)}\big) \\ &= \mathcal{O}(h^{m+1}), \end{aligned} \tag{3.10}$$

where $\xi_i \in [\tau_i, \tau_{i+1}]$. By using Theorems 2.1 and (3.10), we can say that $S_1 = \mathcal{O}(h^{m+1})$. For S_2 by using the Lemma 2.3, we get

$$\begin{aligned} &\int_a^b \Lambda(t_{i,j}, s)e(s)ds - \chi[e]_{i,j} \\ &\leq (b-a)(m+1)\frac{h}{2h'}h \max_{s\in[a,b]} \Big(\frac{\partial\Lambda(t_{i,j}, s)}{\partial s}\underbrace{e(s)}_{\mathcal{O}(h^m)} + \Lambda(t_{i,j}, s)\underbrace{e'(s)}_{\mathcal{O}(h^m)}\Big) \\ &= \mathcal{O}(h^{m+1}). \end{aligned} \tag{3.11}$$

By using (3.11), we obtain

$$\begin{aligned} S_2 &= Q_{\mathcal{A}}(\tilde{z}[e], t_{i,j}) - \chi[e]_{i,j} \\ &= \sum_{k=1}^{m+1} \gamma_{i,j}^k \int_a^b \Lambda(t_{i,k}, s)e(s)ds + \mathcal{O}(h^{m+1}) - \int_a^b \Lambda(t_{i,j}, s)e(s)ds \\ &- \mathcal{O}(h^{m+1}) = \sum_{k=1}^{m+1} \gamma_{i,j}^k \int_a^b \Big(\Lambda(t_{i,k}, s) - \Lambda(t_{i,j}, s)\Big)e(s)ds + \mathcal{O}(h^{m+1}) \\ &= \sum_{k=1}^{m+1} \gamma_{i,j}^k (t_{i,k} - t_{i,j}) \int_a^b \frac{\partial\Lambda}{\partial s}(\bar{\xi}_{i,k}, s)e(s)ds + \mathcal{O}(h^{m+1}) \\ &= \sum_{k=1}^{m+1} \gamma_{i,j}^k (t_{i,k} - t_{i,j}) \frac{\partial\Lambda}{\partial s}(\bar{\xi}_{i,k}, \bar{\zeta}_k)e(\bar{\zeta}_k) \int_a^b 1ds + \mathcal{O}(h^{m+1}) \\ &= (b-a) \sum_{k=1}^{m+1} \gamma_{i,j}^k \underbrace{(t_{i,k} - t_{i,j})}_{\mathcal{O}(h)} \frac{\partial\Lambda}{\partial s}(\bar{\xi}_{i,k}, \bar{\zeta}_k)\underbrace{e(\bar{\zeta}_k)}_{\mathcal{O}(h^m)} + \mathcal{O}(h^{m+1}) \\ &= \mathcal{O}(h^{m+1}), \end{aligned}$$

where $\bar{\xi}_{i,k}$, ζ_k, $\bar{\zeta}_k \in [\tau_i, \tau_{i+1}]$. Therefore we can rewrite (3.9) as

$$(L_{\mathcal{A}}^{(1)}\theta)_{i,j} = a_1(t_{i,j})\theta_{i,j} + \chi[\theta]_{i,j} + \mathcal{O}(h^{m+1}).$$

By using stability of forward Euler scheme, we find

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}).$$

□

3.2. Nonlinear case

Definition 3.2 We define

$$\begin{aligned}
w(t_{i,j}) &:= \int_0^1 F_y\big(t_{i,j}, p(t_{i,j}) + \overline{\varepsilon}_{i,j}\tau, \chi[\pi]_{i,j}\big)d\tau,\\
\overline{w}(t_{i,j}) &:= \int_0^1 F_y\big(t_{i,j}, y(t_{i,j}) + \widehat{\varepsilon}_{i,j}\tau, \chi[\eta]_{i,j}\big)d\tau,\\
R(t_{i,j}) &:= \int_0^1 F_z\Big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j}) + \big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big)\tau\Big)d\tau,\\
\overline{R}(t_{i,j}) &:= \int_0^1 F_z\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j}) + (\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j}))\tau\big)d\tau.
\end{aligned}$$

Remark 3.1 By using the Lipschitz condition for F_y, Lemma 3.2 and Lemma 3.3, we get

$$\begin{aligned}
&|F_y(t_{i,j}, y(t_{i,j}) + \widehat{\varepsilon}_{i,j}\tau, \chi[\eta]_{i,j}) - F_y(t_{i,j}, p(t_{i,j}) + \overline{\varepsilon}_{i,j}\tau, \chi[\pi]_{i,j})|\\
&\leq C_1\Big(\underbrace{|y(t_{i,j}) - p(t_{i,j})|}_{\mathcal{O}(h^m)} + \tau|\underbrace{\widehat{\varepsilon}_{i,j} - \overline{\varepsilon}_{i,j}|}_{\mathcal{O}(h^m)}\Big) + C_2\underbrace{|\chi[\eta]_{i,j} - \chi[\pi]_{i,j}|}_{\mathcal{O}(h^m)} = \mathcal{O}(h^m).
\end{aligned} \tag{3.12}$$

By using (3.12) we obtain

$$\begin{aligned}
\overline{w}(t_{i,j}) - w(t_{i,j}) = \int_0^1 &\Big(F_y\big(t_{i,j}, y(t_{i,j}) + \widehat{\varepsilon}_{i,j}\tau, \chi[\eta]_{i,j}\big)\\
&- F_y\big(t_{i,j}, p(t_{i,j}) + \overline{\varepsilon}_{i,j}\tau, \chi[\pi]_{i,j}\big)\Big)d\tau = \mathcal{O}(h^m).
\end{aligned} \tag{3.13}$$

Then by using (3.13), we can write

$$\begin{aligned}
\overline{w}(t_{i,j})\widehat{\varepsilon}_{i,j} &= w(t_{i,j})\widehat{\varepsilon}_{i,j} + \big(\underbrace{\overline{w}(t_{i,j}) - w(t_{i,j})}_{\mathcal{O}(h^m)}\big)\underbrace{\widehat{\varepsilon}_{i,j}}_{\mathcal{O}(h)}\\
&= w(t_{i,j})\widehat{\varepsilon}_{i,j} + \mathcal{O}(h^{m+1}).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&\overline{R}(t_{i,j}) - R(t_{i,j})\\
&= \int_0^1 \Big(F_z\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j}) + \big(\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})\big)\tau\big)\\
&- F_z\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j}) + \big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big)\Big)d\tau.
\end{aligned}$$

By using (3.11), we can say that

$$|\chi[e]_{i,j} - \widetilde{z}[e](t_{i,j})| = \mathcal{O}(h^{m+1}). \tag{3.14}$$

From the Lipschitz condition for F_z and (3.14), we have

$$\begin{aligned}
&|F_z\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j}) + \big(\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})\big)\tau\big)\\
&- F_z\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j}) + \big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big)|\\
&\leq C_1|\underbrace{y(t_{i,j}) - p(t_{i,j})}_{\mathcal{O}(h^m)}| + C_2\Big(|\underbrace{\widetilde{z}[y](t_{i,j}) - \widetilde{z}[p](t_{i,j})}_{\mathcal{O}(h^m)}|\\
&+ \tau|\underbrace{\big(\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})\big) - \big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big)}_{O(h^{m+1})}|\Big) = \mathcal{O}(h^m).
\end{aligned}$$

Therefore we can say that

$$|\overline{R}(t_{i,j}) - R(t_{i,j})| = \mathcal{O}(h^m). \tag{3.15}$$

By using Lemma 2.3, (3.14) and (3.15), we find

$$\begin{aligned}
&\overline{R}(t_{i,j})\big(\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})\big) = R(t_{i,j})\big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big)\\
&+ R(t_{i,j})\big(\underbrace{\chi[y]_{i,j} - \chi[p]_{i,j} - \widetilde{z}[y](t_{i,j}) + \widetilde{z}[p](t_{i,j})}_{\mathcal{O}(h^{m+1})}\big)\\
&+ \big(\underbrace{\overline{R}(t_{i,j}) - R(t_{i,j})}_{\mathcal{O}(h^m)}\big)\big(\underbrace{\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})}_{\mathcal{O}(h)}\big)\\
&= R(t_{i,j})\big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big) + \mathcal{O}(h^{m+1}).
\end{aligned}$$

When F is nonlinear and $z[\cdot](t)$ is linear we have the following theorem.

Theorem 3.2 *Consider the FID problem (1.1) with boundary condition (1.2), where $F(t, y, z), F_t(t, y, z)$, $F_y(t, y, z)$ and $F_z(t, y, z)$ are Lipschitz-continuous. Also let $z[\cdot](t)$ is linear. Assume that the FID problem has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}),$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

Proof: For nonlinear case, we have

$$\begin{aligned}
(L^{(1)}_{\mathcal{A}}\eta)_{i,j} &= F(t_{i,j}, \eta_{i,j}, \chi[\eta]_{i,j}),\\
(L^{(1)}_{\mathcal{A}}\pi)_{i,j} &= F(t_{i,j}, \pi_{i,j}, \chi[\pi]_{i,j}) + D_{i,j}.
\end{aligned}$$

Thus we can write

$$(L^{(1)}_{\mathcal{A}}\varepsilon)_{i,j} = F(t_{i,j}, \pi_{i,j}, \chi[\pi]_{i,j}) - F(t_{i,j}, \eta_{i,j}, \chi[\eta]_{i,j}) + D_{i,j}. \tag{3.16}$$

By using (3.16), we have

$$\begin{aligned}
(L^{(1)}_{\mathcal{A}}\theta)_{i,j} &= (L^{(1)}_{\mathcal{A}}e)_{i,j} - (L^{(1)}_{\mathcal{A}}\varepsilon)_{i,j} = (L^{(1)}_{\mathcal{A}}p)_{i,j} - (L^{(1)}_{\mathcal{A}}y)_{i,j}\\
&- F(t_{i,j}, \pi_{i,j}, \chi[\pi]_{i,j}) + F(t_{i,j}, \eta_i, \chi[\eta]_{i,j}) - D_{i,j}\\
&= Q_{\mathcal{A}}\big(F(., p, \widetilde{z}[p]), t_{i,j}\big) + D_{i,j} - \mathcal{I}_{\mathcal{A}}\big(F(., y, z[y]), t_{i,j}\big)\\
&- F(t_{i,j}, \pi_{i,j}, \chi[\pi]_{i,j}) + F(t_{i,j}, \eta_{i,j}, \chi[\eta]_{i,j}) - D_{i,j}\\
&= F(t_{i,j}, \eta_{i,j}, \chi[\eta]_{i,j}) + \underbrace{Q_{\mathcal{A}}\big(F(., y, \widetilde{z}[y]), t_{i,j}\big) - \mathcal{I}_{\mathcal{A}}\big(F(., y, z[y]), t_{i,j}\big)}_{\mathcal{O}(h^{m+1})}\\
&- Q_{\mathcal{A}}\big(F(., y, \widetilde{z}[y]), t_{i,j}\big) - F(t_{i,j}, \pi_{i,j}, \chi[\pi]_{i,j}) + Q_{\mathcal{A}}\big(F(., p, \widetilde{z}[p]), t_{i,j}\big)\\
&= -\underbrace{\big(F(t_{i,j}, \pi_{i,j}, \chi[\pi]_{i,j}) - F\big(t_{i,j}, p(t_{i,j}), \chi[p]_{i,j}\big)}_{I_1}\\
&- \underbrace{\Big(F\big(t_{i,j}, p(t_{i,j}), \chi[p]_{i,j}\big) - F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j})\big)\Big)}_{I_2} - F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j})\big)\\
&+ Q_{\mathcal{A}}\big(F(., p, \widetilde{z}[p]), t_{i,j}\big) + \underbrace{F(t_{i,j}, \eta_{i,j}, \chi[\eta]_{i,j}) - F\big(t_{i,j}, y(t_{i,j}), \chi[y]_{i,j}\big)}_{I_3}\\
&+ \underbrace{F\big(t_{i,j}, y(t_{i,j}), \chi[y]_{i,j}\big) - F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j})\big)}_{I_4}\\
&+ F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j})\big) - Q_{\mathcal{A}}\big(F(., y, \widetilde{z}[y]), t_{i,j}\big) + \mathcal{O}(h^{m+1}).
\end{aligned} \tag{3.17}$$

We rewrite I_1, I_2, I_3 and I_4 as follows

$$\begin{aligned}
I_1 &= w(t_{i,j})\overline{\varepsilon}_{i,j} + v(t_{i,j})\chi[\overline{\varepsilon}]_{i,j},\\
I_2 &= R(t_{i,j})\big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big),\\
I_3 &= \overline{w}(t_{i,j})\widehat{\varepsilon}_{i,j} + \overline{v}(t_{i,j})\chi[\widehat{\varepsilon}]_{i,j},\\
I_4 &= \overline{R}(t_{i,j})\big(\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})\big),
\end{aligned}$$

where $w(t_{i,j}), \overline{w}(t_{i,j}), R(t_{i,j})$ and $\overline{R}(t_{i,j})$ are defined in Definition 3.2 and

$$\begin{aligned}
v(t_{i,j}) &:= \int_0^1 F_z\big(t_{i,j}, p(t_{i,j}), \chi[p]_{i,j} + \chi[\overline{\varepsilon}]_{i,j}\tau\big)d\tau,\\
\overline{v}(t_{i,j}) &:= \int_0^1 F_z\big(t_{i,j}, y(t_{i,j}), \chi[y]_{i,j} + \chi[\widehat{\varepsilon}]_{i,j}\tau\big)d\tau.
\end{aligned}$$

Similar to Remark 3.1, we can get

$$\overline{v}(t_{i,j})\chi[\widehat{\varepsilon}]_{i,j} = v(t_{i,j})\chi[\widehat{\varepsilon}]_{i,j} + \mathcal{O}(h^{m+1}). \tag{3.18}$$

Then based on the above discussion, Remark 3.1 and (3.18) we rewrite (3.17) in the following form

$$\begin{aligned}
(L_{\Delta}^{(1)}\theta)_{i,J} &= w(t_{i,j})\theta_{i,j} + v(t_{i,j})\chi[\theta]_{i,j} + F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j})\big) - F\big(t_{i,j}, y(t_{i,j}), z[y](t_{i,j})\big)\\
&+ F\big(t_{i,j}, y(t_{i,j}), z[y](t_{i,j})\big) - F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j})\big) + F\big(t_{i,j}, p(t_{i,j}), z[p](t_{i,j})\big)\\
&- F\big(t_{i,j}, p(t_{i,j}), z[p](t_{i,j}) + Q_{\mathcal{A}}\Big(F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j})\big)\Big)\\
&- Q_{\mathcal{A}}\Big(F\big(t_{i,j}, y(t_{i,j}), z[y](t_{i,j})\big)\Big) + Q_{\mathcal{A}}\Big(F\big(t_{i,j}, y(t_{i,j}), z[y](t_{i,j})\big)\Big)\\
&- Q_{\mathcal{A}}\Big(F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j})\big)\Big) + Q_{\mathcal{A}}\Big(F\big(t_{i,j}, p(t_{i,j}), z[p](t_{i,j})\big)\Big)\\
&- Q_{\mathcal{A}}\Big(F\big(t_{i,j}, p(t_{i,j}), z[p](t_{i,j})\big)\Big) + \mathcal{O}(h^{m+1}).
\end{aligned} \tag{3.19}$$

By using the Lipschitz condition for F and Lemma 2.1, we have

$$\begin{aligned}
&|F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j})\big) - F\big(t_{i,j}, y(t_{i,j}), z[y](t_{i,j})\big)|\\
&\le C|\widetilde{z}[y](t_{i,j}) - z[y](t_{i,j})| = \mathcal{O}(h^{m+1}),
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&|F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j})\big) - F\big(t_{i,j}, p(t_{i,j}), z[p](t_{i,j})\big)|\\
&\le C|\widetilde{z}[p](t_{i,j}) - z[p](t_{i,j})| = \mathcal{O}(h^{m+1}).
\end{aligned} \tag{3.21}$$

Then by using (3.20)-(3.21), we can rewrite (3.19) as

$$(L_{\mathcal{A}}^{(1)}\theta)_{i,j} = w(t_{i,j})\theta_{i,j} + v(t_{i,j})\chi[\theta]_{i,j} + \Phi(t_{i,j}) - Q_{\mathcal{A}}\Big(\Phi(t_{i,j})\Big) + \mathcal{O}(h^{m+1}).$$

where

$$\Phi(t) = F\big(t, p(t), z[p](t)\big) - F\big(t, y(t), z[y](t)\big).$$

By using Taylor expansion, we have

$$\begin{aligned}
|\Phi(t_{i,j}) - \sum_{k=1}^{m+1}\gamma_{i,j}^k\Phi(t_{i,k})| &= |\sum_{k=1}^{m+1}\gamma_{i,j}^k\big(\Phi(t_{i,j}) - \Phi(t_{i,k})\big)|\\
&\le \; Ch\max|\Phi'(t)|.
\end{aligned}$$

We can find

$$
\begin{aligned}
||\Phi'(t)||_\infty &= ||F_t\big(t,p(t),z[p](t)\big) - F_t\big(t,y(t),z[y](t)\big) + F_y\big(t,p(t),z[p](t)\big)p'(t) \\
&- F_y\big(t,y(t),z[y](t)\big)y'(t) + F_z\big(t,p(t),z[p](t)\big)z'[p](t) \\
&- F_z\big(t,y(t),z[y](t)z'[y](t)\big)||_\infty = ||F_t\big(t,p(t),z[p](t)\big) \\
&- F_t\big(t,y(t),z[y](t)\big) + F_y\big(t,p(t),z[p](t)\big)p'(t) - F_y\big(t,y(t),z[y](t)\big)p'(t) \\
&+ F_y\big(t,y(t),z[y](t)\big)p'(t) - F_y\big(t,y(t),z[y](t)\big)y'(t) \\
&+ F_z\big(t,p(t),z[p](t)\big)z'[p](t) - F_z\big(t,y(t),z[y](t)\big)z'[p](t) \\
&+ F_z\big(t,y(t),z[y](t)\big)z'[p](t) - F_z\big(t,y(t),z[y](t)z'[y](t)\big)||_\infty \\
&\le C_1 \underbrace{||p-y||_\infty}_{\mathcal{O}(h^m)} + C_2 \underbrace{||p'-y'||_\infty}_{\mathcal{O}(h^m)} + C_3 || \underbrace{z'[p](t) - z'[y](t)}_{I_5} ||_\infty.
\end{aligned}
$$

We can rewrite I_5 as follows

$$
z'[p](t) - z'[y](t) = \int_a^b \frac{\partial \Lambda}{\partial t}(t,s) \underbrace{e(s)}_{\mathcal{O}(h^m)} ds = \mathcal{O}(h^m).
$$

In this step based on the above discussion, we have

$$
(L_{\mathcal{A}}^{(1)}\theta)_{i,j} = w(t_{i,j})\theta_{i,j} + v(t_{i,j})\chi[\theta]_{i,j} + \mathcal{O}(h^{m+1}).
$$

By stability of forward Euler scheme, we can find

$$
||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}).
$$

□

In this step, we study nonlinear case with nonlinear $z[y](t)$.

Definition 3.3 For nonlinear $z[y](t)$ we define $\overline{\chi}\,[\overline{\varepsilon}\,]_{i,j}$ and $\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j}$ as

$$
\begin{aligned}
\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j} := \chi[\eta]_{i,j} - \chi[y]_{i,j} &= \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Big(K\big(t_{i,j},t_{l,k},\eta_{l,k}\big) - K\big(t_{i,j},t_{l,k},y(t_{l,k})\big)\Big) \\
&= \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\widehat{\varepsilon}_{\,l,k}, \\
\overline{\chi}\,[\overline{\varepsilon}\,]_{i,j} := \chi[\pi]_{i,j} - \chi[p]_{i,j} &= \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Big(K\big(t_{i,j},t_{l,k},\pi_{l,k}\big) - K\big(t_{i,j},t_{l,k},p(t_{l,k})\big)\Big) \\
&= \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\overline{\Omega}_{l,k}(t_{i,j})\overline{\varepsilon}_{\,l,k},
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_{l,k}(t_{i,j}) &:= \int_0^1 K_u\big(t_{i,j},t_{l,k},y(t_{l,k}) + \tau\widehat{\varepsilon}_{\,l,k}\big)d\tau, \\
\overline{\Omega}_{l,k}(t_{i,j}) &:= \int_0^1 K_u\big(t_{i,j},t_{l,k},p(t_{l,k}) + \tau\overline{\varepsilon}_{\,l,k}\big)d\tau.
\end{aligned}
$$

By using the Lipschitz condition for K_u, Theorem 2.1 and Lemma3.2 we can find the following lemma.

Lemma 3.4 *We have*

$$
|\Omega_{l,k}(t_{i,j}) - \overline{\Omega}_{l,k}(t_{i,j})| = \mathcal{O}(h^m).
$$

Lemma 3.5 *We have*

$$\sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\overline{\Omega}_{l,k}(t_{i,j})\overline{\varepsilon}_{l,k} = \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\overline{\varepsilon}_{l,k} + \mathcal{O}(h^{m+1}).$$

Proof: By using Lemma 3.4 we get

$$\sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\overline{\Omega}_{l,k}(t_{i,j})\overline{\varepsilon}_{l,k} = \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\overline{\varepsilon}_{l,k} + \mathcal{O}(h^{m}) \sum_{(l,k)\in\mathcal{B}} \delta_{l,k} \underbrace{\overline{\varepsilon}_{l,k}}_{\mathcal{O}(h)}$$
$$= \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\overline{\varepsilon}_{l,k} + \mathcal{O}(h^{m+1}).$$

□

Lemma 3.6 *When $z[\cdot](t)$ is nonlinear then we have*

$$\overline{\chi}\,[\,\overline{\varepsilon}\,]_{i,j} - \widehat{\chi}\,[\widehat{\varepsilon}]_{i,j} = \mathcal{O}(h^{m}).$$

Proof: From the Lemma 3.5 and Lemma 3.2 one may readily deduce the following result

$$\overline{\chi}\,[\,\overline{\varepsilon}\,]_{i,j} - \widehat{\chi}\,[\widehat{\varepsilon}]_{i,j} = \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\overline{\Omega}_{l,k}(t_{i,j})\overline{\varepsilon}_{l,k} - \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\widehat{\varepsilon}_{l,k}$$
$$= \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})(\overline{\varepsilon}_{l,k} - \widehat{\varepsilon}_{l,k}) + \mathcal{O}(h^{m+1})$$
$$\leq (m+1)(b-a)\frac{h}{h'} \max_{(l,k)\in\mathcal{B}} \Omega_{l,k}(t_{i,j})\underbrace{(\overline{\varepsilon}_{l,k} - \widehat{\varepsilon}_{l,k})}_{\mathcal{O}(h^{m})} + \mathcal{O}(h^{m+1}) = \mathcal{O}(h^{m}).$$

□

Theorem 3.3 *Consider the FID problem (1.1) with boundary condition (1.2), where $F(t,y,z), F_t(t,y,z)$, $F_y(t,y,z)$ and $F_z(t,y,z)$ are Lipschitz-continuous. Also let $z[y](t)$ is nonlinear, i.e, (1.3), where $K(t,s,u)$ and $K_u(t,s,u)$ are Lipschitz-continuous. Assume that the FID problem has a unique and sufficiently smooth solution, then the following estimate holds*

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}),$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

Proof: Similar to the Theorem 3.2, we get

$$\begin{aligned}(L^{(1)}_\Delta\theta)_{i,j} = &-(I_6 + I_7) - F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}[p](t_{i,j})\big) + Q_\Delta\big(F(.,p,\widetilde{z}[p]), t_{i,j}\big)\\ &+ I_8 + I_9 + F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}[y](t_{i,j})\big) - Q_\Delta\big(F(.,y,\widetilde{z}[y]), t_{i,j}\big)\\ &+ \mathcal{O}(h^{m+1}),\end{aligned} \tag{3.22}$$

where

$$\begin{aligned}I_6 &:= w(t_{i,j})\,\overline{\varepsilon}_{i,j} + v^*(t_{i,j})\overline{\chi}\,[\,\overline{\varepsilon}\,]_{i,j},\\ I_8 &:= \overline{w}(t_{i,j})\,\widehat{\varepsilon}_{i,j} + \overline{v}^*(t_{i,j})\widehat{\chi}\,[\widehat{\varepsilon}]_{i,j},\\ I_7 &:= R(t_{i,j})\big(\chi[p]_{i,j} - \widetilde{z}[p](t_{i,j})\big),\\ I_9 &:= \overline{R}(t_{i,j})\big(\chi[y]_{i,j} - \widetilde{z}[y](t_{i,j})\big),\end{aligned}$$

where $w(t_{i,j}), \overline{w}(t_{i,j}), R(t_{i,j})$ and $\overline{R}(t_{i,j})$ are defined in Definition 3.2 and

$$v^*(t_{i,j}) := \int_0^1 F_z\big(t_{i,j}, p(t_{i,j}), \chi[p]_{i,j} + \overline{\chi}\,[\overline{\varepsilon}\,]_{i,j}\tau\big)d\tau,$$
$$\overline{v}^{\,*}(t_{i,j}) := \int_0^1 F_z\big(t_{i,j}, y(t_{i,j}), \chi[y]_{i,j} + \widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j}\tau\big)d\tau.$$

We get

$$\begin{aligned} v^*(t_{i,j}) - \overline{v}^{\,*}(t_{i,j}) = \int_0^1 & F_z\big(t_{i,j}, p(t_{i,j}), \chi[p]_{i,j} + \overline{\chi}\,[\overline{\varepsilon}\,]_{i,j}\tau\big) \\ & - F_z\big(t_{i,j}, y(t_{i,j}), \chi[y]_{i,j} + \widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j}\tau\big)d\tau, \end{aligned}$$

by using Lemma 3.3-3.6 and the Lipschitz condition for F_z, we have

$$\begin{aligned} &|F_z\big(t_{i,j}, p(t_{i,j}), \chi[p]_{i,j} + \overline{\chi}\,[\overline{\varepsilon}\,]_i\tau\big) - F_z\big(t_{i,j}, y(t_{i,j}), \chi[y]_{i,j} + \widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j}\tau\big)| \\ &\le C_1|\underbrace{p(t_{i,j}) - y(t_{i,j})}_{\mathcal{O}(h^m)}| + C_2\big(|\underbrace{\chi[p]_{i,j} - \chi[y]_{i,j}}_{\mathcal{O}(h^m)}| + \tau|\underbrace{\overline{\chi}\,[\overline{\varepsilon}\,]_{i,j} - \widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j}}_{\mathcal{O}(h^m)}|\big) = \mathcal{O}(h^m), \end{aligned}$$

therefore we write

$$\begin{aligned} \overline{v}^{\,*}(t_{i,j})\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j} &= v^*(t_{i,j})\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j} + \big(\underbrace{\overline{v}^{\,*}(t_{i,j}) - v^*(t_{i,j})}_{\mathcal{O}(h^m)}\big)\underbrace{\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j}}_{\mathcal{O}(h)} \\ &= v^*(t_{i,j})\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j} + \mathcal{O}(h^{m+1}). \end{aligned}$$

Also by using Lemma 3.5, we get

$$\widehat{\chi}\,[\widehat{\varepsilon}\,]_{i,j} - \widehat{\chi}\,[\overline{\varepsilon}\,]_{i,j} = \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\theta_{l,k} + \mathcal{O}(h^{m+1}).$$

Then we can rewrite (3.22) as follows

$$\begin{aligned} (L_\Delta^{(1)}\theta)_{i,j} = w(t_{i,j})\theta_i &+ v(t_{i,j}) \sum_{(l,k)\in\mathcal{B}} \delta_{l,k}\Omega_{l,k}(t_{i,j})\theta_{l,k} \\ &+ F\big(t_{i,j}, y(t_{i,j}), \widetilde{z}\,[y](t_{i,j})\big) - F\big(t_{i,j}, p(t_{i,j}), \widetilde{z}\,[p](t_{i,j})\big) \\ &+ Q_\Delta\big(F(., p, \widetilde{z}\,[p]), t_{i,j}\big) - Q_\Delta\big(F(., y, \widetilde{z}\,[y]), t_{i,j}\big) \\ &+ \mathcal{O}(h^{m+1}). \end{aligned}$$

In a similar to the Theorem 3.2, we can complete the proof. □

Similar to the above Theorem, we can prove the following Theorem.

Theorem 3.4 *Consider the FID problem (1.5) with boundary condition (1.2). Also let $z[\cdot](t)$ is nonlinear, i.e, (1.3), where $K(t,s,u)$ and $K_u(t,s,u)$ are Lipschitz-continuous. Assume that the FID problem has a unique and sufficiently smooth solution. Then the following estimate holds*

$$||\theta||_\infty = ||e - \varepsilon||_\infty = \mathcal{O}(h^{m+1}),$$

where e is error, ε is the error estimate and θ is the deviation of the error estimate.

4. Numerical illustration

In this section in order to illustrate the theoretical results, we consider some test problems. Note that we have computed the numerical results by Mathematica-9 programming.

Example 1. To check Theorem 3.1, we consider the FID problem (1.5) with linear $z[\cdot](t)$. In this example we assume that $\beta = 1, \alpha = 0, a_1(t) := t, a_2(t) := -(-1+e)t + t\exp(t^2), \Lambda(t,s) := 2ts$ and $[a,b] := [0,1]$. The exact solution is given by $y(t) = \exp(t^2)$. In Table 1 we choose $m = 4$ and assume that ρ_i $(i = 0,\ldots,5)$ are equidistant point. Also In Table 2 we choose $m = 2$ and $\{\rho_0, \rho_1\, \rho_2, \rho_3\} = \{0, 0.15, 0.80, 1\}$. In this example we choose n collocation intervals of length $1/n$.

Table 1: Numerical results for example 1.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	1.53923e-5	$----$	3.30065e-7	$----$
16	9.75687e-7	3.97965	9.11807e-9	5.17788
32	6.13527e-8	3.99122	2.59974e-10	5.13229
64	3.84530e-9	3.99596	7.67653e-12	5.08177

Table 2: Numerical results for example 1.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	4.38644e-3	$----$	1.89193e-4	$----$
16	1.04811e-3	2.06527	2.80256e-5	2.75504
32	2.54740e-4	2.04069	3.81895e-6	2.87550
64	6.26840e-5	2.02285	4.95540e-7	2.94610

Example 2. This example reveal the Theorem 3.2. We Consider FID problem

$$y'(t) = \big(a_1(t)y(t)\big)^2 + a_2(t) + \int_a^b \Lambda(t,s)y(s)ds,\; t \in [0,1],$$

where $a_1(t) := t$, $\Lambda(t,s) := ts$, $[a,b] := [0,1]$, $\beta = 1$, $\alpha = 2$ and with $a_2(t)$ chosen so that $y(t) = \sin(t)$. In Table 3 we choose n collocation intervals of length $1/n$ and assume that ρ_i $(i = 0,\ldots,4)$ are equidistant point. Also in Table 4 we choose Chebyshev nodes for $\tau_i, (i = 1,\ldots,n-1)$ and $\tau_0 = a$, $\tau_n = b$ and assume that $\{\rho_0, \rho_1\, \rho_2, \rho_3\} = \{0, 0.2, 0.65, 1\}$.

Table 3: Numerical results for example 2 with $m = 3$.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	3.25208e-7	$----$	7.43240e-8	$----$
16	2.14871e-8	3.91982	4.43307e-9	4.06745
32	1.38079e-9	3.95991	2.69761e-10	4.03855
64	8.75068e-11	3.97995	1.66229e-11	4.02044

Table 4: Numerical results for example 2 with $m = 2$.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	2.86845e-4	$----$	3.50259e-6	$----$
16	7.38179e-5	1.95823	4.06108e-7	3.10849
32	1.87431e-5	1.97761	4.93023e-8	3.04214
64	4.71836e-6	1.99000	6.06813e-9	3.02233

Example 3. By using this example we reveal Theorem 3.3. We consider the FID problem (1.1) as

$$y'(t) = \big(a_1(t)y(t)\big)^2 + a_2(t) + z[y](t),$$

where

$$z[y](t) := \int_a^b y^2(s)t^2 s ds,$$

and $a_1(t) := t^2$, $[a, b] := [0, 1]$, $\beta = 1$, $\alpha = 2$ and $a_2(t)$ chosen so that $y(t) = \exp(t^2)$. In Table 5 we choose Chebyshev nodes for $\tau_i, (i = 1, \ldots, n-1)$ and $\tau_0 = 0$, $\tau_n = 1$ also $\{\rho_0, \rho_1\, \rho_2, \rho_3\} = \{0, 0.26, 0.7, 1\}$. In Table 6, ρ_i $(i = 0, \ldots, 5)$ and $\tau_i, (i = 0, \ldots, n)$ are equidistant point.

Table 5: Numerical results for example 3 with $m = 2$.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	3.27350e-3	$----$	4.37835e-4	$----$
16	8.72010e-4	1.90842	5.47554e-5	2.99932
32	2.23511e-4	1.96400	6.79858e-6	3.00969
64	5.64346e-5	1.98569	8.87499e-7	2.93742

Table 6: Numerical results for example 3 with $m = 4$.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	4.35033e-6	$----$	1.07669e-6	$----$
16	3.02492e-7	3.84615	3.39348e-8	4.98769
32	1.98603e-8	3.92894	1.05918e-9	5.00175
64	1.27087e-9	3.96600	3.30302e-11	5.00301

Example 4. In this example we study the following FID problem

$$y'(t) = a_1(t)y(t) + a_2(t) + \int_a^b t^2 s y^2(s)ds,$$

with $a_1(t) := t^3$, $[a, b] := [0, 1]$, $\beta = 3$, $\alpha = 2$ and $a_2(t)$ chosen so that $y(t) = \exp(t)$. This example serve to illustrate the Theorem 3.4. In Table 7, we choose $m = 2$ and assume that τ_i and ρ_i are equidistant point.

Table 7: Numerical results for example 4.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*
8	4.54882e-4	$----$	8.46702e-5	$----$
16	1.26335e-4	1.84824	1.03839e-5	3.02750
32	3.31487e-5	1.93023	1.28381e-6	3.01585
64	8.48154e-6	1.96655	1.59530e-7	3.00853

Example 5. In this example, the numerical results for Algorithm 2.1 is studied. Consider the FID problem

$$y'(t) = te^t + e^t - t + \int_0^1 ty(s)ds,$$
$$y(0) = 0,$$

with exact solution as $y(t) = te^t$. The results for this problem are given in Table 8. In this table, e^* denotes the infinity norm between the improved collocation method and exact solution. For this results, $m = 2$, $\{\rho_0, \rho_1\, \rho_2, \rho_3\} = \{0, 0.15, 0.80, 1\}$ and n collocation intervals of length $1/n$ are chosen. By using this results, we can see that the order of convergence collocation method ($\mathcal{O}(h^3)$) is one unit higher than the order of convergence the collocation method ($\mathcal{O}(h^2)$). Then, these results indicate improved method efficiency.

Table 8: Numerical results for example 5.

n	$\|\|e\|\|_\infty$	*Order*	$\|\|\theta\|\|_\infty$	*Order*	$\|\|e^*\|\|_\infty$	*Order*
2	2.80944e-2	$----$	1.15153e-3	$----$	4.39762e-3	$----$
4	6.22619e-3	2.17386	1.57056e-4	2.87420	4.65292e-4	3.24052
8	1.43204e-3	2.12028	2.11810e-5	2.89043	5.35936e-5	3.11800
16	3.40758e-4	2.07125	2.76866e-6	2.93551	6.43337e-6	3.05841
32	8.29227e-5	2.03891	3.54254e-7	2.96633	7.88137e-7	3.02906
64	2.04401e-5	2.02036	4.45819e-8	2.99025	9.75325e-8	3.01449
128	5.07308e-6	2.01047	5.35966e-9	3.05624	1.21306e-8	3.00724

Example 6. As a last example, the results of the proposed algorithm are compared with the results of methods in [31,32,33]. Consider the FID problem

$$y'(t) = 1 - \frac{t}{3} + \int_0^1 tsy(s)ds,$$
$$y(0) = 0,$$

with exact solution as $y(t) = t$. we choose $n = 10, m = 2$ and assume that τ_i and ρ_i are equidistant point. The results of the proposed algorithm are shown in Table 9 and compared. Based on the results, it is evident that the proposed method exhibits better accuracy in comparison to the suggested methods.

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Table 9: Absolute errors for Example 6.

t	method in [31]	method in [32]	method in [33]	proposed method
0.1	2.1794e-4	1.6667e-3	3.7900e-6	4.3021e-16
0.2	6.3855e-4	6.0939e-3	1.5160e-5	6.3837e-16
0.3	7.9137e-4	1.3202e-2	3.41100e-5	1.6653e-16
0.4	2.1559e-2	2.2914e-2	6.0640e-5	1.6654e-16
0.5	4.9936e-3	3.5158e-2	9.4750e-5	4.4408e-16
0.6	2.2173e-2	6.6965e-2	1.3644e-4	2.2204e-16
0.7	1.0565e-4	7.1243e-2	1.8571e-4	1.1102e-16
0.8	1.4323e-3	8.6398e-2	2.4256e-4	1.1101e-15
0.9	2.0775e-2	1.0810e-1	3.0699e-4	1.1103e-15

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