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More on the dual notion of n-absorbing submodules

Fatemeh Ranjbar Hamghavandi and Mahdieh Ebrahimpour*

ABSTRACT: Let n be a positive integer. In this paper, the dual notion of n-absorbing submodules is studied in more detail from a functional point of view. Some related results and useful examples concerning this class of submodules are investigated. Also, we answer Question 2.5 in [6].

Key Words: Prime ideals, prime submodules, n-absorbing ideals, n-absorbing submodules, n-second submodules.

Contents

1. Introduction

Throughout this paper, R is a commutative ring with $1 \neq 0$ (unless explicitly stated that it can be without unity). Duality has an ancient history in mathematics and physics. It is applicable in different fields of mathematics such as algebra and geometry.

In this paper, we first remind the dual of some classical concepts and then we study the dual of some of their extended notions. To this end, we first need to review some definitions.

A proper ideal P of R is said to be prime if $ab \in P$ implies $a \in P$ or $b \in P$, where $a, b \in R$. Some generalizations of prime ideals were studied in [2], [8], [10] and [12]. The primeness concept was extended to modules as follows: a proper submodule P of an R-module M is said to be prime, if $r \in R$ and $x \in M$ with $rx \in P$ implies that $r \in (P :_R M) = \{r \in R \mid rM \subseteq P\}$ or $x \in P$. It is not difficult to verify that if P is a prime submodule of M, then $(P :_R M)$ is a prime ideal of R. In another viewpoint, a proper submodule N of an R-module M is a prime submodule if for each $a \in R$ the homomorphism $a. : \frac{M}{N} \to \frac{M}{N}$, that operates by multiplication, is either injective or zero. This implies that $Ann_R(\frac{M}{N}) = p$ is a prime ideal of R and N is called a p-prime submodule. The concept of second submodule which is a dual notion of prime submodule (in a certain sense) was introduced in [20]. A non-zero submodule N of M is said to be a second submodule if for each $a \in R$ the homomorphism $a. : N \to N$, that operates by multiplication, is either surjective or zero. This implies that $Ann_R(N) = p$ is a prime ideal of R and N is said to be p-second. An R-module M is said to be prime, if zero submodule is a prime submodule of M. Also, M is a second R-module, if M is a second submodule of itself. So the concept of second R-modules is just the dual notion of prime R-modules.

These concepts and some of their generalizations have been deeply studied recently ([9], [11], [15], [17], [19], [16], [18] and [20]). The dual notions also attracted many researchers ([3-7] and [20]).

Let n be a positive integer. A proper submodule P of an R-module M is called n-absorbing, if $a_1 \cdots a_n x \in P$ implies either $a_1 \cdots a_n \in (P :_R M)$ or $a_1 \cdots \widehat{a_i} \cdots a_n x \in P$ for some $1 \leq i \leq n$, where $a_1, \cdots, a_n \in R$ and $x \in M$ (note that $a_1 \cdots \widehat{a_i} \cdots a_n$ means $a_1 \cdots a_{i-1} a_{i+1} \cdots a_n$), see [9]. This concept is a generalization of the concept of n-absorbing ideals, that was introduced in [1]. Note that M is said to be an n-absorbing R-module, if zero is an n-absorbing submodule of M.

* Corresponding author.
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A proper submodule K of an R-module M is said to be completely irreducible, if $K = \cap_{i\in I} N_i$ implies that $K = N_i$ for some $i \in I$, where $\{N_i \mid i \in I\}$ is a family of submodules of M. Let N, L be two submodules of M. To show that $N \subseteq L$, it is enough to prove that if K is a completely irreducible submodule of M with $L \subseteq K$, then $N \subseteq K$, see [3, 2.1]. So a non-zero submodule N of M is a second submodule if and only if for every $a \in R$ and every completely irreducible submodule K of M, $aN \subseteq K$ implies either $N \subseteq K$ or $a \in Ann_R(N)$.

In [6], the authors used this result and proposed two concepts n-absorbing second and strongly n-absorbing second submodules for the dual notion of n-absorbing submodules.

A non-zero submodule N of an R-module M is said to be n-absorbing second (strongly n-absorbing second), if for every completely irreducible submodule (submodule) K of M, $a_1 \cdots a_n N \subseteq K$ implies either $a_1 \cdots a_n \in Ann_R(N)$ or $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq K$ for some $1 \leq i \leq n$, where $a_1, \cdots, a_n \in R$.

Evidently, every strongly n-absorbing second submodule is an n-absorbing second submodule. In [6, Question 2.5], the authors have asked if the converse is also true? By an example, we show that the answer is negative.

In this paper, we first present a functional method, which gives us an equivalent condition to the concept of n-absorbing submodules. By using this method, we can specify the functional dual notion of n-absorbing submodules (in a certain sense), which we call that n-second submodules. A non-zero R-module M is an n-second R-module, if M is an n-second submodule of itself. So the concept of n-second R-modules is just the dual notion of n-absorbing R-modules.

After introducing the concept of n-second submodules, an equivalent condition to this concept is presented. Using this equivalent condition, it can be shown that a submodule N of M is n-second if and only if N is strongly n-absorbing second, that was the second suggestion in [6], see the previous paragraphs. So all the results in [6], on n-absorbing second and strongly n-absorbing second submodules hold for n-second submodules. In this paper, some other results and several useful examples concerning these classes of submodules are presented. Also, n-second submodules of some well-known modules and some functorial results on the classes of all n-absorbing and all n-second R-modules are investigated. The set of all zero-divisors of an R-module M is denoted by $Z_R(M)$. Also, $L \subset N$ means $L \subseteq N$ and $L \neq N$.

2. n-second submodules

At the beginning of this section, we present a functional method, which gives us an equivalent condition to the concept of n-absorbing submodules. For this, the following new concepts are used.

Remark 2.1 *Let R be a commutative ring, M an R-module, N a proper submodule of M, n a positive integer, $a_1, \cdots, a_n \in R$ and $a_i^* : a_1 \cdots \widehat{a_i} \cdots a_n \frac{M}{N} \to \frac{M}{N}$, defined by $a_i^*(a_1 \cdots \widehat{a_i} \cdots a_n(x+N)) = a_1 \cdots a_n x + N$ for every $x \in M$, where $1 \leq i \leq n$. We say that the family $\{a_i^* \mid 1 \leq i \leq n\}$ is injective, if $a_1 \cdots a_n x \in N$ implies $a_1 \cdots \widehat{a_i} \cdots a_n x \in N$ for some $1 \leq i \leq n$, where $x \in M$. Also, we say that this family is zero if each a_i^* is zero.*

It is clear that if one of a_i^*'s is injective, then the family $\{a_i^* \mid 1 \leq i \leq n\}$ is injective. In Proposition 2.1, we show that the converse is true for $n = 2$.

Proposition 2.1 *Let R be a commutative ring, M an R-module, N a proper submodule of M and $a, b \in R$. If the family $\{a^*, b^*\}$ is injective, then either a^* is injective or b^*.*

Proof: Assume that neither a^* is injective nor b^*. So there exists $m \in M$ such that $abm \in N$ and $am \notin N$. Thus $bm \in N$, by assumption. Similarly, there exists an $m' \in M$ with $abm' \in N$ and $bm' \notin N$ and so $am' \in N$. If $m + m' \in N$, then $bm + bm' = b(m + m') \in N$ and so $bm' \in N$, which is a contradiction. Hence $m + m' \notin N$. We have $ab(m + m') \in N$. So $am + am' = a(m + m') \in N$ or $bm + bm' = b(m + m') \in N$, by assumption, which is a contradiction. □

It can be seen that a proper submodule of an R-module M is prime if and only if the homomorphism $a. : \frac{M}{N} \to \frac{M}{N}$, that operates by multiplication, is either injective or zero for every $a \in R$. For n-absorbing submodules, we have the following extended result.

Theorem 2.1 *Let R be a commutative ring, M an R-module and n a positive integer. A proper submodule N of M is an n-absorbing submodule of M if and only if the family $\{a_i^* \mid 1 \leq i \leq n\}$ of R-module homomorphisms is either injective or zero for every $a_1, \cdots, a_n \in R$.*

Proof: Let N be an n-absorbing submodule of M, $a_1, \cdots, a_n \in R$, $x \in M$ and $a_1 \cdots a_n x \in N$. Since N is n-absorbing, we have either $a_1 \cdots a_n \in (N :_R M)$ or $a_1 \cdots \widehat{a_i} \cdots a_n x \in N$ for some $1 \leq i \leq n$. So the family $\{a_i^* \mid 1 \leq i \leq n\}$ is either zero or injective by definition.

Conversely, let N be a proper submodule of M, $a_1, \cdots, a_n \in R, x \in M$ and $a_1 \cdots a_n x \in N$. If the family $\{a_i^* \mid 1 \leq i \leq n\}$ is zero, then $a_1 \cdots a_n \in (N :_R M)$. Otherwise, the family $\{a_i^* \mid 1 \leq i \leq n\}$ is injective. So there exists an $1 \leq i \leq n$ such that $a_1 \cdots \widehat{a_i} \cdots a_n x \in N$. Therefore, N is an n-absorbing submodule of M. □

A non-zero submodule N of an R-module M is said to be second if for every $a \in R$, the homomorphism $a. : N \to N$, that operates by multiplication, is either surjective or zero, see [20]. Next, we expand this concept and present a dual notion of n-absorbing submodules (in a certain sense).

Definition 2.1 *Let R be a commutative ring (not necessarily with unity), M an R-module and n a positive integer. We say that a non-zero submodule N of M is an n-second submodule of M if for every $a_1, a_2, \cdots, a_n \in R$, there exists an $1 \leq i \leq n$ such that the homomorphism $a_i^{**} : N \to a_1 \cdots \widehat{a_i} \cdots a_n N$, defined by $a_i^{**}(x) = a_1 \cdots a_n x$ for every $x \in N$, is either surjective or zero. So 1-second submodules are exactly second submodules.*

Example 2.1 (1) *Let V be a vector space over a field F. It is clear that every non-zero subspace of V is an n-second submodule of V, for every positive integer n.*

(2) *Set n be a positive integer, $p_1, \cdots, p_{n+1}$ are distinct positive prime integers, $R = p_1 \cdots p_n \mathbb{Z}$, $M = \mathbb{Z}_{p_{n+1}}$ as an R-module and $a_1, \cdots, a_n \in R$. If $p_{n+1} \mid a_1 \cdots a_n$, then $a_1 \cdots a_n \in Ann_R(M)$. Otherwise, $a_1 \cdots a_n M = M = a_1 \cdots \widehat{a_i} \cdots a_n$ for every $1 \leq i \leq n$. Thus a_i^{**} is either zero or surjective and so M is an n-second submodule of itself.*

(3) *Let p be a positive prime integer, $M = \mathbb{Z}_{p^\infty}$ as a $p\mathbb{Z}$-module, n, m two positive integers with $m \leq n$, and $N = \langle \frac{1}{p^m} + \mathbb{Z} \rangle$. Let $a_1, \cdots, a_n \in p\mathbb{Z}$. So $a_i = pt_i$ for some $t_i \in \mathbb{Z}$ and every $1 \leq i \leq n$. Hence $a_1 \cdots a_n N = p^n t_1 \cdots t_n N = 0$. Thus a_i^{**} is zero for every $1 \leq i \leq n$ and so N is an n-second submodule of M.*

Let R be a commutative ring, M an R-module and n, m two positive integers with $n \leq m$. It is clear that every n-second submodule of M is an m-second submodule. In the following, it turns out that the converse is not true in general.

Example 2.2 *Set m, n be two positive integers with $m \langle n$, p a positive prime integer and $R = p\mathbb{Z}$.*

(1) *Let $M = \mathbb{Z}_{p^\infty}$ as an R-module and $N = \langle \frac{1}{p^n} + \mathbb{Z} \rangle$. We know that N is an n-second submodule of M by Example 2.1. Let $a_1 = \cdots = a_m = p$. Clearly, $p^m N \neq 0$. If the homomorphism p^{**} is surjective, then $p^m N = p^{m-1} N$ and so there exists a $k \in p\mathbb{Z}$ such that $\frac{1}{p^{n-m+1}} - \frac{p^m k}{p^n} \in \mathbb{Z}$ which is a contradiction. So N is not an m-second submodule of M.*

(2) *Let $M = \mathbb{Z}_{p^n}$ as an R-module. Then for every $a_1, \cdots, a_n \in R$, $a_1 \cdots a_n M = 0$ and so M is an n-second submodule of itself. Let $a_1 = \cdots = a_m = p$. Then $a_1 \cdots a_m M \neq 0$. If a_i^{**} is surjective for some $1 \leq i \leq m$, then $p^m M = p^{m-1} M$ and so $\mathbb{Z}_{p^{n-m}} \cong \mathbb{Z}_{p^{n-m+1}}$ which is a contradiction. Therefore, M is not an m-second submodule of itself.*

Example 2.3 *Let m, n be two positive integers with $m+1 \langle n$, F a field, $S = \langle x_1, \cdots, x_n \rangle$ as an ideal of the multiple polynomial ring $R = F[x_1, \cdots, x_n]$, $T = \langle \{x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} \mid i_1 + \cdots + i_n = n, 0 \leq i_1, \cdots, i_n \leq n\} \rangle$ as an ideal of R, $M = \frac{S}{T}$ as an S-module and $N = M$. It is easy to show that for every $f_1, \cdots, f_n \in S$, $f_1 \cdots f_n \in Ann_S(N)$. So N is an n-second submodule of M. Let $g_i = x_i$ for every $1 \leq i \leq m$. Then $g_1 \cdots g_m \notin Ann_S(N)$. If g_i^{**} is surjective for some $1 \leq i \leq m$, then $x_1 \cdots \widehat{x_i} \cdots x_m N = x_1 \cdots x_m N$. So $x_1 \cdots x_m - x_1 \cdots x_m f \in T$ for some $f \in S$, which is impossible. Therefore, N is not an m-second submodule of M.*

It is known that a proper submodule N of an R-module M is said to be completely irreducible, if $N = \cap_{i\in I} N_i$ implies that $N = N_i$ for some $i \in I$, where $\{N_i \mid i \in I\}$ is a family of submodules of M. For example, every proper $\mathbb{Z}$-submodule of $\mathbb{Z}_{p^\infty}$ is completely irreducible, where p is a positive prime integer. Also, it can be observe that a non-zero $\mathbb{Z}$-submodule of $\mathbb{Z}$ is completely irreducible if and only if it is primary.

Remark 2.2 *Let R be a commutative ring and M a non-zero R-module.*

(1) *It is known that every proper submodule of M is an intersection of completely irreducible submodules of M by [17].*

(2) *Let N, L be two submodules of M. To show $N \subseteq L$, it is enough to prove that if K is a completely irreducible submodule of M with $L \subseteq K$, then $N \subseteq K$ by part* (1).

One can show that a non-zero submodule N of an R-module M is a second submodule if and only if for every $a \in R$ and every completely irreducible submodule K of M, $aN \subseteq K$ implies either $a \in Ann_R(N)$ or $N \subseteq K$ by Remark 2.2(2). In Theorem 2.2, we extend this result to n-second submodules.

Theorem 2.2 *Let R be a commutative ring, M an R-module, N a non-zero submodule of M and n a positive integer. Then N is an n-second submodule of M if and only if for every $a_1, a_2, \cdots, a_n \in R$ and every completely irreducible submodules $K_1, \cdots, K_n$ of M, $a_1 \cdots a_n N \subseteq \cap_{j=1}^n K_j$ implies either $a_1 \cdots a_n \in Ann_R(N)$ or $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq \cap_{j=1}^n K_j$ for some $1 \le i \le n$.*

Proof: Let N is an n-second submodule of M and $K_1, \cdots, K_n$ completely irreducible submodules of M with $a_1 \cdots a_n N \subseteq \cap_{j=1}^n K_j$. Since N is an n-second submodule, there exists an $1 \le i \le n$ such that the homomorphism a_i^{**} is either zero or surjective. If a_i^{**} is zero, then $a_1 \cdots a_n \in Ann_R(N)$. If a_i^{**} is surjective, then $a_1 \cdots \widehat{a_i} \cdots a_n N = a_1 \cdots a_n N \subseteq \cap_{j=1}^n K_j$.

Conversely, assume that N is not an n-second submodule of M. Hence there exist $a_1, \cdots, a_n \in R$ such that a_i^{**} is neither zero nor surjective for every $1 \le i \le n$. So $a_1 \cdots a_n N \ne 0$ and $a_1 \cdots a_n N \ne a_1 \cdots \widehat{a_i} \cdots a_n N$ for every $1 \le i \le n$. Thus there exist completely irreducible submodules $K_1, \cdots, K_n$ of M such that $a_1 \cdots a_n N \subseteq K_i$ and $a_1 \cdots \widehat{a_i} \cdots a_n N \not\subseteq K_i$ for every $1 \le i \le n$ by Remark 2.2(2). Hence $a_1 \cdots a_n N \subseteq \cap_{j=1}^n K_j$ and $a_1 \cdots \widehat{a_i} \cdots a_n N \not\subseteq \cap_{j=1}^n K_j$ for every $1 \le i \le n$, which is a contradiction. □

Theorem 2.3 *Let R be a commutative ring, n a positive integer, M an R-module and N a non-zero submodule of M. Then the following statements are equivalent;*

(1) *N is an n-second submodule of M.*

(2) *For every $a_1, \cdots, a_n \in R$ and every submodule K of M, $a_1 \cdots a_n N \subseteq K$ implies $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq K$ for some $1 \le i \le n$, or $a_1 \cdots a_n \in Ann_R(N)$.*

(3) *For every $a_1, \cdots, a_n \in R$, $a_1 \cdots \widehat{a_i} \cdots a_n N = a_1 \cdots a_n N$ for some $1 \le i \le n$, or $a_1 \cdots a_n \in Ann_R(N)$.*

Proof: (1) ⇒ (2) Let $a_1, \cdots, a_n \in R$ and K a submodule of M with $a_1 \cdots a_n N \subseteq K$. Assume that $a_1 \cdots a_n \notin Ann_R(N)$. So for every completely irreducible submodule L of M with $K \subseteq L$, there exists an $1 \le i \le n$ such that $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq L$ by Theorem 2.2. If there exists an $1 \le i_0 \le n$ such that $a_1 \cdots \widehat{a_{i_0}} \cdots a_n N \subseteq L$ for every completely irreducible submodule L of M with $K \subseteq L$, then $a_1 \cdots \widehat{a_{i_0}} \cdots a_n N \subseteq K$, because every proper submodule of M is an intersection of completely irreducible submodules of M by Remark 2.2(1). So we are done. Otherwise, for every $1 \le i \le n$, there exists a completely irreducible submodule L_i of M with $K \subseteq L_i$ such that $a_1 \cdots \widehat{a_i} \cdots a_n N \not\subseteq L_i$. Since $a_1 \cdots a_n N \subseteq K \subseteq \cap_{i=1}^n L_i$, there exists a $1 \le j \le n$ such that $a_1 \cdots \widehat{a_j} \cdots a_n N \subseteq \cap_{i=1}^n L_i$ by Theorem 2.2. Hence $a_1 \cdots \widehat{a_j} \cdots a_n N \subseteq L_j$ which is a contradiction.

(2) ⇒ (1) It is clear by Theorem 2.2.

(2) ⇒ (3) Let $a_1, \cdots, a_n \in R$ and $K = a_1 \cdots a_n N$. Then $a_1 \cdots a_n N \subseteq K$ implies $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq K$ for some $1 \le i \le n$ and so $a_1 \cdots \widehat{a_i} \cdots a_n N = a_1 \cdots a_n N$, or $a_1 \cdots a_n \in Ann_R(N)$ by hypothesis.

(3) ⇒ (2) Let $a_1, \cdots, a_n \in R$ and K a submodule of M with $a_1 \cdots a_n N \subseteq K$. Then $a_1 \cdots \widehat{a_i} \cdots a_n N = a_1 \cdots a_n N \subseteq K$ for some $1 \le i \le n$, or $a_1 \cdots a_n \in Ann_R(N)$ by hypothesis. □

Corollary 2.1 *Let R be a commutative ring, M an R-module and n a positive integer. Then a submodule N of M is n-second if and only if N is a strongly n-absorbing second submodule of M.*

Proof: It is clear by Theorem 2.3. □

Corollary 2.2 *Let R be a commutative ring, M an R-module and n a positive integer. If N is an n-second submodule of M, then $Ann_R(N)$ is an n-absorbing ideal of R.*

Proof: The result follows from Corollary 2.1, and [6, Theorem 2.12].

□

Let n be a positive integer. If N is an n-second submodule of an R-module M with $p = Ann_R(N)$, then we say that N is a p-n-second submodule of M.

It is clear that every strongly n-absorbing second (n-second) submodule is an n-absorbing second submodule. In [6], Question 2.5, the authors have asked if the converse is true in general? In Example 2.4, we answer this question.

Example 2.4 *Let $n \geq 2$ be a positive integer, $p_1 = 2, p_2, \cdots, p_{n+2}$ distinct positive prime integers, $m = p_1 \cdots p_{n+2}$ and $M = \mathbb{Z}_m$ as a $\mathbb{Z}$-module. It is easy to show that every submodule of M is of the form $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_{n+2}^{\alpha_{n+2}} \mathbb{Z}_m$, where $0 \leq \alpha_j \leq 1$ for every $1 \leq j \leq n+2$. Let $H_k = p_k \mathbb{Z}_m$ for every $1 \leq k \leq n+2$. We can easily show that the only completely irreducible submodules of M are $H_1, \cdots, H_{n+2}$.*

Let $a_1, a_2 \in \mathbb{Z}$, with $a_1 a_2 H_1 \subseteq H_t$ for some $1 \leq t \leq n+2$. So $p_t \mid a_1$ or $p_t \mid a_2$. Hence either $a_1 H_1 \subseteq H_t$ or $a_2 H_1 \subseteq H_t$. Therefore, H_1 is a 2-absorbing second and so an n-absorbing second submodule of M.

It is clear that $p_2 \cdots p_{n+1} H_1 \subseteq \cap_{j=2}^{n+1} H_j$. But there is no $2 \leq i \leq n+1$ such that $p_2 \cdots \widehat{p_i} \cdots p_{n+1} H_1 \subseteq \cap_{j=2}^{n+1} H_j \subseteq H_i$, because $p_i \not| p_1 \cdots \widehat{p_i} \cdots p_{n+1}$. Also, $p_2 \cdots p_{n+1} \not\subseteq Ann_{\mathbb{Z}}(H_1)$. Therefore, H_1 is not an n-second submodule of M. So it is not also a strongly n-absorbing second submodule of M by Corollary 2.1.

Let R be a commutative ring, $n \geq 2$ a positive integer, M an R-module and $a_1, \cdots, a_{n-1} \in R$. Put $\omega_{\{a_1, \cdots, a_{n-1}\}}(N) = \{a \in R \mid a^{**} : N \to a_1 \cdots a_{n-1} N$ is not surjective$\}$. For $n = 1$, put $\omega(N) = \{a \in R \mid a^{**} : N \to N$ is not surjective$\}$.

Lemma 2.1 *Let R be a commutative ring, n a positive integer, M an R-module and N a non-zero submodule of M. Then the following statements are equivalent;*

(1) *N is an n-second submodule of M.*

(2) *$\omega_{\{a_1, \cdots, a_{n-1}\}}(N) = Ann_R(a_1 \cdots a_{n-1} N)$ for every $a_1, \cdots, a_{n-1} \in R$ with $a_1 \cdots a_{n-1} \notin Ann_R(N)$.*

Proof: For $n = 1$, it is clear that N is a second submodule if and only if $\omega(N) = Ann_R(N)$. Now, let $n \geq 2$.

(1) ⇒ (2) Let $a_1, \cdots, a_{n-1} \in R$ with $a_1 \cdots a_{n-1} \notin Ann_R(N)$ and $a \in \omega_{\{a_1, \cdots, a_{n-1}\}}(N)$. So a^{**} is not surjective. Since N is n-second, we have a^{**} is zero and so $a \in Ann_R(a_1 \cdots a_{n-1} N)$.

Let $a \in Ann_R(a_1 \cdots a_{n-1} N)$. Thus a^{**} is zero. If a^{**} is surjective, then $0 = a a_1 \cdots a_{n-1} N = a_1 \cdots a_{n-1} N$ which is a contradiction. Therefore, $a \in \omega_{\{a_1, \cdots, a_{n-1}\}}(N)$.

(2) ⇒ (1) Let $a_1, \cdots, a_n \in R$, $i \in \{1, \cdots, n\}$ and a_i^{**} is not zero. So $a_i \notin Ann_R(a_1 \cdots \widehat{a_i} \cdots a_n N)$. Hence $a_i \notin \omega_{\{a_1, \cdots, \widehat{a_i}, \cdots, a_n\}}(N)$ by hypothesis. So a_i^{**} is surjective. Therefore, N is an n-second submodule of M. □

Remark 2.3 *Let R be a commutative ring (not necessarily with unity) and n a positive integer. We say that an R-module M is n-divisible if for every $x \in M$ and every $a_1, \cdots, a_n \in R$ with $a_1 \cdots a_n \neq 0$, there exists an $1 \leq i \leq n$ and a $y \in M$ such that $a_1 ... a_n y = a_1 \cdots \widehat{a_i} \cdots a_n x$. For example, if R is an integral domain and K the field of fractions of R, then every direct sum of K is an n-divisible R-module. Clearly, if M is a non-zero 1-divisible R-module, then $Ann_R(M) = 0$.*

Let R be a commutative ring and n, m two positive integers with $m \leq n$. It is easy to show that every m-divisible R-module is n-divisible. In Example 2.5, we show that the converse is not true in general.

Example 2.5 *Let R be the polynomial ring $F[x]$, where F is a field, m, n two positive integers with $m \langle n$, $T = \frac{Rx}{Rx^n}$ and $M = T$ as a T-module. Let $a_1, \cdots, a_n \in T$. Then $a_1 \cdots a_n \in Rx^n$. Hence M is an n-divisible T-module by definition.*

Let $a_1 = \cdots = a_m = x + Rx^n$. Then there is no $f + Rx^n \in M$ such that $x^m f + Rx^n = x^{m-1}(x + Rx^n)$. Hence M is not an m-divisible T-module.

Theorem 2.4 *Let R be a commutative ring, n a positive integer, M an R-module and N a non-zero submodule of M with $p = Ann_R(N)$. Then the following statements are equivalent;*

(1) N is a p-n-second submodule of M.

(2) N is an n-divisible $\frac{R}{p}$-module.

Proof: (1) $\Rightarrow$ (2) Let $x \in N$ and $\overline{a_i} = a_i + p \in \frac{R}{p}$, $1 \leq i \leq n$ such that $\overline{a_1} \cdots \overline{a_n} \neq 0_{\frac{R}{p}}$. So $a_1 \cdots a_n \notin p = Ann_R(N)$. Since N is a p-n-second submodule of M, there exists an $1 \leq i \leq n$ such that a_i^{**} is surjective and so there exists a $y \in N$ such that $a_1 \cdots a_n y = a_1 \cdots \widehat{a_i} \cdots a_n x$. Thus $\overline{a_1} \cdots \overline{a_n} y = \overline{a_1} \cdots \widehat{\overline{a_i}} \cdots \overline{a_n} x$ and so N is an n-divisible $\frac{R}{p}$-module.

(2) $\Rightarrow$ (1) Let $a_1, \cdots, a_n \in R$ with $a_1 \cdots a_n \notin Ann_R(N) = p$. So $\overline{a_1} \cdots \overline{a_n} \neq 0_{\frac{R}{p}}$. Since N is an n-divisible $\frac{R}{p}$-module, for every $x \in N$ there exists a $y \in N$ such that $\overline{a_1} \cdots \overline{a_n} y = \overline{a_1} \cdots \widehat{\overline{a_i}} \cdots \overline{a_n} x$ and so $a_1 \cdots a_n y = a_1 \cdots \widehat{a_i} \cdots a_n x$ for some $1 \leq i \leq n$. Hence a_i^{**} is surjective. So N is a p-n-second R-module. □

Theorem 2.5 *Let $n \geq 2$ be a positive integer, R a commutative ring, M an R-module and N an n-second submodule of M. Then the following hold;*

(1) If K is a submodule of M with $N \not\subseteq K$, then $(K :_R N)$ is an n-absorbing ideal of R.

(2) If $Ann_R(N)$ is an $(n-1)$-absorbing ideal of R, then $(K :_R N)$ is an $(n-1)$-absorbing ideal of R for every submodule K of M with $N \not\subseteq K$.

(3) $(\cap_{i=1}^n L_i :_R N)$ is an $(n-1)$-absorbing ideal of R for all completely irreducible submodules $L_1, \cdots, L_n$ of M with $N \not\subseteq \cap_{i=1}^n L_i$ if and only if $Ann_R(N)$ is an $(n-1)$-absorbing ideal of R.

Proof: (1) Since $N \not\subseteq K$, we have $(K :_R N)$ is a proper ideal of R. Let $a_1, \cdots, a_{n+1} \in R$ and $a_1 \cdots a_{n+1} \in (K :_R N)$. Since N is an n-second submodule of M, we have either $a_1 \cdots a_n N = 0$ and so $a_1 \cdots a_n \in (K :_R N)$ or $a_1 \cdots \widehat{a_i} \cdots a_n N = a_1 \cdots a_n N$ and so $a_1 \cdots \widehat{a_i} \cdots a_n a_{n+1} \in (K :_R N)$ for some $1 \leq i \leq n$ by Theorem 2.2.

(2) Since $N \not\subseteq K$, we have $(K :_R N)$ is a proper ideal of R. Let $a_1, \cdots, a_n \in R$ and $a_1 \cdots a_n \in (K :_R N)$. So $a_1 \cdots a_n N \subseteq K$. Since N is an n-second submodule of M, we have either $a_1 \cdots a_n N = 0$ or $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq K$ for some $1 \leq i \leq n$ by Theorem 2.3. If $a_1 \cdots a_n N = 0$, then there exists a $1 \leq j \leq n$ such that $a_1 \cdots \widehat{a_j} \cdots a_n N = 0 \subseteq K$ and so $a_1 \cdots \widehat{a_j} \cdots a_n \in (K :_R N)$, because $Ann_R(N)$ is an $(n-1)$-absorbing ideal of R. Therefore, $(K :_R N)$ is an $(n-1)$-absorbing ideal of R.

(3) Let $a_1, \cdots, a_n \in R$ with $a_1 \cdots a_n \in Ann_R(N)$.

Assume that $a_1 \cdots \widehat{a_i} \cdots a_n \notin Ann_R(N)$ for every $1 \leq i \leq n$. So there exist completely irreducible submodules L_i of M such that $a_1 \cdots \widehat{a_i} \cdots a_n N \not\subseteq L_i$ for every $1 \leq i \leq n$, because every proper submodule of M is an intersection of completely irreducible submodules by Remark 2.2(1). Since $(\cap_{i=1}^n L_i :_R N)$ is an $(n-1)$-absorbing ideal of R and $a_1 \cdots a_n \in (\cap_{i=1}^n L_i :_R N)$, there exists an $1 \leq i \leq n$ such that $a_1 \cdots \widehat{a_i} \cdots a_n N \subseteq \cap_{i=1}^n L_i \subseteq L_i$ which is a contradiction.

Conversely, it is clear by part (2). □

3. n-second submodules of some special modules

In this section, we investigate how n-secondness is transferred between some related modular structures.

Theorem 3.1 *Let R be a commutative ring, n a positive integer, M an R-module and L, N two submodules of M with $L \subset N$. If N is an n-second submodule of M, then $\frac{N}{L}$ is an n-second submodule of the R-module $\frac{M}{L}$.*

Proof: Let $a_1, \cdots, a_n \in R$. Since N is an n-second submodule of M, there exists an $1 \leq i \leq n$ such that $a_i^{**} : N \to a_1 \cdots \widehat{a_i} \cdots a_n N$ is either surjective or zero. Hence $a_i^{**} : \frac{N}{L} \to a_1 \cdots \widehat{a_i} \cdots a_n \frac{N}{L}$ is also either surjective or zero. So $\frac{N}{L}$ is an n-second submodule of $\frac{M}{L}$.

□

Theorem 3.2 *Let R be a commutative ring, S a multiplicatively closed subset of R, n a positive integer and M an R-module. If N is an n-second submodule of M with $N_S \neq 0$, then N_S is an n-second submodule of the R_S-module M_S.*

Proof: Let N is an n-second submodule of M and $\frac{a_1}{s_1}, \cdots, \frac{a_n}{s_n} \in R_S$. Since N is an n-second submodule, there exists an $1 \leq i \leq n$ such that a_i^{**} is either surjective or zero. Thus $\frac{a_i}{s_i}^{**}$ is also either surjective or zero. Therefore, N_S is an n-second submodule of M_S.

□

Theorem 3.3 *Let (R, m) be a commutative local ring, n a positive integer and M an R-module. Then the following statements are equivalent;*
(1) *N is an n-second submodule of M.*
(2) *N_p is an n-second submodule of the R_p-module M_p for every prime ideal p of R with $N_P \neq 0$.*
(3) *N_m is an n-second submodule of the R_m-module M_m.*

Proof:
(1) $\Rightarrow$ (2) It is clear by Theorem 3.2.
(2) $\Rightarrow$ (3) It is clear, because m is a prime ideal of R with $N_m \neq 0$.
(3) $\Rightarrow$ (1) It is clear, because $R \setminus m$ is the set of all unit elements of R. □

Theorem 3.4 *Let m, n be two positive integers, R a commutative ring, M_i an R-module and N_i a submodule of M_i for every $1 \leq i \leq m$, $M := M_1 \times \cdots \times M_m$ and $N := N_1 \times \cdots \times N_m$ an n-second submodule of the R-module M. If $N_i \neq 0$, then N_i is an n-second submodule of M_i, where $1 \leq i \leq m$.*

Proof: It is clear by the definition of n-second submodules. □

Let R_i be a commutative ring, M_i an R_i-module for every $1 \leq i \leq m$, and $R := R_1 \times \cdots \times R_m$. We know that every submodule of the R-module $M = M_1 \times \cdots \times M_m$ is of the form $N = N_1 \times \cdots \times N_m$, where N_i is a submodule of M_i for every $1 \leq i \leq m$. In this case, we call N_i's the components of N.

Theorem 3.5 *Let m, n be two positive integers, R_i a commutative ring, M_i an R_i-module for every $1 \leq i \leq m$, $R = R_1 \times \cdots \times R_m$ and $M = M_1 \times \cdots \times M_m$. If N is an n-second submodule of the R-module M, then every non-zero component N_j of N is an n-second submodule of the R_j-module M_j, where $1 \leq j \leq m$.*

Proof: Let $j \in \{1, \cdots, m\}$ with $N_j \neq 0$ and $a_1, \cdots, a_n \in R_j$.

Then $(0, \cdots, 0, a_k, 0, \cdots, 0) \in R$, where a_k is at the j-th place for every $1 \leq k \leq n$. Since $N = N_1 \times \cdots \times N_m$ is an n-second submodule of M, there exists an $1 \leq i \leq n$ such that $(0, \cdots, 0, a_i, 0, \cdots, 0)^{**}$ is either surjective or zero. So a_i^{**} is also either surjective or zero. Therefore, N_j is an n-second submodule of M_j. □

Lemma 3.1 *Let R be a commutative ring, n a positive integer and M an R-module. If I is an n-second ideal of R with $I \not\subseteq Ann_R(M)$, then IM is an n-second submodule of M.*

Proof: The proof is straightforward. □

Remark 3.1 *Let R be a commutative ring. An R-module M is called a multiplication module if every submodule N of M is of the form IM for some ideal I of R. It is easy to show that $N = (N :_R M)M$.*

Corollary 3.1 *Let R be a commutative ring, n a positive integer and M a faithful multiplication R-module. If every non-zero ideal of R is an n-second ideal, then every non-zero submodule of M is an n-second submodule.*

Proof: It is clear by Lemma 3.1, and Remark 3.1. □

Remark 3.2 *Let R be a commutative ring. An R-module M is said to be a comultiplication R-module (the dual notion of multiplication R-module) if every submodule N of M is of the form $(0 :_M I)$ for some ideal I of R, see [7].*

Lemma 3.2 *Let R be a commutative ring, n a positive integer and M a faithful comultiplication R-module. Then $\omega_{\{a_1,\cdots,a_{n-1}\}}(M) = Z_R(a_1 \cdots a_{n-1}R)$ for every $a_1, \cdots, a_{n-1} \in R$ (For $n = 1$, $\omega(M) = Z_R(R)$).*

Proof: $(\Rightarrow)$ Let $a_1, \cdots, a_{n-1} \in R$ and $a \in \omega_{\{a_1,\cdots,a_{n-1}\}}(M)$. So a^{**} is not surjective. Since M is comultiplication, there exists an ideal I of R such that $aa_1 \cdots a_{n-1}M = (0 :_M I)$. Thus $Iaa_1 \cdots a_{n-1}M = 0$. Since M is faithful, we have $a(Ia_1 \cdots a_{n-1}) = Iaa_1 \cdots a_{n-1} = 0$ and so $a \in Z_R(a_1 \cdots a_{n-1}R)$. Note that if $Ia_1...a_{n-1} = 0$, then $a_1 \cdots a_{n-1}M \subseteq (0 :_M I) = aa_1 \cdots a_{n-1}M$ and so a^{**} is surjective which is a contradiction.

$(\Leftarrow)$ Let $a \in Z_R(a_1 \cdots a_{n-1}R)$. So there exists a $0 \neq c \in R \setminus Ann_R(a_1 \cdots a_{n-1})$ such that $aa_1 \cdots a_{n-1}c = 0$. Hence we have $(aa_1 \cdots a_{n-1}c)M = (cR)(aa_1 \cdots a_{n-1}M) = 0$. This implies that $aa_1 \cdots a_{n-1}M \subseteq (0 :_M cR) \neq a_1 \cdots a_{n-1}M$, because M is faithful and $c \notin Ann_R(a_1 \cdots a_{n-1})$. Hence a^{**} is not surjective and so $a \in \omega_{\{a_1,\cdots,a_{n-1}\}}(M)$. □

Corollary 3.2 *Let R be an integral domain, n a positive integer and M a faithful comultiplication R-module. Then M is an n-second submodule of itself.*

Proof: We can conclude the assertion by Lemma 3.2. □

Let R be a commutative ring. An R-module M is said to be a weak comultiplication module if every second submodule N of M is of the form $(0 :_M I)$ for some ideal I of R, or M does not have any second submodule, see [5].

Remark 3.3 *Let R be a commutative ring, n a positive integer and M an R-module. We denote the set of all n-second submodules of M by n-sec(M) and we say that M is an n-comultiplication module, if n-sec$(M) = \emptyset$ or every $N \in n$-sec(M) is of the form $(0 :_M I)$ for some ideal I of R. So 1-comultiplication modules are just weak comultiplication modules.*

Let n, m be two positive integers with $n \leq m$. It is clear that every n-second submodule of M is an m-second submodule. So every m-comultiplication R-module is n-comultiplication. In Examples 2.2 and 2.3, we showed that the converse is not true in general.

It is clear that every comultiplication module is an n-comultiplication module for every positive integer n. Next, we show that the converse is not true in general.

Example 3.1 *Let n be a positive integer. We first show that $\mathbb{Z}$ as a $\mathbb{Z}$-module, does not have any n-second submodule. Let $N = m\mathbb{Z}$ is a non-zero submodule of $\mathbb{Z}$ and $a_1, \cdots, a_n \in \mathbb{Z} \setminus \{0, 1, -1\}$. Then $a_1 \cdots a_n m\mathbb{Z} \neq 0$. If $a_1 \cdots \widehat{a_i} \cdots a_n m\mathbb{Z} = a_1 \cdots a_n m\mathbb{Z}$ for some $1 \leq i \leq n$, then a_i is a unit which is a contradiction. Hence a_i^{**} is neither zero nor surjective for every $1 \leq i \leq n$. Therefore, $\mathbb{Z}$ is an n-comultiplication module by definition. We have $(0 :_{\mathbb{Z}} m\mathbb{Z}) = 0$ for every positive integer m and $(0 :_{\mathbb{Z}} 0) = \mathbb{Z}$. So for every non-zero proper submodule N of $\mathbb{Z}$, there is not any ideal I of $\mathbb{Z}$ such that $(0 :_{\mathbb{Z}} I) = N$. Therefore, $\mathbb{Z}$ is not a comultiplication $\mathbb{Z}$-module.*

4. Some categorical results on n-absorbing and n-second R-modules

In this section, some functorial results on n-absorbing modules and their dual notion, i.e., n-second modules are presented.

Remark 4.1 *Let R be a commutative ring, n a positive integer and M an R-module. We say that M is an n-absorbing R-module, if zero is an n-absorbing submodule of M. Also, we say that M is an n-second R-module, if M is an n-second submodule of itself. For example, every torsion free R-module is n-absorbing and so every integral domain R as an R-module is n-absorbing. Also, every vector space V over a field F is both n-absorbing and n-second F-module.*

Theorem 4.1 *Let R be a commutative ring and n a positive integer. Then every finitely generated n-second R-module is an n-absorbing R-module.*

Proof: Let M is a finitely generated n-second R-module, $a_1, \cdots, a_n \in R$ and $x \in M$ with $a_1 \cdots a_n x = 0$. Assume that $a_1 \cdots a_n \notin Ann_R(M)$. Since M is an n-second R-module, there exists an $1 \leq i \leq n$ such that $a_1 \cdots \widehat{a_i} \cdots a_n M = a_1 \cdots a_n M$. Let $M' = a_1 \cdots \widehat{a_i} \cdots a_n M$. So M' is a finitely generated R-module and $(a_i)M' = M'$. Hence there exists an $r \in (a_i)$ such that $(1-r)M' = 0$ by [13, 2.2, Proposition 2]. Let $r = sa_i$ for some $s \in R$. Then $(1 - sa_i)a_1 \cdots \widehat{a_i} \cdots a_n x = 0$ and so $a_1 \cdots \widehat{a_i} \cdots a_n x = sa_1 \cdots a_n x = 0$. Therefore, M is an n-absorbing R-module. □

Theorem 4.2 *Let R be a commutative ring, n a positive integer, ς the category of R-modules and T a linear, right exact and covariant functor over ς. If M is an n-second R-module with the following conditions;*

(1) $T(M) \neq 0$,

(2) *There exists an epimorphism $F_r : T(rM) \to rT(M)$ for every $r \in R$,*

(3) $F_{a_1 \cdots \widehat{a_i} \cdots a_n} o T(a_i^{**} : M \to a_1 \cdots \widehat{a_i} \cdots a_n M)$ *is equal to* $a_i^{**} : T(M) \to a_1 \cdots \widehat{a_i} \cdots a_n T(M)$ *for every $a_1, \cdots, a_n \in R$ and every $1 \leq i \leq n$,*

then $T(M)$ is also an n-second R-module.

Proof: For every $a_1, \cdots, a_n \in R$, there exists an $1 \leq i \leq n$ such that $a_i^{**} : M \to a_1 \cdots \widehat{a_i} \cdots a_n M$ is either surjective or zero. So $a_i^{**} : T(M) \to a_1 \cdots \widehat{a_i} \cdots a_n T(M)$ is also either surjective or zero by hypothesis. Therefore, $T(M)$ is n-second. □

Corollary 4.1 *Let R be a commutative ring and n a positive integer. If M is an n-second R-module, then $M \otimes_R N$ is also an n-second R-module for every R-module N with $M \otimes_R N \neq 0$.*

Proof: The result follows from the fact that $- \otimes_R N$ is a right exact covariant functor and the conditions (2) and (3) of Theorem 4.2 are clearly satisfied for it. □

Remark 4.2 *Let R be a commutative ring. Recall that the following conditions are equivalent;*

(1) *P is a projective R-module.*

(2) *If $M \to N \to 0$ is an exact sequence of R-modules, then $Hom_R(P, M) \to Hom_R(P, N) \to 0$ is also an exact sequence of R-modules, see [14].*

Corollary 4.2 *Let R be a commutative ring and n a positive integer. If M is an n-second R-module, then $Hom_R(N, M)$ is also an n-second R-module for every projective R-module N with $Hom_R(N, M) \neq 0$.*

Proof: Clearly, $Hom_R(N, -)$ is a right exact covariant functor and we can easily show that the conditions (2) and (3) of Theorem 4.2 are satisfied for it. Hence the result is verified by Remark 4.2. □

Theorem 4.3 *Let R be a commutative ring, n a positive integer, ς the category of R-modules and T a linear, left exact and contravariant functor over ς. If M is an n-second R-module with the following conditions;*

(1) $T(M) \neq 0$,

(2) *There exists a monomorphism $F_r : rT(M) \to T(rM)$ for every $r \in R$,*

(3) $T(a_i^{**} : M \to a_1 \cdots \widehat{a_i} \cdots a_n M) o F_{a_1 \cdots \widehat{a_i} \cdots a_n}$ *is equal to* $a_i^* : a_1 \cdots \widehat{a_i} \cdots a_n T(M) \to T(M)$ *for every* $a_1, \cdots, a_n \in R$ *and every* $1 \leq i \leq n$,

then $T(M)$ is an n-absorbing R-module.

Proof: For every $a_1, \cdots, a_n \in R$, there exists an $1 \leq i \leq n$ such that $a_i^{**} : M \to a_1 \cdots \widehat{a_i} \cdots a_n M$ is either surjective or zero. Hence $a_i^* : a_1 \cdots \widehat{a_i} \cdots a_n T(M) \to T(M)$ is either injective or zero by hypothesis. So the family $\{a_i^* : a_1 \cdots \widehat{a_i} \cdots a_n \ T(M) \to T(M) \mid 1 \leq i \leq n\}$ is either injective or zero. Therefore, $T(M)$ is n-absorbing. □

Corollary 4.3 *Let R be a commutative ring and n a positive integer. If M is an n-second R-module, then $Hom_R(M, N)$ is an n-absorbing R-module for every R-module N with $Hom_R(M, N) \neq 0$.*

Proof: It is known that $Hom_R(-, N)$ is a left exact contravariant functor and it is easy to show that the conditions (2) and (3) of Theorem 4.3 are satisfied for it. So the result can be concluded. □

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F. Ranjbar Hamghavandi and M. Ebrahimpour,
Department of Mathematics,
Vali-e-Asr University of Rafsanjan,
Iran.
E-mail address: `fatemeh.ranjbar@stu.vru.ac.ir`
E-mail address: `m.ebrahimpour@vru.ac.ir`