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Zeros of harmonic trinomials with complex parameter

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ABSTRACT: In this paper, we study the number of zeros of harmonic trinomials with complex parameter of the form $p_a(z) = z^n + a\bar{z}^k - 1$, $a \in \mathbb{C}, n > k$, and $gcd(n,k) = 1$. We are interested about two problems, first problem about the location of this roots, seconde problem about the number and the location of roots of the harmonic polynomials.

Key Words: Polynomial, harmonic trinomial, Wilmshurst's conjecture.

Contents

1. Introduction

Lewy's [8] proved that $f = h + \overline{g}$ (where h and g are analytic) is locally univalent and sense-preserving if and only if $h'(z) \neq 0$ and the dilatation function ω of f, defined by

$$\omega(z) = \frac{g'(z)}{h'(z)},$$

satisfies $|\omega(z)| < 1$. The dilatation function $\omega(z)$ can be viewed as a measure of how far $f(z)$ is from being analytic $\omega(z) \equiv 0$ for analytic functions, and $|\omega(z)| \leq k < 1$ is the well-studied class of quasiconformal functions [6] .

The multiplicity of a zero z_0 of a complex-valued harmonic function is defined via its power series expansion about the zero. That is, let

$$f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{i=r}^{\infty} a_i(z - z_0)^i + \overline{(b_0 + \sum_{i=s}^{\infty} b_i(z - z_0)^i)} \qquad (1.1)$$

where $a_r \neq 0$ and $b_s \neq 0$.

1. If z_0 is in the sense-preserving region of the plane,then $r \geq s$, and the order of the zero is r.
2. If z_0 is in the sense-reversing region of the plane, then $s \geq r$, and the order of the zero $-s$.
3. If $p = h + \overline{g}$ is a harmonic polynomial for which the degree of h is n, the degree of g is m, and $m \leq n$, then Sheil-Small [11] conjectured that the maximum number of zeros of p is n^2. Wilmshurst [12] proved this conjecture and provided an example to show that n^2 is sharp.

Peretz and Schmid [10] also proved the conjecture, while Bshouty et al. [3] provided another example to establish the sharpness of the bound. Wilmshurst further conjectured that in the more restrictive case in which $1 \leq m \leq n - 1$, p has at most $m(m - 1) + 3n - 2$ zeros.

Khavinson and Swiatek [4] looked at the subclass of harmonic polynomials $p(z) = h(z) - \overline{z}$ related to gravitational lensing. They showed that if the degree of h is $n > 1$, then

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the number of zeros is bounded by $3n-2$, which satisfies Wilmshurst's conjecture.

Melman [9] investigated trinomials of the form $q(z) = z^n - az^k - 1$, where $1 \leq k \leq n-1$, $n \geq 3$, $a \in C$, and $gcd(n,k) = 1$. He gleaned information relating to the location of the zeros of q. In a similar fashion, Brilleslyper and Schaubroeck [1], [2], considered the family of trinomials $p(z) = z^n + z^k - 1$, where $1 \leq k \leq n-1$, $n \geq 2$, and derived a formula for the number of zeros of p located on the unit circle.

M. Brillesl Yper, J. Brooks, M. Dorff, R. Howell, and L. Schaubroeck [13] considered the family of harmonic trinomials $p_c(z) = z^n + c\overline{z}^k - 1$, where $1 \leq k \leq n-1$, $n \geq 3$, $c \in \mathbb{R}^+$, and $gcd(n,k) = 1$.

Much work has been done examining the similarities and differences between analytic and harmonic functions. Many familiar results for analytic functions hold for complex-valued harmonic functions with only slight modifications. In this paper we are interested about two problems, first problem, about the location of this roots, seconde problem, about the number and the location of roots of the harmonic polynomials.

So we are interested about this two familly of trinomials

$$\begin{aligned} p_a(z) &= z^n + a\overline{z}^k - 1, \\ q_a(z) &= z^n + az^k - 1, \\ where,\ 1 &\leq k \leq n-1, n \geq 3, a \in \mathbb{C}, and\ gcd(n,k) = 1. \end{aligned}$$

2. Preliminary

Definition 2.1 *[5] A hypocycloid centered at the origin is the curve traced by a fixed point on a circle of radius b rolling inside a larger origin-centered circle of radius a. The curve is given by the parametric equations:*

$$\begin{aligned} x(\theta) &= (a-b)\cos(\theta) + b\cos(\frac{a-b}{b}\theta), \\ y(\theta) &= (a-b)\sin(\theta) - b\sin(\frac{a-b}{b}\theta). \end{aligned}$$

If $\frac{a}{b}$ is written in reduced form as $\frac{p}{q} \in \mathbb{Q}$, then the hypocycloid has p cusps, and each arc connects cusps that are q units away from each other in a counterclockwise direction. Such a hypocycloid is called a (p,q) hypocycloid, and the range of θ values to trace the the entire hypocycloid is $0 \leq \theta \leq 2\pi q$.

We show that $p_\alpha(\Gamma_\alpha)$ is a hypocycloid of type $(n+k, n)$.

Definition 2.2 *[5] The argument principle. A harmonic function $f = h + \overline{g}$, where f and g are analytic functions, is called sense-preserving at z_0 if the Jacobian*

$$J_h(z) = \left|f'(z)\right|^2 - \left|g'(z)\right|^2 > 0,$$

for every z in some punctured neighborhood of z_0.

We also say that h is sense-reversing if $\overline{h}$ is sense-preserving at z_0. If h is neither sensepreserving nor sense-reversing at z_0, then z_0 is called singular and necessarily (but not sufficiently) $J_h(z_0) = 0$.

Remark 2.1 *Note that for harmonic functions*

$$z - \overline{p(z)}, degp > 1$$

Then, z_0 is sensepreserving, reversing or singular if and only if $\left|p'(z_0)\right|$ is less than 1, greater than 1 or equal to 1, respectively.

-If Γ is an oriented closed curve and F does not vanish on Γ, then the notation

$$\Delta_\Gamma \arg F(z),$$

means the increment of the argument of $F(z)$ along Γ.

Theorem 2.1 *[7] Let $F(z)$ be a complex harmonic polynomial with complex coefficients where*

$$F(z) = P_n(z) + \overline{Q_m(z)},\ n > m,$$

then $F(z) = 0$ has at most $n^2 - n$ solutions, in the set $Rez \neq 0$ and $Imz \neq 0$.

Where $P_n(z) = z^n + (z-1)^n$, $Q_m(z) = z^m - (z-1)^m$, and $F(z) = P_n(z) + \overline{Q_m(z)}$ $(n, m \in \mathbb{N})$.

Theorem 2.2 *[13] Let $p_c(z) = z^n + c\overline{z}^k - 1$, where $1 \leq k \leq n-1$, $n \geq 3$, $c \in \mathbb{R}^+$, and $gcd(n,k) = 1$ and let $N = [\frac{k}{2}] + 1$. There exist N critical values c_j, with $0 < c_1 < c_2 < .... < c_N$, such that*

1. *If $0 \leq c \leq c_1$, then $p_c(z)$ has n distinct zeros.*
2. *If $c_j < c < c_{j+1}$ for some $1 \leq j \leq N-1$, then $p_c(z)$ has $n + 4j - 2$ distinct zeros.*
3. *If $c > c_{N,}$ then $p_c(z)$ has $n + 2k$ distinct zeros.*

Example 2.1 *Consider*

$$p_c(z) = z^5 + c\overline{z}^3 - 1.$$

Where $N = 2$, $c_1 = 1.26$ and $c_2 < 1.96$.

1. *If $c = 1 < c_1$, the trinomial $p_c(z)$ has $n = 5$ zeros. $|z| = a = \left(\frac{3}{5}\right)^{\frac{1}{2}}$.*
2. *If $c = 1.5$, $c_1 < c < c_2$, then $p_c(z)$ has $n = 5 + 4(1) - 2 = 7$ zeros. $|z| = b = \left(\frac{4.5}{5}\right)^{\frac{1}{2}}$.*
3. *Finally if $c = 3 > c_2$, then the number of zeros has increased to $5 + 2(3) = 11$. $|z| = e = \left(\frac{9}{5}\right)^{\frac{1}{2}}$.*

Remark 2.2 *The number of zeros of $p_\alpha(z) = z^n + \alpha\overline{z}^k - 1$ increases from n to $n + 2k$ as α increases. The α-values at which the number of zeros increases will be called the critical α-values.*

A straightforward computation gives that $|\omega(z)| = 1$ if and only if $|z| = (\frac{\alpha k}{n})^{\frac{1}{n-k}}$, with $p_\alpha(z)$ being sense-reversing on the interior of the critical circle Γ_α and sense-preserving on its exterior.

Where:

$$\Gamma_\alpha = \left\{ z : |z| = R_\alpha(\frac{\alpha k}{n})^{\frac{1}{n-k}} \right\}.$$

3. Main results

Lemma 3.1 *Let $p_\alpha(z) = z^n + \alpha\overline{z}^k - 1$, where $1 \leq k \leq n-1$, $n \geq 3$, $\alpha \in \mathbb{R}^+$, and $gcd(n,k) = 1$ and let $N = [\frac{k}{2}] + 1$, for R sufficiently large, the winding number of the image of $|z| = R$ under $p_\alpha(z)$ around the origin is n. Thus, for all α, the sum of the orders of the zeros of $p_\alpha(z)$ is n.*

Proof. *From a standard Rouche-type argument by comparing*

$$\left|p_\alpha(R\ e^{it}) - R^n e^{int}\right|,$$

with $\left|R^n e^{int}\right|$ for sufficiently large R. □

Lemma 3.2 *All zeros of $p_\alpha(z) = z^n + \alpha\overline{z}^k - 1$ have order 1 or -1.*

Proof. Let z_0 be a zero of

$$\begin{aligned} p_\alpha(z) &= h + \overline{g} \\ &= (z^n - 1) + \alpha\overline{z_k}. \end{aligned}$$

Then $z_0 \neq 0$ since $p_\alpha(0) = -1$.

Since h and g are polynomials, the series expansions of h and g about z_0 are finite series.

From (1.1),

$$a_1 = nz_0^{n-1} \quad \text{and} \quad b_1 = \alpha k z_0^{k-1}$$

neither of zero. Thus the order of the zero at z_0 is either $+1$ if $|z_0| > R\alpha$, or -1 if $|z_0| < R\alpha$. □

Lemma 3.3 *The image $p_\alpha(\Gamma_\alpha)$ of the critical circle is an $(n+k,n)$ hypocycloid centered at $z=-1$. The value of α affects only the size of the hypocycloid.*

Proof. We have

$$p_\alpha(R_\alpha\ e^{it}) = R_\alpha^n e^{int} + \alpha R_\alpha^k e^{-ikt} - 1,$$

real part is

$$u(R_\alpha\ e^{it}) = e^{\frac{n}{n-k}}\left[(\frac{k}{n})^{\frac{n}{n-k}}\cos nt + (\frac{k}{n})^{\frac{k}{n-k}}\cos kt\right] - 1,$$

and imaginary part is

$$v(R_\alpha\ e^{it}) = e^{\frac{n}{n-k}}\left[(\frac{k}{n})^{\frac{n}{n-k}}\sin nt - (\frac{k}{n})^{\frac{k}{n-k}}\sin kt\right].$$

We notice

$$\begin{aligned} a &= \alpha^{\frac{n}{n-k}}(\frac{k}{n})^{\frac{k}{n-k}}(\frac{n+k}{n}), \\ b &= \alpha^{\frac{n}{n-k}}(\frac{k}{n})^{\frac{k}{n-k}}, \\ nt &= \phi. \end{aligned}$$

We have

$$\begin{aligned} a-b &= \alpha^{\frac{n}{n-k}}(\frac{k}{n})^{\frac{n}{n-k}}, \\ \frac{a-b}{b} &= \frac{k}{n}, \\ kt &= \frac{a-b}{b}\phi. \end{aligned}$$

By definition (2.1), we see that the equations for u and v describe a hypocycloid centered at $z=-1$ instead of the origin. We observe that

$$\frac{a}{b} = \frac{n+k}{n}$$

does not depend upon the constant α, and the entire hypocycloid is traced for $0 \le t \le 2\pi$. □

Lemma 3.4 *[13] The hypocycloid $p_\alpha(\Gamma_\alpha)$ has $N = \left[\frac{k}{2}\right]+1$ distinct intersections with the real axis to the right of its center. These intersections, in turn, correspond to the N critical α-values α_j , with $0 < \alpha_1 < \ldots < \alpha_N$, at which $p_\alpha(\Gamma_\alpha)$ intersects the origin. The winding number H_α, of $p_\alpha(\Gamma_\alpha)$ around the origin is as follows:*
-If $0 \le \alpha \le \alpha_1$, then $H_\alpha = 0$.
-If $\alpha_j < \alpha < \alpha_{j+1}$ for some $1 \le j \le N-1$, then $H_\alpha = -2j+1$.
-If $\alpha > \alpha_N$, then $H_\alpha = -k$.

Theorem 3.1 *Let*

$$p_a(z) = z^n + a\overline{z}^k - 1,$$

where $1 \le k \le n-1$, $n \ge 3$, $a \in \mathbb{C}$, $a = s + i\beta$, $\alpha \in \mathbb{R}_+^*$ and $gcd(n,k) = 1$ and let $N = [\frac{k}{2}]+1$.
There exist N critical values a_j, with $0 < |a_1| < |a_2| < \ldots. < |a_N|$, such that

1. *If $0 \le |a| \le |a_1|$, then $p_a(z)$ has n distinct zeros.*
2. *If $|a_j| < |a| < |a_{j+1}|$ for some $1 \le j \le N-1$, then $p_a(z)$ has $n+4j-2$ distinct zeros.*
3. *If $|a| > |a_N|$, then $p_a(z)$ has $n+2k$ distinct zeros.*

Proof. Let

$$p_a(z) = z^n + a\overline{z}^k - 1,$$

where

$$a \in \mathbb{C},\ a = s + i\beta, s \in \mathbb{R}^*.$$

Then

$$\begin{aligned} p_a(z) &= z^n + \alpha\overline{z}^k + i\beta\overline{z}^k - 1 \\ &= z^n + s\overline{z}^k - 1 + i\beta\overline{z}^k \\ &= p_s(z) + i\beta\overline{z}^k. \end{aligned}$$

Where

$$p_s(z) = z^n + s\overline{z}^k - 1.$$

I)-If $\beta = 0$,

$$p_a(z) = p_s(z).$$

According to [theorem 2 [13]],

1) If $0 \le s \le s_1$, then $p_s(z)$ has n distinct zeros.
2) If $s_j < s < s_{j+1}$ for some $1 \le j \le N-1$, then $p_s(z)$ has $n + 4j - 2$ distinct zeros.
3) If $s > s_N$, then $p_s(z)$ has $n + 2k$ distinct zeros.

II)- If $\beta \neq 0$

Let z_0 the zero of $p_s(z)$,

$$z_0^n + s\overline{z_0}^k - 1 = 0 \Longrightarrow \overline{z_0}^k = \frac{1 - \overline{z_0}^n}{s}$$

then

$$i\beta\overline{z_0}^k = \frac{i\beta}{s}(1 - \overline{z_0}^n).$$

And

$$\begin{aligned} p_a(z_0) &= z_0^n + s\overline{z_0}^k + i\beta\overline{z_0}^k - 1 \\ &= z_0^n + s\overline{z_0}^k - 1 + i\beta\overline{z_0}^k \\ &= p_s(z_0) + i\beta\overline{z_0}^k \\ &= i\beta\overline{z_0}^k. \end{aligned}$$

If z_0 is the zero of $p_a(z)$,

$$\begin{aligned} p_a(z_0) &= i\beta\overline{z_0}^k \\ &= \frac{i\beta}{\alpha}(1 - \overline{z_0}^n) \\ &= 0. \end{aligned}$$

Then

$$\begin{aligned} 1 - \overline{z_0}^n &= 0 \Longrightarrow \overline{z_0}^n = 1 \\ &\Longrightarrow \overline{z_0^n} = 1 \\ &\Longrightarrow \overline{z_0} = 1\ or - 1 \\ &\Longrightarrow z_0 = 1\ or - 1. \end{aligned}$$

By Lemma (3.1) and the argument principle for harmonic functions, the sum of the orders of the zeros of $p_a(z)$ is always n, and by Lemma (3.2), all zeros of $p_a(z)$ have order 1 or -1.

We apply the argument principle to say that the number of zeros in the sense-preserving region of the plane must be $n + |1 - 2j| = n + 2j - 1$. Thus the total number of zeros is $n + 2(2j - 1) = n + 4j - 2$.

Since $H_a = -k\ for\ |a| > |a_n|$, we have that $p_a(z)$ has $n + 2k$ zeros in this case. □

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