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Generalized Nörlund Means of Complex Uncertain Variables

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ABSTRACT: In this article we have introduced the notion of generalized Nörlund mean associated with sequence of complex uncertain variables. The generalized Nörlund mean is defined based on two Nörlund means. We have established some results on the relationship of two regular Nörlund means. The focus is to show that there is a third mean, which is stronger than the each of two Nörlund means considerd.

Key Words: Uncertain variable, complex uncertain sequence, A-limitable, regular method.

Contents

1. Introduction

In modern life, human nature reflects information that is ambiguous or imprecise in nature. B. Liu developed uncertainty theory [6] in 2009 to deal with this type of human uncertainty. Later on Peng [11] introduced complex uncertain variables as a generalization of uncertainty theory and further research on the relevant subject has been initiated by Chen et al. [1], Tripathy and Nath [4], Tripathy and Dowari [15], Tripathy and Baruah [14] and many others. In the field of complex uncertain theory, transformation plays a crucial rule, and Nörlund mean and Riesz mean are two essential types of transformation in this regard. The notion of Nörlund mean and Riesz mean can be found in many book on summability theory, one may refer to the monograph by Peterson [12]. Further in the field of complex uncertain sequence the development of Nörlund mean, Riesz mean and their regularities are induced by Tripathy and Dowari [15] and Saha et al. [13], Datta and Tripathy [3], Dowari andTripathy [5], Nathand Tripathy [9,10],Tripathy and Nath [16] and You [17].

2. Definition and Preliminaries

Definition 2.1 *[7] Let $\mathcal{L}$ be σ- algebra on a non-empty set Γ. A set function $\mathcal{M}$ on Γ is called an uncertain measure if it satisfies the following axioms:*

Axiom 1 (Normality Axiom). $\mathcal{M}\{\Gamma\}=1$;

Axiom 2 (Duality Axiom). $\mathcal{M}\{\Lambda\}+\mathcal{M}\{\Lambda^c\}=1$, *for every* $\Lambda \in \mathcal{L}$;

Axiom 3 (Subadditivity Axiom). *For every countable sequence of* $\{\Lambda_j\} \in \mathcal{L}$, *we have*

$$\mathcal{M}\{\bigcup_{j=1}^{\infty} \Lambda_j\} \leq \sum_{j=1}^{\infty} \mathcal{M}(\Lambda_j).$$

The triplet $(\Gamma, \mathcal{L}, \mathcal{M})$ is called an uncertainty space, and each element Λ in $\mathcal{L}$ is called an event. In order to obtain an uncertain measure of compound events, a product uncertain measure is defined by [7] as follows:

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Axiom 4 (Product Axiom). *Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be an uncertain space for $k = 1, 2, 3, \ldots$. The product uncertain measure $\mathcal{M}$ is an uncertain measure satisfying*

$$\mathcal{M}\{\prod_{j=1}^{\infty} \Lambda_j\} = \bigwedge_{j=1}^{\infty} \mathcal{M}(\Lambda_j),$$

where Λ_k are arbitrarily chosen events from Γ_k, for k = 1, 2 ,3,... respectively.

Definition 2.2 *[7]. An uncertain variable ξ is a measurable function from an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\}$=$\{\gamma \in \Gamma : \xi(\gamma) \in B\}$ is an event.*

Definition 2.3 *[7]. The uncertainty distribution ϕ of an uncertain variable ξ is defined by $\phi(x) = \mathcal{M}\{\xi \leq x\}$, for any real x.*

Definition 2.4 *[8]. The expected value operator of an uncertain variable ξ is defined by $E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq r\}dr - \int_{-\infty}^{0} \mathcal{M}\{\xi \leq r\}dr$, provided that at least one of the two integrals is finite.*

Definition 2.5 *[11]. A complex uncertain variable is a measurable function ξ from an uncertain space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of complex numbers, i.e., for any Borel set B of complex numbers, the set $\{\xi \in B\}$=$\{\gamma \in \Gamma : \xi(\gamma) \in B\}$ is an event.*

Definition 2.6 *[1]. The complex uncertain sequence $\{\xi_n\}$ is said to be convergent almost surely (a.s.) to ξ if there exists an event Λ with $\mathcal{M}\{\Lambda\} = 1$ such that*

$\lim_{n\to\infty} \|\xi_n(\gamma) - \xi(\gamma)\| = 0$, *for every $\gamma \in \Lambda$. In that case it is written as $\xi_n \to \xi$, a.s.*

Definition 2.7 *[1]. The complex uncertain sequence $\{\xi_n\}$ is said to be convergent in measure to ξ if for any $\varepsilon \geq 0$, $\lim_{n\to\infty} \mathcal{M}\{\|\xi_n - \xi)\| \geq \varepsilon\} = 0$.*

Definition 2.8 *[1]. The complex uncertain sequence $\{\xi_n\}$ is said to be convergent in mean to ξ if $\lim_{n\to\infty} E[\|\xi_n - \xi\|] = 0$.*

Definition 2.9 *[1]. Let $\phi_1, \phi_2, \phi_3, \ldots\ldots$ be the complex uncertainty distributions of complex uncertain variables $\xi_1, \xi_2, \xi_3 \ldots..$ respectively. Then the complex uncertain sequence $\{\xi_n\}$ is convergent in distribution to ξ if $\lim_{n\to\infty} \phi_n(c) = \phi(c)$, for all $c \in \mathbb{C}$, at which $\phi(c)$ is continuous.*

Definition 2.10 *[1]. The complex uncertain sequence $\{\xi_n\}$ is said to be convergent uniformly almost surely (u.a.s.) to ξ if there exists a sequence of events $\{E'_k\}$, $\mathcal{M}\{E'_k\} \to 0$, such that $\{\xi_n\}$ converges uniformly to ξ in $\Gamma - E'_k$, for any fixed $k \in \mathbb{N}$.*

Definition 2.11 *[15]. Let $\{p_n\}$ be a sequence of non-negative real numbers, those are not all 0 and in particular $p_1 > 0$. Then the Nörlund mean of the complex uncertain sequence $\{\xi_n\}$ is given by*

$$\tau_n(\gamma) = \frac{\xi_n(\gamma)p_1 + \ldots + \xi_1(\gamma)p_n}{P_n},$$

where $\gamma \in \Gamma$ for the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ and $P_n = p_1 + p_2 + \ldots + p_n$, for n = 1, 2, . . .

Definition 2.12 *[2]. If $A = (a_{mn})$ is a real matrix and $\{\xi_n\}$ be the complex uncertain sequence then the matrix transformation of $\{\xi_n\}$ is defined by another sequence $\{\tau_m\}$ if it exists, where $\tau_m(\gamma) = \sum_{n=1}^{\infty} a_{mn}\xi_n(\gamma)$.*

Definition 2.13 *[12]. For any two limitation methods A and B, B is said to be stronger than A if every A-limitable sequence is also B- limitable to the same limit as A and it has been denoted by $A \subseteq B$.*

Lemma 2.1 *[12]. A necessary and sufficient condition for the existence of the $A-$transformation for all sequences is that $\sum\limits_{n=1}^{\infty} |a_{mn}|$ converges for all $m \in \mathbb{N}$.*

Lemma 2.2 *[12]. The matrix $A = (a_{mn})$ transforming between sequence spaces is called regular if and only if the following conditions holds:*

(a) $\sum\limits_{n=1}^{\infty} |a_{mn}| < B$, *for every m where B is a constant.*

(b) $\forall n$, $\lim\limits_{n\to\infty} a_{mn} = 0$.

(c) $\sum\limits_{n=1}^{\infty} a_{mn} = 1$, *as $m \to \infty$.*

3. Main Result

In this section A-transformation of a complex uncertain variable is defined where it is an important part of finding a regular matrix which transforms a complex uncertain sequence into a mean known as Nörlund mean. It has also been established that for every pair of Nörlund means there exist another third mean stronger than both of the Nörlund means.

Theorem 3.1 *If (N, p_n) and (N, q_n) are two regular Nörlund means of the complex uncertain sequence $\{\xi_n(\gamma)\}$, then there always exist another third mean (N, r_n) which will include both (N, p_n) and (N, q_n).*

Proof: Suppose that $\{\tau_n^1(\gamma)\}$ and $\{\tau_n^2(\gamma)\}$ be respectively the (N, p_n) and (N, q_n) means of the complex uncertain sequence $\{\xi_n(\gamma)\}$ and let $\{\tau_n(\gamma)\}$ be the (N, r_n) Nörlund mean of the sequence $\{\xi_n(\gamma)\}$ generated by the sequence $r_n = p_n q_1 + p_{n-1} q_2 + ... + p_1 q_n$.

Then, we have

$$\begin{aligned}\tau_n(\gamma) &= \tfrac{p_1q_1\xi_n(\gamma)+(p_1q_2+p_2q_1)\xi_{n-1}(\gamma)+...+(p_nq_1+p_{n-1}q_2+...+p_1q_n)\xi_1(\gamma)}{p_1q_1+(p_1q_2+p_2q_1)+...+(p_1q_n+p_2q_{n-2}+...+p_nq_1)}\\ &= \tfrac{p_1(q_1\xi_n(\gamma)+q_2\xi_{n-1}(\gamma)+...+q_n\xi_1(\gamma))+...+p_n(q_1\xi_1(\gamma))}{p_1(q_1+...+q_n)+...+p_nq_1}\\ &= \tfrac{p_1Q_n\tau_n^2(\gamma)+...+p_nQ_1\tau_1^2(\gamma)}{p_1Q_n+...+p_nQ_1},\end{aligned}$$

which is the A-transformation of $\{\tau_n^2(\gamma)\}$, where the matrix $A = (a_{mn})$ is given by

$$\begin{aligned}a_{mn} &= \tfrac{p_{m-n+1}Q_n}{\sum\limits_{k=1}^{m} p_{m-k+1}Q_k}, \text{ for } n \leq m;\\ &= 0, \text{ for } n > m.\end{aligned}$$

Also, we have $\sum a_{mn} = 1$, for $m = 1, 2, ...$

Now, we have

$$\sum_{k=1}^{m} p_{m-k+1}Q_k = p_mQ_1 + ... + p_1Q_m > p_mQ_1 + ... + p_1Q_1 > BP_m, \text{ for } (0 < q_1 \leq B).$$

Moreover, for a fixed value of n, $0 \leq \lim\limits_{m\to\infty} \tfrac{p_{m-n+1}Q_n}{\sum\limits_{k=1}^{m} p_{m-k+1}Q_k} < \lim\limits_{m\to\infty} \tfrac{p_{m-n+1}Q_n}{BP_m} = 0$,

which implies, $A = (a_{mn})$ is a regular matrix and thus it implies that if $\tau_n^2(\gamma)$ converges then $\tau_n(\gamma)$ converges to the same limit.

Hence, $(N, r_n) \supseteq (N, q_n)$.

In a similar way it can be shown that $(N, r_n) \supseteq (N, p_n)$ and hence (N, r_n) contains both (N, p_n) and (N, q_n). $\square$

Theorem 3.2 *Let $\{\tau_n^1(\gamma)\}$ and $\{\tau_n^2(\gamma)\}$ be respectively the (N, p_n) and (N, q_n) means of the complex uncertain sequence $\{\xi_n(\gamma)\}$. Then, $(N, p_n) \subseteq (N, q_n)$ if and only if there exists one B for which $|a_1|P_n + ... + |a_n|P_1 \leq BQ_n$ for every n and also $\lim_{n\to\infty} \frac{a_n}{Q_n} = 0$.*

Proof: We have,

$$\sum_{n=1}^{\infty} Q_n \tau_n^2(\gamma) x^{n-1} = \sum_{n=1}^{\infty} (q_1\xi_n(\gamma) + ... + q_n\xi_1(\gamma))x^{n-1}$$
$$= q(x)\xi(x),$$

where $q(x) = \sum_{n=1}^{\infty} q_n x^{n-1}, \xi(x) = \sum_{n=1}^{\infty} \xi_n(\gamma)x^{n-1}$ and x is sufficiently small.

Similarly, $\sum_{n=1}^{\infty} P_n \tau_n^1(\gamma) x^{n-1} = p(x)\xi(x)$.

Therefore,

$$\frac{\sum_{n=1}^{\infty} Q_n \tau_n^2(\gamma) x^{n-1}}{\sum_{n=1}^{\infty} P_n \tau_n^1(\gamma) x^{n-1}} = \frac{q(x)}{p(x)}$$
$$= a(x)$$
$$= \sum_{n=1}^{\infty} a_n x^{n-1}.$$

This implies, $Q_n\tau_n^2(\gamma) = a_n P_1 \tau_1^1(\gamma) + ... + a_1 P_n \tau_n^1(\gamma)$.

$$\implies \tau_n^2(\gamma) = \frac{a_n P_1 \tau_1^1(\gamma) + ... + a_1 P_n \tau_n^1(\gamma)}{Q_n}.$$

Hence, $\tau_n^2(\gamma)$ can be expressed as $\sum_{n=1}^{\infty} a_{nm}\tau_m^1(\gamma)$,

where

$$a_{nm} = \frac{a_{n-m+1}P_m}{Q_n} \text{ if } m \leq n,$$
$$= 0, \text{ if } m > n.$$

It is very easy to verify that the matrix $A = (a_{nm})$ is regular, since

$$\lim_{n\to\infty} a_{nm} = \lim_{n\to\infty} \frac{a_{n-m+1}P_m}{Q_n}$$
$$= \lim_{n\to\infty} \frac{a_{n-m+1}P_m}{Q_{n-m+1}}$$
$$= P_m \lim_{n\to\infty} \frac{a_{n-m+1}}{Q_{n-m+1}}$$
$$= P_m \lim_{n\to\infty} \frac{a_n}{Q_n}$$
$$= 0, \text{ for every } m.$$

Also, by the given hypothesis we have,

$$\sum_{m=1}^{\infty}|a_{nm}| = \frac{|a_1|P_n+...+|a_n|P_1}{Q_n} \leq B, \text{ and } \sum_{m=1}^{\infty} a_{nm} = 1,$$

which directly implies $a = (a_{nm})$ is regular.

On the contrary part, $\tau_n^2(\gamma) = \sum_{n=1}^{\infty} a_{nm}\tau_m^1(\gamma)$,

where $a_{nm} = \frac{a_{n-m+1}P_m}{Q_n}$ if $m \leq n$
$= 0$ if $m > n$.
Since, $A = (a_{nm})$ is regular, therefore $\lim\limits_{n\to\infty} a_{nm} = 0$.

$$\implies \lim_{n\to\infty} \frac{a_{n-m+1}P_m}{Q_n} = 0.$$

$$\implies P_m \lim_{n\to\infty} \frac{a_{n-m+1}}{Q_{n-m+1}} = 0.$$

$$\implies \lim_{n\to\infty} \frac{a_n}{Q_n} = 0.$$

By the hypothesis, we have

$$\sum_{m=1}^{\infty}|a_{nm}| \leq B$$

$$\implies |a_1|P_n + ... + |a_n|P_1 \leq BQ_n.$$ □

Conclusion. In this article we studied on generalized N$\ddot{o}$rlund mean on the complex uncertain sequences. We have established some inclusion theorems. The concepts established in the article can be applied for further study in the direction of different kind of means and their generalizations. The findings of the paper can also be used in the field of data analysis.

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Availability of data and material (data transparency). Not Applicable.

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