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Weakly μ-Countable compactness and Weakly $\mu\mathcal{H}$-Countable compactness

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ABSTRACT: In this paper, we introduce and study the notions of weakly μ-countably compact spaces in generalized topology and weakly $\mu\mathcal{H}$-countably compact spaces with respect to hereditary class $\mathcal{H}$. Several of their properties and relations are presented here. In addition, some preservation properties of functions are studied and investigated.

Key Words: generalized topology, hereditary class, μ-countable cvovering, weakly μ-countably compact, weakly $\mu\mathcal{H}$-countably compact.

Contents

1. Introduction and preliminaries

According to [2], a generalized topology (briefly GT) on a nonempty set X is a subset μ of the power set $\exp X$ such that $\emptyset \in \mu$ and the union of the elements of an arbitrary subset of μ belongs to μ. We call the pair (X, μ) a generalized topological space (briefly GTS) on X. The elements of μ are called μ-open sets and their complements are called μ-closed sets. If A is a subset of a GTS (X, μ), then the μ-closure of A, $c_\mu(A)$, is the intersection of all μ-closed sets containing A and the μ-interior of A, $i_\mu(A)$, is the union of all μ-open sets contained in A (see [2] and [3]). Clearly, A is μ-closed if and only if $A = c_\mu(A)$, $c_\mu(A)$ is the smallest μ-closed set containing A, $i_\mu(A)$ is the largest μ-open set contained in A. In this paper, we consider $X = \bigcup \mathcal{U}$ for any cover $\mathcal{U}$ of μ-open subsets of X. In particular, X is called a weakly μ-compact space if for every μ-open cover of X, the finite union of the μ-closures of whose members covers X, see [10]. More generalizations of these notions with respect to a hereditary class $\mathcal{H}$ can be seen in [5,1,7]. A hereditary class $\mathcal{H}$ is a nonempty subset of the power set $\exp X$ that satisfies the following property: if $A \in \mathcal{H}$ and $B \subset A$, then $B \in \mathcal{H}$, see [4]. A hereditary class $\mathcal{H}$ is called an ideal if $\mathcal{H}$ satisfies the additional condition: for $A, B \in \mathcal{H}$ implies $A \cup B \in \mathcal{H}$, see [6]. Given a generalized topological space (X, μ) with a hereditary class $\mathcal{H}$, for a subset A of X, the generalized local function of A with respect to $\mathcal{H}$ and μ [4] is defined as $A^* = \{x \in X \mid A \cap V \notin \mathcal{H} \text{ for any } V \in \mu \text{ containing } x\}$. For $A \subseteq X$, define $c^*_\mu(A) = A \cup A^*$ and the family $\mu^* = \{A \subset X : X \setminus A = c^*(X \setminus A)\}$ is a GT on X which is finer than μ. The elements of μ^* are called μ^*-open sets and their complement are called μ^*-closed sets. We call $(X, \mu, \mathcal{H})$ a hereditary generalized topological space and briefly we denote it by HGTS. In this paper, we define the notions of weakly μ-countably compact and weakly $\mu\mathcal{H}$-countably compact spaces as generalizations of weakly μ-compact and weakly $\mu\mathcal{H}$-compact ($\mu\mathcal{H}$-countably compact) spaces, respectively. Finally, in this section, we recall the following definitions and facts for their importance in the material of the present paper.

Definition 1.1 [10] Let A be a subset of a GTS (X, μ). Then A is said to be:

1. μ-regular open if $A = i_\mu(c_\mu(A))$;

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2. μ-regular closed if $X \setminus A$ is μ-regular open.

Definition 1.2 [10] Let (X, μ) be a GTS and A be a subset of X. A point $x \in X$ is called a $\theta\mu$-accumulation point of A if $c_\mu(U) \cap A \neq \emptyset$ for every μ-open subset U of X containing x. The set of all $\theta\mu$-accumulation points of A is called the θ_μ-closure of A and is denoted by $(c_\mu)_\theta(A)$. In addition, A is said to be μ_θ-closed if $(c_\mu)_\theta(A) = A$. The complement of a μ_θ-closed set is said to be μ_θ–open.

Definition 1.3 [10] A GTS (X, μ) is said to be μ-regular if for each μ-open subset U of X and each $x \in U$, there exist a μ-open subset V of X and a μ-closed subset F of X such that $x \in V \subset F \subset U$.

Proposition 1.1 *[10] Let (X, μ) be a μ-disconnected GTS. Then (X, μ) is weakly μ-compact if and only if every μ-clopen set is weakly μ-compact.*

Proposition 1.2 *[9] Let (X, μ) and (Y, ν) be GTSs and σ be the generalized product topology on $X \times Y$. The projection function $P_X : (X \times Y, \sigma) \to (X, \mu)$ $\big($resp. $P_Y : (X \times Y, \sigma) \to (Y, \nu)\big)$ is (σ, μ)-continuous $\big($resp. (σ, ν)-continuous $\big)$.*

Lemma 1.1 *[10] Let A and B be subsets of a GTS (X, μ) such that $A \subset B$. Then $c_{\mu_B}(A) = c_\mu(A) \cap B$.*

Definition 1.4 [8] Let (X, μ) and (Y, ν) be two GTSs. A function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-continuous if for every $V \in \nu$, $f^{-1}(V) \in \mu$

Definition 1.5 [11] Let (X, μ) and (Y, ν) be two GTSs. A function $f : (X, \mu) \to (Y, \nu)$ is said to be (μ, ν)-open if for every $U \in \mu$, $f(U) \in \nu$

2. weakly μ-countably compact and weakly $\mu\mathcal{H}$-countably compact spaces

In this section we define and study the notions of weakly μ-countably compact and weakly $\mu\mathcal{H}$-countably compact spaces. Most of the results in this section are proved with respect to weakly $\mu\mathcal{H}$-countably compact spaces. By taking $\mathcal{H} = \{\emptyset\}$, we get directly the results for weakly μ-countably compact spaces.

Definition 2.1 A GTS (X, μ) is said to be weakly μ-countably compact if for every countable μ-open cover $\{V_\lambda : \lambda \in \Delta\}$ of X, there is a finite subset Δ_0 of Δ such that $X = \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}$.

Definition 2.2 A HGTS $(X, \mu, \mathcal{H})$ is said to be weakly $\mu\mathcal{H}$-countably compact or weakly μ-countably compact with respect to a hereditary class $\mathcal{H}$ if for every countable μ-open cover $\{V_\lambda : \lambda \in \Delta\}$ of X, there is a finite subset Δ_0 of Δ such that $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$.

It is clear that a GTS (X, μ) is weakly μ-countably compact if and only if $(X, \mu, \{\emptyset\})$ is weakly $\mu\{\emptyset\}$-countably compact. And if a GTS (X, μ) is a weakly μ-countably compact space, then a HGTS $(X, \mu, \mathcal{H})$ is a weakly $\mu\mathcal{H}$-countably compact space. The following example shows that the converse is not true in general.

Example 2.1 *Let $\mu = \{V \subseteq (0, 1) : V \text{ is uncountable}\} \cup \{\emptyset\}$ be a GT on $X = (0, 1)$ and $\mathcal{H} = \{X \setminus V : V \in \mu\}$ be a hereditary class on X. The family $\{V_n = (\frac{1}{n}, 1) : n \in \mathbb{N}\}$ is a countable μ-open cover of X. For any finite subset Δ_0 of $\mathbb{N}$ and $i \in \Delta_0$, we have $c_\mu(V_i) = V_i$. Therefore, $X \neq \bigcup_{i \in \Delta_0} c_\mu(V_i)$, so (X, μ) is not weakly μ-countably compact. Now, let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ-open cover of X and Δ_0 be any finite subset of Δ. Since $X \setminus V_\lambda \in \mathcal{H}$ and $X \setminus \bigcup_{\lambda \in \Delta_0} c_\mu(V_\lambda) \subseteq X \setminus V_\lambda \in \mathcal{H}$. This means $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact.*

Theorem 2.2 *Let μ be a GT on a nonempty set X and H_f be the set of all finite subsets of X. A GTS (X, μ) is weakly μ-countably compact if and only if the $(X, \mu, \mathcal{H}_f)$ is a weakly $\mu\mathcal{H}_f$-countably compact space.*

Proof: ($\Rightarrow$): It is obvious.
($\Leftarrow$): Suppose $(X, \mu, \mathcal{H}_f)$ is weakly $\mu\mathcal{H}_f$-countably compact and let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ-open cover of X. By assumption, there is a finite subset Δ_0 of Δ such that $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}_f$. So, $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} = \{x_1, x_2, ..., x_n\}$. For each x_i, $1 \leq i \leq n$, there is V_{λ_i} such that $x_i \in V_{\lambda_i}$. Thus $X = \big(\bigcup\limits_{\lambda\in\Delta_0} c_\mu(V_\lambda)\big) \cup \big(\bigcup\limits_{i=1}^{n} c_\mu(V_{\lambda_i})\big)$. Hence, (X, μ) is weakly μ-countably compact. □

Theorem 2.3 *For a HGTS $(X, \mu, \mathcal{H})$, the following statements are equivalent:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact;*
2. *for any countable μ-regular open cover $\{U_\lambda : \lambda \in \Delta\}$ of X, there is a finite subset Δ_0 of Δ such that $X \setminus \bigcup\{c_\mu(U_\lambda) : \lambda \in \Delta_0\}) \in \mathcal{H}$.*

Proof: (1) $\Rightarrow$ (2): It is obvious form the definition.
(2) $\Rightarrow$ (1): Let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ-open cover of X. Then the family $\{U_\lambda = i_\mu(c_\mu(V_\lambda)) : \lambda \in \Delta\}$ is a countable μ-regular open cover of X. From assumption, there is a finite subset Δ_0 of Δ such that $X \setminus \bigcup\{c_\mu(U_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$. But $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \subseteq X \setminus \bigcup\{c_\mu(U_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$ which means $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$ and so $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact. □

Theorem 2.4 *A GTS (X, μ) is weakly μ-countably compact if and only if for any countable μ-regular open cover $\{U_\lambda : \lambda \in \Delta\}$ of X, there exits a finite subset Δ_0 of Δ such that $X = \bigcup\{c_\mu(U_\lambda) : \lambda \in \Delta_0\}$.*

Theorem 2.5 *A HGTS $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact if and only if for any countable family $\{F_\lambda : \lambda \in \Delta\}$ of μ-closed subsets of X having the property that $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} \notin \mathcal{H}$ for any finite subset Δ_0 of Δ, then $\bigcap\{F_\lambda : \lambda \in \Delta\} \neq \emptyset$.*

Proof: Assume that $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact and $\{F_\lambda : \lambda \in \Delta\}$ is a countable family of μ-closed subsets of X such that $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} \notin \mathcal{H}$ for any finite subset Δ_0 of Δ. Suppose that $\bigcap\{F_\lambda : \lambda \in \Delta\} = \emptyset$, then $\{X \setminus F_\lambda : \lambda \in \Delta\}$ is a countable μ-open cover of X. From assumption, there is a finite subset Δ_0 of Δ such that $X \setminus \bigcup\{c_\mu(X \setminus F_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$. Hence $\bigcap\{i_\mu(F_\mu) : \lambda \in \Delta_0\} \in \mathcal{H}$ which contradicts the assumption. Thus $\bigcap\{F_\lambda : \lambda \in \Delta\} \neq \emptyset$.
Conversely, let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ-open cover of X. Assume that for any finite subset Δ_0 of Δ, we have $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \notin \mathcal{H}$, and so $\bigcap\{i_\mu(X \setminus V_\lambda) : \lambda \in \Delta_0\} \notin \mathcal{H}$. By assumption, $\bigcap\{(X \setminus V_\lambda) : \lambda \in \Delta\} \neq \emptyset$, this is contrary to the fact that $\{V_\lambda : \lambda \in \Delta\}$ is a μ-open cover of X. Thus $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact. □

Theorem 2.6 *A GTS (X, μ) is a weakly μ-countably compact space if and only if for any countable family $\{F_\lambda : \lambda \in \Delta\}$ of μ-closed subsets of X having the property that $\bigcap\{i_\mu(F) : \lambda \in \Delta_0\} \neq \emptyset$ for any finite subset Δ_0 of Δ, then $\bigcap\{F : \lambda \in \Delta\} \neq \emptyset$.*

Theorem 2.7 *The following properties are equivalent for a HGTS $(X, \mu, \mathcal{H})$:*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact;*
2. *for any countable family of $\{F_\lambda : \lambda \in \Delta\}$ of μ-closed subsets of X such that $\bigcap\{F_\lambda : \lambda \in \Delta\} = \emptyset$, there exists a finite subset Δ_0 of Δ such that $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$;*
3. *for any countable family of $\{F_\lambda : \lambda \in \Delta\}$ of μ-regular closed subsets of X such that $\bigcap\{F_\lambda : \lambda \in \Delta\} = \emptyset$, there exists a finite subset Δ_0 of Δ such that $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$.*

Proof: (1) $\Rightarrow$ (2): Let $\{F_\lambda : \lambda \in \Delta\}$ be a countable family of μ-closed subsets of X such that $\bigcap\{F_\lambda : \lambda \in \Delta\} = \emptyset$, which means that $\{X \setminus F_\lambda : \lambda \in \Delta\}$ is a countable μ-open cover of X. By assumption, there is a finite subset Δ_0 of Δ such that $X \setminus \bigcup\{c_\mu(X \setminus F_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$, and so $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$.
(2) $\Rightarrow$ (3): It is obvious.
(3) $\Rightarrow$ (1): Let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ-open cover of X. Then the family $\{i_\mu(c_\mu(V_\lambda)) : \lambda \in \Delta\}$ is a countable μ-regular open cover of X, which means that, $\{c_\mu(i_\mu(X \setminus V_\lambda)) : \lambda \in \Delta\}$ is a countable family of μ-regular closed subsets of X and $\bigcap\{c_\mu(i_\mu(X \setminus V_\lambda)) : \lambda \in \Delta\} = \emptyset$. By assumption, there is a finite subset Δ_0 of Δ such that $\bigcap\{i_\mu(c_\mu(i_\mu(X \setminus V_\lambda))) : \lambda \in \Delta_0\} \in \mathcal{H}$. Since $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} = \bigcap\{i_\mu(X \setminus V_\lambda) : \lambda \in \Delta_0\} \subseteq \bigcap\{i_\mu(c_\mu(i_\mu(X \setminus V_\lambda))) : \lambda \in \Delta_0\}$ and $\mathcal{H}$ is a hereditary class, we get $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$ and this completes the proof. □

Theorem 2.8 *The following properties are equivalent for a GTS (X, μ):*

1. *(X, μ) is weakly μ-countably compact;*
2. *for any countable family of $\{F_\lambda : \lambda \in \Delta\}$ of μ-closed subsets of X such that $\bigcap\{F_\lambda : \lambda \in \Delta\} = \emptyset$, there is a finite subset Δ_0 of Δ such that $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} = \emptyset$;*
3. *for any countable family of $\{F_\lambda : \lambda \in \Delta\}$ of μ-regular closed subsets of X such that $\bigcap\{F_\lambda : \lambda \in \Delta\} = \emptyset$, there is a finite subset Δ_0 of Δ such that $\bigcap\{i_\mu(F_\lambda) : \lambda \in \Delta_0\} = \emptyset$.*

A subset A of a GTS (X, μ) is μ_θ-open if and only if for each $x \in A$, there exists a μ-open set U such that $x \in U \subseteq c_\mu(U) \subseteq A$ (see [10]). By using this notion we get the following results.

Theorem 2.9 *If a HGTS $(X, \mu, \mathcal{H})$ is a weakly $\mu\mathcal{H}$-countably compact, then for every countable μ_θ-open cover $\{V_\lambda : \lambda \in \Delta\}$ of X there is a finite Δ_0 subset of Δ such that $X \setminus \bigcup\{V_\lambda : \lambda \in \Delta_0\} \in \mathcal{H}$.*

Proof: Assume that $(X, \mu, \mathcal{H})$ is a weakly $\mu\mathcal{H}$-countably compact space and let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ_θ-open cover of X. Then for each $x \in X$, there exists $\lambda_x \in \Delta$ such that $x \in V_{\lambda_x}$. Since V_{λ_x} is μ_θ-open, there exists a μ-open set U_{λ_x} such that $x \in U_{\lambda_x} \subseteq c_\mu(U_{\lambda_x}) \subseteq V_{\lambda_x}$. Hence $\{U_{\lambda_x} : x \in X\}$ is a countable μ-open cover of X. By assumption on the space $(X, \mu, \mathcal{H})$, there exists a finite subset Δ_0 of Δ such $X \setminus \bigcup\{c_\mu(U_{\lambda_x}) : \lambda_x \in \Delta_0\} \in \mathcal{H}$. Since, $c_\mu(U_{\lambda_x}) \subseteq V_{\lambda_x}$ for every $\lambda_x \in \Delta_0$ which means that $X \setminus \bigcup\{V_{\lambda_x} : \lambda_x \in \Delta_0\} \subseteq X \setminus \bigcup\{c_\mu(U_{\lambda_x}) : \lambda_x \in \Delta_0\}$ which implies $X \setminus \bigcup\{V_{\lambda_x} : \lambda_x \in \Delta_0\} \in \mathcal{H}$. □

Theorem 2.10 *If a GTS (X, μ) is a weakly μ-countably compact, then for every countable μ_θ-open cover $\{V_\lambda : \lambda \in \Delta\}$ of X there is a finite Δ_0 subset of Δ such that $X = \bigcup\{V_\lambda : \lambda \in \Delta_0\}$.*

Lemma 2.1 *[10] A GTS (X, μ) is a μ-regular if and only if every μ-open subset of X is μ_θ-open.*

Corollary 2.1 *Let $(X, \mu, \mathcal{H})$ be a μ-regular HGTS. Then $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact if and only if $(X, \mu, \mathcal{H})$ is $\mu\mathcal{H}$-countably compact.*

Proof: The proof follows from Theorem 2.9 and Lemma 2.1. □

Corollary 2.2 *Let (X, μ) be a μ-regular GTS. Then (X, μ) is weakly μ-countably compact if and only if $(X, \mu,)$ is μ-countably compact.*

Proof: The proof follows from Theorem 2.10 and Lemma 2.1. □

Corollary 2.3 *If a HGTS $(X, \mu^*, \mathcal{H})$ is a weakly $\mu^*\mathcal{H}$-countably compact space, then $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact.*

3. Functions on weakly $\mu\mathcal{H}$-countably compact spaces

We study the effect of functions on weakly μ-countably compact spaces and weakly $\mu\mathcal{H}$-countably compact spaces.

Lemma 3.1 *[10] A function $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ is (μ, ν)-continuous if and only if $c_\mu(f^{-1}(B)) \subset f^{-1}(c_\nu(B))$ for every subset B of Y.*

Theorem 3.1 *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous function. If A is weakly $\mu\mathcal{H}$-countably compact relative to X, then $f(A)$ is weakly $\nu f(\mathcal{H})$-countably compact relative to Y.*

Proof: Let $\{V_\lambda : \lambda \in \Delta\}$ be any countable ν-open cover of $f(A)$. Since f is (μ, ν)-continuous, $\{f^{-1}(V_\lambda) : \lambda \in \Delta\}$ is a countable μ-open cover of A. By assumption, A is weakly $\mu\mathcal{H}$-countably compact relative to X and so there is a finite subset Δ_0 of Δ such that $A \setminus \bigcup\{c_\mu\big(f^{-1}(V_\lambda)\big) : \lambda \in \Delta_0\} \in \mathcal{H}$. By applying Lemma 3.1, we get $A \setminus \bigcup\{f^{-1}(c_\nu(V_\lambda)) : \lambda \in \Delta_0\} \subseteq A \setminus \bigcup\{c_\mu\big(f^{-1}(V_\lambda)\big) : \lambda \in \Delta_0\}$ which means $f(A) \setminus f\big(\bigcup\{f^{-1}\big(c_\nu(V_\lambda)\big) : \lambda \in \Delta_0\}\big) \subseteq f\big(A \setminus \{f^{-1}\big(\bigcup c_\nu(V_\lambda)\big) : \lambda \in \Delta_0\}\big)$ and so $f(A) \setminus \bigcup\{c_\nu(V_\lambda) : \lambda \in \Delta_0\} \in f(\mathcal{H})$. Thus $f(A)$ is weakly $\nu f(\mathcal{H})$-countably compact relative to Y. □

Corollary 3.1 *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous surjective function. If $(X, \mu, \mathcal{H})$ is a weakly $\mu\mathcal{H}$-countably compact space, then (Y, ν) is weakly $\nu f(\mathcal{H})$-countably compact.*

Theorem 3.2 *Let $f : (X, \mu) \to (Y, \nu)$ be a (μ, ν)-continuous function. If A is weakly μ-countably compact relative to X, then $f(A)$is weakly ν-countably compact compact relative to Y.*

Corollary 3.2 *Let $f : (X, \mu) \to (Y, \nu)$ be a (μ, ν)-continuous surjective function. If (X, μ) is weakly μ-countably compact, then (Y, ν) is weakly ν-countably compact.*

Proposition 3.1 *Let $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous surjection and $(X, \mu, \mathcal{H})$ be a weakly $\mu\mathcal{H}$-countably compact space. If Y is a finite space, then (Y, ν) is weakly ν-countably compact.*

Proof: Suppose that $(X, \mu, \mathcal{H})$ is a weakly $\mu\mathcal{H}$-countably compact space and $f : (X, \mu, \mathcal{H}) \to (Y, \nu)$ be a (μ, ν)-continuous surjection. By Corollary 3.1A, $(Y, \nu, f(\mathcal{H}))$ is weakly $\nu f(\mathcal{H})$-countably compact. Since Y is finite, $f(\mathcal{H})$ is the class of finite subsets and by Theorem 2.2, (Y, ν) is weakly ν-countably compact. □

Corollary 3.3 *Let $f : (X, \mu) \to (Y, \nu, \mathcal{H})$ be a (μ, ν)-open bijective function. If $(Y, \nu, \mathcal{H})$ is a weakly $\nu\mathcal{H}$-countably compact space, then $\big(X, \mu, f^{-1}(\mathcal{H})\big)$ is a weakly $\mu f^{-1}(\mathcal{H})$-countably compact space.*

Proof: The proof follows directly from the fact that if $f : (X, \mu) \to (Y, \nu)$ is a (μ, ν)-open bijective function, then $f^{-1} : (Y, \nu) \to (X, \mu)$ is (ν, μ)-continuous. Applying Corollary 3.1A completes the proof. □

The following corollary follows directly from Proposition 1.2.

Corollary 3.4 *Let (X, μ) and (Y, ν) be GTSs and σ be the generalized product topology on $X \times Y$. If $X \times Y$ is weakly σ-countably compact, then (X, μ) is weakly μ-countably compact and (Y, ν) is weakly ν-countably compact.*

Proof: Since the projection function $P_X : (X \times Y, \sigma) \to (X, \mu)$ $\big($*resp.* $P_Y : (X \times Y, \sigma) \to (Y, \nu)\big)$ is (σ, μ)-continuous $\big($resp. (σ, ν)-continuous $\big)$, then by applying Corollary 3.2A, we obtain that (X, μ) is weakly μ-countably compact (resp. (Y, ν) is weakly ν-countably compact). □

4. Weakly $\mu\mathcal{H}$-countable compact sets relative to a space

Next we define the weakly μ-countably compact and weakly $\mu\mathcal{H}$-countably compact subsets relative to a space and study their interesting properties.

Definition 4.1 A subset A of GTS (X,μ) is said to be weakly μ-countably compact relative to X if for every countable μ-open cover $\{V_\lambda : \lambda \in \Delta\}$ of A, there is a finite subset Δ_0 of Δ such that $A \subseteq \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}$.

Definition 4.2 Let $(X,\mu,\mathcal{H})$ be a HGTS. A subset A of X is said to be weakly $\mu\mathcal{H}$-countably compact relative to X if for every countable μ-open cover $\{V_\lambda : \lambda \in \Delta\}$ of A, there is a finite subset Δ_0 of Δ such that $A \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$.

Theorem 4.1 *Let $(X,\mu,\mathcal{H})$ be weakly $\mu\mathcal{H}$-countably compact. If F is a μ-clopen subset of X, then F is weakly $\mu\mathcal{H}$-countably compact relative to X.*

Proof: Let F be a μ-clopen subset of X and $\{V_\lambda : \lambda \in \Delta\}$ be any countable μ-open cover of F. Since $X \setminus F$ is μ-open, the collection $\{V_\lambda : \lambda \in \Delta\} \cup \{X \setminus F\}$ is a countable μ-open cover of X and $(X,\mu,\mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact. Therefore, there is a finite subset Δ_0 of Δ such that $X \setminus \big[\big(\bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \cup \{c_\mu(X \setminus F)\}\big] \in \mathcal{H}$. Since $c_\mu(X \setminus F) = X \setminus F$, then $X \setminus \big[\big(\bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \cup \{X \setminus F\}\big] = \big(X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \cap F \in \mathcal{H}$ which means that $F \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$. Hence F is weakly $\mu\mathcal{H}$-countably compact relative to X. □

Theorem 4.2 *Let (X,μ) be weakly μ-countably ccompact. If F is a μ-clopen subset of X, then F is weakly μ-countably compact compact relative to X.*

Theorem 4.3 *Let $(X,\mu,\mathcal{H})$ be a HGTS. If every proper μ-regular closed subset of X is weakly $\mu\mathcal{H}$-countably compact relative to X, then $(X,\mu,\mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact.*

Proof: Let $\{V_\lambda : \lambda \in \Delta\}$ be any countable μ-open cover of X. Pick $\lambda_0 \in \Delta$ such that $V_{\lambda_0} \neq \emptyset$. So, the set $X \setminus i_\mu(c_\mu(V_{\lambda_0}))$ is proper μ-regular closed. By assumption, there is a finite subset Δ_0 of Δ such that $\big(X \setminus i_\mu(c_\mu(V_{\lambda_0}))\big) \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0 \setminus \{\lambda_0\}\} \in \mathcal{H}$. Therefore, $X \setminus \big[\big(\bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0 \setminus \{\lambda_0\}\}\big) \cup c_\mu(V_{\lambda_0})\big] \subset \big(X \setminus i_\mu(c_\mu(V_{\lambda_0}))\big) \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0 \setminus \{\lambda_0\}\}$ which means $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \subset \big(X \setminus i_\mu(c_\mu(V_{\lambda_0}))\big) \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0 \setminus \{\lambda_0\}\}$. Hence $X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$ and $(X,\mu,\mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact. □

Theorem 4.4 *Let (X,μ) be a GTS. If every proper μ-regular closed subset of X is weakly μ-countably compact relative to X, then (X,μ) is weakly μ-countably compact.*

Proposition 4.1 *Let A and B be subsets of a HGTS $(X,\mu,\mathcal{H})$. If A is μ-clopen and B is weakly $\mu\mathcal{H}$-countably compact relative to X, then $A \cap B$ is weakly $\mu\mathcal{H}$-countably compact relative to X.*

Proof: Let $\{V_\lambda : \lambda \in \Delta\}$ be a countable μ-open cover of $A \cap B$. Then $\{V_\lambda : \lambda \in \Delta\} \cup \{X \setminus A\}$ is a countable μ-open cover of B. From assumption, there is a finite subset Δ_0 of Δ such that $B \setminus \big[\big(\bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \cup \{c_\mu(X \setminus A)\}\big] \in \mathcal{H}$. Since $c_\mu(X \setminus A) = X \setminus A$, then $B \setminus \big[\big(\bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \cup \{X \setminus A\}\big] = B \cap \big[X \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big] \cap A = (A \cap B) \setminus \big(\bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \in \mathcal{H}$. Therefore, $A \cap B$ is weakly $\mu\mathcal{H}$-countably compact relative to X. □

Proposition 4.2 *Let A and B be subsets of a GTS (X,μ). If A is μ-clopen and B is weakly μ-countably compact compact relative to X, then $A \cap B$ is weakly μ-countably compact compact relative to X.*

Theorem 4.5 *Let $(X,\mu,\mathcal{H})$ be a HGTS and $\mathcal{H}$ be an ideal on X. If A and B are weakly $\mu\mathcal{H}$-countably compact relative to X, then $A \cup B$ is weakly $\mu\mathcal{H}$-countably compact relative to X.*

Proof: Let A and B be weakly $\mu\mathcal{H}$-countably compact relative to X and let $\{V_\lambda : \lambda \in \Delta\}$ be any countable μ-open cover of $A \cup B$. By assumption, there exist two finite subsets Δ_0 and Δ_1 of Δ and $H_0, H_1 \in \mathcal{H}$ such that $\big(A \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\}\big) \subseteq H_0 \in \mathcal{H}$ and $\big(B \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_1\}\big) \subseteq H_1 \in \mathcal{H}$. Since $\big((A \cup B) \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0 \cup \Delta_1\}\big) \subseteq \big((A \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\})\big) \cup \big((B \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_1\})\big) \subseteq H_0 \cup H_1$ and $\mathcal{H}$ is an ideal, we get $(A \cup B) \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0 \cup \Delta_1\} \in \mathcal{H}$. This means $A \cup B$ is weakly $\mu\mathcal{H}$-countably compact compact relative to X. □

Theorem 4.6 *Let (X, μ) be a GTS. If A and B are weakly μ-countably compact relative to X, then $A \cup B$ is weakly μ-countably compact relative to X.*

A GTS (X, μ) is said to be μ-connected if X can not be expressed as the union of two disjoint nonempty μ-open sets. In the opposite case, (X, μ) is said to be μ-disconnected, or equivalently, (X, μ) has a proper nonempty μ-clopen set (see [10]).

Proposition 4.3 *Let $(X, \mu, \mathcal{H})$ be a μ-disconnected HGTS and $\mathcal{H}$ be an ideal on X. The following statements are equivalent.*

1. *$(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact.*
2. *Every μ-clopen subset of X is weakly $\mu\mathcal{H}$-countably compact relative to X.*

Proof: (1) $\Rightarrow$ (2): The proof follows from Theorem 4.1.
(2) $\Rightarrow$ (1): Since $(X, \mu, \mathcal{H})$ is μ-disconnected, X has a partition $\{A, B\}$ such that A and B are μ-clopen sets. Since A and B are weakly $\mu\mathcal{H}$-countably compact relative to X. Thus by Theorem 4.5, $(X, \mu, \mathcal{H})$ is weakly $\mu\mathcal{H}$-countably compact. □

Proposition 4.4 *Let (X, μ) be a μ-disconnected GTS. Then the following statements are equivalent.*

1. *(X, μ) is weakly μ-countably compact.*
2. *Every μ-clopen subset of X is weakly μ-countably compact relative to X.*

Let $(X, \mu, \mathcal{H})$ be a HGTS and $A \subseteq X$, $A \neq \emptyset$. The family $\mathcal{H}_A = \{H \cap A : H \in \mathcal{H}\}$ forms a hereditary class on A and (A, μ_A) is the subspace of X on A and μ_A is a generalized topology on A.

Proposition 4.5 *Let A and B be subsets of a HGTS $(X, \mu, \mathcal{H})$ and $A \subseteq B$. If A is weakly $\mu_B\mathcal{H}_B$-countably compact relative to B, then A is weakly $\mu\mathcal{H}$-countably relative to X.*

Proof: Assume that A is weakly $\mu_B\mathcal{H}_B$-countably compact relative to B and let $\{V_\lambda : \lambda \in \Delta\}$ be any countable cover of A by μ-open sets of X. Then the family $\{V_\lambda \cap B : \lambda \in \Delta\}$ is a countable μ_B-open cover of A. From assumption, there is a finite subset Δ_0 of Δ such that $A \setminus \bigcup\{c_{\mu_B}(V_\lambda \cap B) : \lambda \in \Delta_0\} \in \mathcal{H}_B$. Since $c_{\mu_B}(V_\lambda \cap B) = c_\mu(V_\lambda \cap B) \cap B \subset c_\mu(V_\lambda)$, then $A \backslash \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \subset A \backslash \bigcup\{c_{\mu_B}(V_\lambda \cap B) : \lambda \in \Delta_0\}$. Since $\mathcal{H}_B$ is a hereditary class and $\mathcal{H}_B \subset \mathcal{H}$, we get $A \setminus \bigcup\{c_\mu(V_\lambda) : \lambda \in \Delta_0\} \in \mathcal{H}$, which means that A is weakly $\mu\mathcal{H}$-countably compact relative to X. □

Proposition 4.6 *Let A and B be subsets of a GTS (X, μ) and $A \subseteq B$. If A is weakly μ_B-countably compact relative to B, then A is weakly μ-countably compact relative to X.*

Corollary 4.1 *Let A be a subset of a HGTS $(X, \mu, \mathcal{H})$. If A is weakly $\mu_A\mathcal{H}_A$-countably compact, then A is weakly $\mu\mathcal{H}$-countably compact relative to X.*

Corollary 4.2 *Let A be a subset of a GTS (X, μ). If A is weakly μ_A-countably compact, then A is weakly μ-countably compact relative to X.*

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