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Solutions of $d(n) = d(\varphi(n))$ where n has four different prime divisors

Bouhadjar Slimane* and Bellaouar Djamel

ABSTRACT: For a positive integer n, let $d(n)$, $\varphi(n)$ and $\omega(n)$ denote the number of positive divisors of n, the Euler's phi function of n and the number of different prime divisors of n, respectively. In this paper, we focus on positive integers n such that $d(n) = d(\varphi(n))$ with $\omega(n) = 4$.

Key Words: Diophantine equations, number of positive divisors, Euler's function, number of distinct prime divisors.

Contents

1. Introduction

Let n be a positive integer and let $d(n)$ be the divisor function, which counts the number of positive divisors of n. That is,

$$d(n) = \sum_{d|n} 1.$$

Recall that $d(n) = 2$ if and only if n is prime and that $d(n)$ is prime if and only if $n = p^{q-1}$, where p and q are both prime. The Euler's phi function $\varphi(n)$ counts the number of positive integers up to n that are coprime to n. That is,

$$\varphi(n) = \sum_{\substack{(k,n)=1 \\ k \leq n}} 1.$$

It is well-known that for the natural number $n \geq 2$ with canonical representation $n = q_1^{a_1} q_2^{a_2} \dots q_k^{a_k}$ (where $k, a_1, \dots, a_k$ are positive integers and $q_1, q_2, \dots, q_k$ are different primes), we have

$$\begin{aligned} d(n) &= (a_1 + 1)(a_2 + 1) \dots (a_k + 1), \\ \varphi(n) &= q_1^{a_1 - 1}(q_1 - 1) q_2^{a_2 - 1}(q_2 - 1) \dots q_k^{a_k - 1}(q_k - 1). \end{aligned}$$

* Corresponding author

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Many problems in number theory may be reduced to finding the intersection of two sequences of positive integers (see, e.g [8], [9], [11]). Recently, De Koninck [6, page 78] studied the multiplicative functions $d(n)$ and $\varphi(n)$ for which the equation $\varphi(d(n)) = d(\varphi(n))$ has infinitely many solutions. On the other hand, Sándor [13, pages 110-111] gave all solutions of the following equation

$$d(n) = \varphi(n). \tag{1.1}$$

In fact, Sándor proved that $1, 3, 8, 10, 24$ and 30 are the only solutions of (1.1), while $\varphi(n) > d(n)$ for $n \geq 31$.

The current paper is a continuation of [1], [2], [3], [4] and [5], where the proofs are all on the elementary side and depend on long case by case analysis type arguments. In fact, in [4] the authors studied the comparison between the value of the divisor function to its value at Euler's function. More precisely, based on (1.1), they investigated the solutions n of the equation

$$d(n) = d(\varphi(n)), \tag{1.2}$$

where n has at most three different prime divisors. As a continuation of the subsequent paper [4], we determine all solutions n of the above equation that have four different prime divisors.

Before proceeding, we introduce the notations to state the problem that we study. Let us denote by S the set of positive integer solutions n for the equation (1.2), that is,

$$S = \{n \in \mathbb{N} : d(n) = d(\varphi(n))\}.$$

With the help of a computer, we find that the first elements of S are 1, 3, 14, 15, 22, 28, 44, 46, 50, 56, 68, 70, 78, 88, 92, 94, ..., 3045365504, ... Also, the first few odd elements of S are 1, 3, 15, 255, 65535, 77805, 161595, 331695, 575025, 664335, ... Further, to make the work more interesting we define for any $k \geq 1$ the subset $S_k \subseteq S$ by $S_k = S \cap \mathbb{W}_k$, where $\mathbb{W}_k = \{n \in \mathbb{N} : \omega(n) = k\}$ and $\omega(n)$ denotes the number of distinct prime factors of n. So the purpose of the present paper is to characterize the elements of S_4.

Recall that a positive integer of the form $F_n = 2^{2^n} + 1$ is called a *Fermat number*. The numbers $F_0 = 3$, $F_1 = 5$, $F_2 = 17$, $F_3 = 257$, and $F_4 = 65537$ are the only known Fermat primes. It is well-known that if $p = 2^k + 1$ is a prime then $k = 2^n$ for some $n \geq 0$. Moreover, a prime p is said to be a *Sophie Germain prime* [7] if $2p + 1$ is also a prime, in which case, the later prime is called *safe prime*. It has been conjectured that there are infinitely many Sophie Germain primes, but this remains unproved. For details, see [10]. Note that these special primes are the factors of the elements of S_1 and S_2, as we see in the following theorem:

Theorem 1.1 (see [4]) *The only elements of S_1 are 1 and 3 and the only elements of S_2 are 15, $2^{2^i-2}F_i$ where F_i is a Fermat prime with $i \geq 2$, $2^a q$ where $a \geq 1$ and $q \geq 7$ is a safe prime, and $2^{(2^i-1)j-1}F_i^j$ where F_i is a Fermat prime with $i \geq 1$ and $j \geq 2$.*

As a consequence of Theorem 1.1, the set S_2 is infinite since $2^a \cdot 7 \in S_2$ for every $a \geq 1$. In addition, in [4], the authors characterized the elements of S_3. That is, they found all solutions that have three different prime divisors. Similarly, the authors found the set S_3 infinite and proved a necessary condition for a number n to be in S and some corollaries.

Proposition 1.1 (necessary condition, [4]) *Let $k \geq 2$ and let $q_1 < q_2 < \ldots < q_k$ be primes. Consider the number $n = q_1^{a_1} q_2^{a_2} \ldots q_k^{a_k}$, where $a_1, a_2, \ldots, a_k$ are positive integers with $a_k \geq 2$. If $n \in S_k$, then*

$$a_k \mid \prod_{i=1}^{k-1}(a_i + 1). \tag{1.3}$$

Noting that (1.3) is not a sufficient condition for a number n to be in S_k. In fact, the numbers $n_1 = 2 \cdot 3^2$ and $n_2 = 3 \cdot 5^2$ are the smallest natural numbers that satisfy (1.3) for $k = 2$ with the fact that $d(n_1) > d(\varphi(n_1))$ and $d(n_2) < d(\varphi(n_2))$. Similarly, $n_1 = 2 \cdot 3 \cdot 5^2$ and $n_2 = 2 \cdot 3 \cdot 11^2$ are the smallest natural numbers that satisfy (1.3) for $k = 3$, where $d(n_1) > d(\varphi(n_1))$ and $d(n_2) < d(\varphi(n_2))$. Moreover, the importance of Proposition 1.1 is that for $k = 4$, if a_4 does not divide $(a_1+1)(a_2+1)(a_3+1)$ then $n = q_1^{a_1} q_2^{a_2} q_3^{a_3} q_4^{a_4} \notin S_4$.

Corollary 1.1 (see [4]) *Let k be a positive integer with $k \leq 4$. If $n = q_0 q_1 \dots q_k$, where $q_0, q_1, \dots, q_k$ are primes with $q_0 < \dots < q_k$, then $d(n)/d(\varphi(n)) \neq 1$ except for $(q_0, q_1, \dots, q_k) = (F_0, F_1, \dots, F_k)$.*

Corollary 1.2 (see [4]) *Let $k \geq 2$. Assume that there exist infinitely many Sophie Germain primes (or equivalently infinitely many safe primes), then there are infinitely many even positive integers n such that $n \in S_{k+2}$.*

Corollary 1.3 (see [4]) *Assume that $2 \leq k \leq 7$. The Sophie Germain prime conjecture implies that there exist infinitely many square-free numbers n such that $n \in S_k$.*

Recall that a natural number n is k *powerful* prime power (see [12, page 32]) if exponents of exactly k distinct prime divisors of n are greater than 1. For example, in [4] the infinite families $\{2 \cdot 3^y \cdot 13,\ y \geq 2\}$ and $\{2 \cdot 5^y \cdot 11,\ y \geq 2\}$ are in S_3, where every element has one powerful prime power[1]. However, for the infinite family of solutions $\left\{2^2 \cdot 3^y \cdot 7^3,\ y \geq 2\right\}$ their elements have three powerful prime powers.

The aim of the present paper is to characterize all positive integers of the form $n = q_1^a q_2^b q_3^c q_4^d$ such that $d(n) = d(\varphi(n))$, where q_1, q_2, q_3, q_4 are distinct primes with $2 \leq q_1 < q_2 < q_3 < q_4$ and a, b, c, d are positive integers. Precisely, among these numbers, we find infinitely many solutions having k powerful prime powers $(1 \leq k \leq 4)$. For this purpose, consider the case when n is square-free and we distinguish four cases when n has one powerful prime power, six cases when n has two powerful prime powers, four cases when n has three powerful prime powers and one case when n has four powerful prime powers. We also consider separately the parity of n. Assume first that n is odd, that is, $n = q_1^a q_2^b q_3^c q_4^d \in S_4$, where $3 \leq q_1 < q_2 < q_3 < q_4$ are primes and a, b, c, d are positive integers. Then we put

$$\begin{cases} q_1 - 1 = 2^x m_1 \\ q_2 - 1 = 2^y q_1^{\alpha_1} m_2 \\ q_3 - 1 = 2^z q_1^{\alpha_2} q_2^{\alpha_4} m_3 \\ q_4 - 1 = 2^t q_1^{\alpha_3} q_2^{\alpha_5} q_3^{\alpha_6} m_4 \end{cases} \tag{1.4}$$

where $\alpha_i \geq 0$ and $x, y, z, t, m_i \geq 1$ with $(2q_1q_2q_3q_4, m_1m_2m_3m_4) = 1$. In the case when n is even, i.e., $q_1 = 2$ we put

$$\begin{cases} q_2 - 1 = 2^x m_1 \\ q_3 - 1 = 2^y q_2^{\alpha_1} m_2 \\ q_4 - 1 = 2^z q_2^{\alpha_2} q_3^{\alpha_3} m_3 \end{cases} \tag{1.5}$$

where $\alpha_i \geq 0$ and $x, y, z, m_i \geq 1$ with $(2q_2q_3q_4, m_1m_2m_3) = 1$. In both cases, we put $m = \prod m_i$. Thus in order to prove that n satisfies (1.2), it suffices to confirm that the exponents of n and the above variables satisfy one of the following Diophantine equations:

$$(a+1)(b+1)(c+1)(d+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5)(c+\alpha_6)\, d \cdot d(m), \tag{1.6}$$

and

$$(a+1)(b+1)(c+1)(d+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(c+\alpha_3)\, d \cdot d(m). \tag{1.7}$$

Now, we are in a position to state our main theorem.

Theorem 1.2 (Main Theorem) *Let $n \in \mathbb{W}_4$. Then*

1. *S_4 has no elements of the form*

$$\begin{matrix} q_1q_2q_3^cq_4, & q_1q_2q_3q_4^d, & q_1q_2q_3^cq_4^d, & q_1q_2^bq_3q_4^d, & q_1q_2^bq_3^cq_4, & q_1^aq_2^bq_3q_4^d, \\ q_1^aq_2q_3^cq_4^d, & q_1^aq_2^bq_3^cq_4 & q_1q_2^bq_3^cq_4^d, & 2q_2^bq_3^cq_4^d\ , & q_1^aq_2^bq_3^cq_4^d. & \end{matrix} \tag{1.8}$$

[1] A prime power is a positive integer power of a single prime number.

2. *Only square-free numbers in* S_4 *are* $F_0F_1F_2F_3$, $2\cdot5\cdot11\cdot17$, $2\cdot5\cdot7\cdot17$, $2\cdot5\cdot17\cdot(2p+1)$, $2\cdot3\cdot(2p+1)(2q+1)$ *and* $2\cdot3\cdot(2p+1)\left(2p^2+1\right)$.
3. S_4 *has some elements of the form* $q_1^a q_2 q_3 q_4$ *and* $q_1 q_2^b q_3 q_4$.
4. S_4 *has some elements of the form* $2^a q_2 q_3 q_4$, $2q_2^b q_3 q_4$ *and* $2q_2 q_3^c q_4$. *Moreover, the only solutions of the form* $2q_2q_3q_4^d$ *are* $2\cdot3\cdot5\cdot13^2$, $2\cdot3\cdot5\cdot7^4$ *and* $2\cdot3\cdot5\cdot257^2$.
5. S_4 *has some elements of the form* $q_1^a q_2 q_3 q_4^d$, $q_1^a q_2 q_3^c q_4$, $q_1^a q_2^b q_3 q_4$ *and* $2^a q_2 q_3^c q_4^d$.
6. *Elements of the form* $2q_2^b q_3^c q_4$ *of* n *in* S_4 *are* $2\cdot3^x\cdot5^4\cdot7$, $2\cdot3^4\cdot5^x\cdot11$, $2\cdot3^3\cdot5^2\cdot17$, $2\cdot3^2\cdot5^3\cdot17$ $2\cdot3^{5(\alpha_1+\alpha_2)-6}\left(2\cdot3^{\alpha_1}+1\right)^2\left(2^2.3^{\alpha_2}+1\right)$ *and* $2\cdot3^{(\alpha_1+\alpha_2-1)c-1}\left(2\cdot3^{\alpha_1}+1\right)^c\left(2\cdot3^{\alpha_2}+1\right)$.
7. *Elements of the form* $2q_2^b q_3 q_4^d$ *of* n *in* S_4 *are* $2\cdot3^{(\alpha_1+\alpha_2-1)d-1}\left(2\cdot3^{\alpha_1}+1\right)\left(2\cdot3^{\alpha_2}+1\right)^d$, $n=2\cdot3^3\cdot5\cdot17^2$, $2\left(2^x+1\ \right)^{\frac{5(\alpha_1+\alpha_2)d-4d-4}{4-d}}\left(2^y\left(2^x+1\right)^{\alpha_1}+1\right)\left(2^z\left(2^x+1\right)^{\alpha_2}+1\right)^d$, *where* $x+y+z=4$ *and* $d\in\{2,3,4\}$, $2\cdot3^2\cdot5\cdot17^3$ *and* $2\cdot\left(2^x+1\ \right)^b\left(2^y+1\right)\left(2^z\left(2^x+1\right)+1\right)^2$.
8. *Elements of the form* $2q_2q_3^cq_4^d$ *of* n *in* S_4 *are* $2\cdot3\cdot5^3\cdot17^2$, $2\cdot3\cdot5^2\cdot17^3$, $2\cdot3\cdot5^x\cdot11^4$, $2\cdot3\cdot5^{5x-6}\left(2\cdot5^x+1\right)^2$ *and* $2\cdot3\cdot5^{15x-16}\left(2\cdot5^x+1\right)^3$.
9. S_4 *has infinitely many elements of the form* $2^a q_2^b q_3 q_4$, $2^a q_2 q_3^c q_4$, $2^a q_2 q_3 q_4^d$, $2^a q_2^b q_3^c q_4$, $2^a q_2^b q_3 q_4^d$ *and* $2^a q_2^b q_3^c q_4^d$.

2. Proof of Theorem 1.2

First of all, let us find all numbers n that have four distinct prime factors but $n\notin S_4$.

2.1. Proof of Part 1

Let n be one of the forms stated in (1.8). For the sake of contradiction, assume that $n\in S_4$ and apply (1.4) and (1.6).

1) $n=q_1q_2q_3^cq_4$. We get $8(c+1)=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)(\alpha_4+\alpha_5+1)(c+\alpha_6)\cdot d(m)$. When $\alpha_6\geq1$ or $d(m)\geq2$ there are no solutions. When $\alpha_6=0$ and $d(m)=1$, we have

$$8(c+1)=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)(\alpha_4+\alpha_5+1)\,c.$$

We also see that $\alpha_1+\alpha_2+\alpha_3=0$ or $\alpha_4+\alpha_5=0$. When $\alpha_1+\alpha_2+\alpha_3=0$ and $\alpha_4+\alpha_5\geq1$, we have

$$8(c+1)=(x+y+z+t+1)(\alpha_4+\alpha_5+1)\,c.$$

This is impossible since $y\geq2$ and $z\geq4$. When $\alpha_1+\alpha_2+\alpha_3\geq1$ and $\alpha_4+\alpha_5=0$, we also have

$$c\,(7-(x+y+z+t))=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)-8.$$

Likewise, if $\alpha_1+\alpha_2+\alpha_3=1$ then $x+y+z+t\geq8$ which is impossible. Similarly, if $\alpha_1+\alpha_2+\alpha_3\geq2$, then we get a contradiction.

2) $n=q_1q_2q_3q_4^d$. We can easily show that this case does not provide solutions. In fact, if $d(n)=d(\varphi(n))$ then $8(d+1)=d(A)\,d\geq16d$, where $A=(q_1-1)(q_2-1)(q_3-1)(q_4-1)$. This is impossible. Thus, the equation (1.2) has no positive integer solutions of the desired form.

3) $n=q_1q_2q_3^cq_4^d$. We obtain

$$4(c+1)(d+1)=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)(\alpha_4+\alpha_5+1)(c+\alpha_6)\,d\cdot d(m).$$

Clearly, $d(m)$ cannot be ≥2. Then $m=1$ and hence

$$4(c+1)(d+1)=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)(\alpha_4+\alpha_5+1)(c+\alpha_6)\,d.$$

If $\alpha_1+\alpha_2+\alpha_3\geq1$ or $\alpha_4+\alpha_5\geq1$, then we get a contradiction. Thus $\alpha_i=0$ $(1\leq i\leq5)$. Hence, $4(c+1)(d+1)=(x+y+z+t+1)(c+\alpha_6)\,d$. Note that α_6 cannot be ≥1 since q_i is a Fermat prime for $i=1,2,3$. Thus, q_4 is also a Fermat prime and this is impossible.

4) $n = q_1 q_2^b q_3 q_4^d$ or $n = q_1^a q_2^b q_3 q_4^d$. The proof of these cases is similar to the proof of Case **3)**.

5) Let $n = q_1^a q_2^b q_3 q_4^d$. Applying (1.4) and (1.6) we have

$$2(a+1)(b+1)(d+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5)(\alpha_6+1)\, d \cdot d(m).$$

Note that $d(m)$ cannot be ≥ 2. Then $m = 1$, and so

$$2(a+1)(b+1)(d+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5)(\alpha_6+1)\, d.$$

Also $\alpha_6 = 0$ and hence $2(a+1)(b+1)(d+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5)\, d$. There are two possibilities: $\alpha_1+\alpha_2+\alpha_3 = 0$ and $\alpha_4+\alpha_5 \geq 1$ or $\alpha_1+\alpha_2+\alpha_3 \geq 1$ and $\alpha_4+\alpha_5 = 0$.

i) $\alpha_1+\alpha_2+\alpha_3 = 0$ and $\alpha_4+\alpha_5 \geq 1$. We obtain that $x+y+z+t \geq 5$, a contradiction since $6ab > 2(a+1)(d+1)$.

ii) $\alpha_1+\alpha_2+\alpha_3 \geq 1$ and $\alpha_4+\alpha_5 = 0$. We obtain

$$2(a+1)(b+1)(d+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)\, bd.$$

This is impossible since $a+\alpha_1+\alpha_2+\alpha_3 \geq a+1$ and $(x+y+z+t+1)\, bd > 2(b+1)(d+1)$.

6) $n = q_1^a q_2 q_3^c q_4^d$ or $q_1^a q_2^b q_3^c q_4$ or $q_1 q_2^b q_3^c q_4^d$. The proof of these cases is similar to the proof of **5)**.

7) Assume that $n = 2q_2^b q_3^c q_4^d$. By (1.5) and (1.7), we have

$$2(b+1)(c+1)(d+1) = (x+y+z+1)(b+\alpha_1+\alpha_2)(c+\alpha_3)\, d \cdot d(m).$$

We see that $d(m) = 1$, otherwise the above equation is not true. Therefore,

$$2(b+1)(c+1)(d+1) = (x+y+z+1)(b+\alpha_1+\alpha_2)(c+\alpha_3)\, d. \tag{2.1}$$

Also, the inequalities $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 \geq 1$ cannot hold simultaneously. Then we have three possibilities:

- $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \geq 1$. From (2.1), we can immediately deduce that $2(b+1)(d+1) > 5bd$, which is impossible.
- $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 = 0$. If $\alpha_1 = 0$ or $\alpha_2 = 0$, then by (2.1), $2(c+1)(d+1) > 5cd$. Impossible. If $\alpha_1, \alpha_2 \geq 1$, This gives $4cd < 2(c+1)(d+1)$, which is a contradiction for $(c,d) \neq (2,2)$, $(3,2)$ and $(2,3)$. But in these points the equation $2(b+1)(c+1)(d+1) = (x+y+z+1)(b+\alpha_1+\alpha_2)\, cd$ does not hold.
- $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Similarly, we find that $8bcd < 2(b+1)(c+1)(d+1)$, which is impossible.

8) Let $n = q_1^a q_2^b q_3^c q_4^d$, then by (1.4) and (1.6) we get

$$(a+1)(b+1)(c+1)(d+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5)(c+\alpha_6)\, d \cdot d(m).$$

Notice that $d(m)$ cannot be ≥ 2, otherwise $(a+1)(b+1)(c+1)(d+1) \geq 10abcd$, which is not true. Moreover, if one of the inequalities $\alpha_1+\alpha_2+\alpha_3 \geq 1$, $\alpha_4+\alpha_5 \geq 1$ and $\alpha_6 \geq 1$ holds, then exactly one of the inequalities: $(b+1)(c+1)(d+1) \geq 5bcd$, $(a+1)(b+1)(d+1) \geq 5abd$ and $(a+1)(c+1)(d+1) \geq 5acd$ holds; but this is impossible. Thus, $m = 1$ and $\alpha_i = 0$ for $1 \leq i \leq 6$. Hence, q_i $(1 \leq i \leq 4)$ is a Fermat prime from which it follows that $(a+1)(b+1)(c+1)(d+1) \geq 16abcd$, which is impossible as well. The proof of Part 1 is finished.

2.2. Proof of Part 2

By [4, Corollary 2.2.1], we make here some minor revisions when $n \in S_4$ is square-free. In fact, by Corollary 1.1, if $n \in S_4$ is odd square-free, then $n = F_0 F_1 F_2 F_3$. Moreover, if $n = 2q_2 q_3 q_4 \in S_4$, then $16 = d((q_2-1)(q_3-1)(q_4-1))$. We will now consider separately three cases $(q_2-1)(q_3-1)(q_4-1) = 2^{15}$, $(q-1)(r-1)(s-1) = 2^7 \cdot m$ with m is odd prime and $(q_2-1)(q_3-1)(q_4-1) = 2^3 \cdot m$ with m is odd positive integer such that $d(m) = 4$. In the case when $(q_2-1)(q_3-1)(q_4-1) = 2^{15}$. It follows that

$2^{k_1}+2^{k_2}+2^{k_3}=15$ for some $k_3>k_2>k_1\geq 0$, this is a contradiction. When $(q_2-1)(q_3-1)(q_4-1)=2^7\cdot m$, where m is prime. We must therefore have $n=2\cdot 5\cdot 7\cdot 17$ or $n=2\cdot 5\cdot 11\cdot 17$ or $n=2\cdot 5\cdot 17\cdot(2p+1)$, where p is a prime such that $2p+1$ is also prime. When $(q_2-1)(q_3-1)(q_4-1)=2^3\cdot m$, where m is an odd positive integer with $d(m)=4$. We draw a similar conclusion to deduce that $n=2\cdot 3\cdot(2p+1)(2q+1)$, where p,q are prime numbers such that $2p+1$ and $2q+1$ are also prime or $n=2\cdot 3\cdot(2p+1)\left(2p^2+1\right)$, where p is a prime such that $2p+1$ and $2p^2+1$ are also prime.

This completes the proof of Part 2.

2.3. Proof of Part 3

First, assume that $n=q_1^a q_2 q_3 q_4$. Define $A=(q_1-1)(q_2-1)(q_3-1)(q_4-1)$. Since q_1 is prime, we distinguish two cases:

Case 1. Assume that $(A,q_1)=1$. In fact, we see that $8(a+1)=d(A)\,a\geq 16a$, which is impossible.

Case 2. Assume now that $(A,q_1)=q_1$. From (1.4) and (1.6), we can derive the equation

$$8(a+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)\,d(m).$$

We observe that $m=1$, and so $8(a+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)$. Or, equivalently, we see that $a(7-(x+y+z+t))=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3)-8$. Thus,t $4\leq x+y+z+t\leq 6$. Then the solutions are given by

$$n=(2^x+1)^{\frac{(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3)-8}{7-(x+y+z+t)}}\left(2^y\cdot(2^x+1)^{\alpha_1}+1\right)\left(2^z\cdot(2^x+1)^{\alpha_2}+1\right)\left(2^t\cdot(2^x+1)^{\alpha_3}+1\right).$$

In particular, if $x+y+z+t=6$ then $x\in\{1,2\}$ and $y,z,t\in\{1,2,3\}$. Also, if $x+y+z+t=5$, then $x,y,z,t\in\{1,2\}$. While, if $x+y+z+t=4$ then $(x,y,z,t)=(1,1,1,1)$. Here, 3 must be divides $5(\alpha_1+\alpha_2+\alpha_3)-8$ since $a=\frac{5(\alpha_1+\alpha_2+\alpha_3)-8}{3}$. Hence,

$$n=3^{\frac{5(\alpha_1+\alpha_2+\alpha_3)-8}{3}}\left(2\cdot 3^{\alpha_1}+1\right)\left(2\cdot 3^{\alpha_2}+1\right)\left(2\cdot 3^{\alpha_3}+1\right). \tag{2.2}$$

For example, for $\alpha_1=1$, $\alpha_2=2$ and $\alpha_3=4$ we get $n=3^9\cdot 7\cdot 19\cdot 163$.

Second, assume that $n=q_1q_2^bq_3q_4\in S_4$. We have by (1.4) and (1.6) that

$$8(b+1)=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)(b+\alpha_4+\alpha_5)(\alpha_6+1)\cdot d(m).$$

Clearly, $d(m)$ cannot be ≥ 2. From this, it easily follows that

$$8(b+1)=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3+1)(b+\alpha_4+\alpha_5)(\alpha_6+1).$$

Following are the only cases to study.

i) $\alpha_1+\alpha_2+\alpha_3=0$, $\alpha_4+\alpha_5=0$ and $\alpha_6\geq 1$. Here we obtain $8(b+1)=(x+y+z+t+1)(\alpha_6+1)\,b$, and this is impossible.

ii) $\alpha_1+\alpha_2+\alpha_3=0$, $\alpha_4+\alpha_5\geq 1$ and $\alpha_6=0$. We get $8(b+1)=(x+y+z+t+1)(b+\alpha_4+\alpha_5)$, and so $b(7-(x+y+z+t))=(x+y+z+t+1)(\alpha_4+\alpha_5)-8$. If $\alpha_4+\alpha_5=1$, this is impossible since at least three of the primes q_1,q_2,q_3,q_4 are Fermat numbers. If $\alpha_4+\alpha_5\geq 2$, then $4\leq x+y+z+t\leq 6$. For $x+y+z+t=4$ there are no solutions. For $x+y+z+t=5$, we have

$$n=3\cdot 5^{\frac{5(\alpha_4+\alpha_5)-8}{3}}\left(2\cdot 5^{\alpha_4}+1\right)\left(2\cdot 5^{\alpha_5}+1\right), \tag{2.3}$$

where $2\cdot 5^{\alpha_4}+1$ and $2\cdot 5^{\alpha_4}+1$ are primes with $\alpha_4<\alpha_5$ and 3 divides $5(\alpha_4+\alpha_5)-8$. For $x+y+z+t=6$ we also obtain $n=(2^x+1)(2^y+1)^{7(\alpha_4+\alpha_5)-8}\left(2^z(2^y+1)^{\alpha_4}+1\right)\left(2^t(2^y+1)^{\alpha_5}+1\right)$. For example, for $x=z=\alpha_4=1$, $y=t=2$ and $\alpha_5=24$ we have $n=3\cdot 5^{39}\cdot 11\cdot 119\,209\,289\,550\,781\,251$.

iii) $\alpha_1+\alpha_2+\alpha_3\geq 1$, $\alpha_4+\alpha_5=0$ and $\alpha_6=0$. We can write the given expression as follows:

$$b(7-(x+y+z+t))=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3)-8.$$

When $\alpha_1+\alpha_2+\alpha_3=1$, there are no solution since three of the primes q_1,q_2,q_3,q_4 are Fermat numbers. But, when $\alpha_1+\alpha_2+\alpha_3\geq 2$, we see that $4\leq x+y+z+t\leq 6$ and therefore,

$$n=(2^x+1)\left(2^y(2^x+1)^{\alpha_1}+1\right)^{\frac{(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3)-8}{7-(x+y+z+t)}}\left(2^z(2^x+1)^{\alpha_2}+1\right)\left(2^t(2^x+1)^{\alpha_3}+1\right).$$

2.4. Proof of Part 4

We will prove that our equation has solutions of the form $2^a q_2q_3q_4$, $2q_2^bq_3q_4$, $2q_2q_3^cq_4$ and $2q_2q_3q_4^d$.

A. Assume that $n=2^aq_2q_3q_4\in S_4$. By (1.5) and (1.7), we have

$$8(a+1)=(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)\cdot d(m).$$

- When $d(m)=1$. From the above equality, we get

$$n=2^{\frac{(\alpha_1+\alpha_2+1)(\alpha_3+1)(x+y+z)-8}{8-(\alpha_1+\alpha_2+1)(\alpha_3+1)}}(2^x+1)\left(2^y(2^x+1)^{\alpha_1}+1\right)\left(2^z(2^x+1)^{\alpha_2}\left(\left(2^y(2^x+1)^{\alpha_1}+1\right)\right)^{\alpha_3}+1\right).$$

  In particular, if $(x,y,z)=(1,1,2)$ and $(\alpha_1,\alpha_2,\alpha_3)=(1,1,0)$ we get $n=2^2\cdot3\cdot7\cdot29$.

- When $d(m)=2$. It follows that $4(a+1)=(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)$. We have only the following possibilities: $\alpha_1+\alpha_2=0$ or $\alpha_3=0$. For example, if $\alpha_1+\alpha_2=0$, then $4(a+1)=(a+x+y+z)(\alpha_3+1)$. For $x=y=z=\alpha_3=1$ and $a=2$ we have $2^2\cdot3\cdot5\cdot71$.

- $d(m)=3$. Here $\alpha_1+\alpha_2=0$ or $\alpha_3=0$. In the first case we get

$$n=2^{\frac{3(\alpha_3+1)(x+y+z)-8}{8-3(\alpha_3+1)}}(2^x+1)(2^y+1)\left(2^z\cdot p^2+1\right).$$

  In particular, for $\alpha_3=0$ we have $n=2^{\frac{3(x+y+z)-8}{5}}(2^x+1)(2^y+1)\left(2^z\cdot p^2+1\right)$.

- If $7\ge d(m)\ge4$, then $\alpha_1=\alpha_2=\alpha_3=0$ and so $a=\frac{d(m)(x+y+z)-8}{8-d(m)}$. For $d(m)=4$, the solutions are:

$$n=2^{x+y+z-2}(2^x+1)(2^y+1)\left(2^z\cdot p^3+1\right).\tag{2.4}$$

B. Let $n=2q_2^bq_3q_4\in S_4$. By (1.5), (1.7), $8(b+1)=(x+y+z+1)(b+\alpha_1+\alpha_2)(\alpha_3+1)\cdot d(m)$. We now consider the following two cases.

Case 1. If $\alpha_1+\alpha_2\ge1$, then $m=1$ so $8(b+1)=(x+y+z+1)(b+\alpha_1+\alpha_2)(\alpha_3+1)$. Thus, we have $\alpha_3=0$. Then $b(7-(x+y+z))=(x+y+z+1)(\alpha_1+\alpha_2)-8$, and so $3\le x+y+z\le7$. Thus the solutions are given by

$$n=2(2^x+1)^{\frac{(x+y+z+1)(\alpha_1+\alpha_2)-8}{7-(x+y+z)}}\left(2^y(2^x+1)^{\alpha_1}+1\right)\left(2^z(2^x+1)^{\alpha_2}+1\right),\tag{2.5}$$

where $7-(x+y+z)$ divides $(x+y+z+1)(\alpha_1+\alpha_2)-8$.

Case 2. If $\alpha_1=\alpha_2=0$. Then $8(b+1)=(x+y+z+1)b(\alpha_3+1)\cdot d(m)$. Here we must have $\alpha_3=0$ and $d(m)\ge1$ or $\alpha_3\ge1$ and $d(m)=1$.

- $\alpha_3=0$ and $d(m)\ge1$. Therefore, $b((x+y+z+1)\cdot d(m)-8)=8$. In this case the primes q_2,q_3 and q_4 are Fermat numbers and hence the last equation is not true.
- $\alpha_3\ge1$ and $d(m)=1$. That is, $8(b+1)=(x+y+z+1)b(\alpha_3+1)$. Then $\alpha_3=1$ and so $b(x+y+z-3)=4$. Here we have two facts:

i) For $b=2$ we have $x+y+z=5$. There are no solutions.

ii) For $b=4$ we have $x+y+z=4$, which gives rise the solution $n=2\cdot3^4\cdot5\cdot11$.

C. Let $n=2q_2q_3^cq_4\in S_4$. As above, we have $8(c+1)=(x+y+z+1)(\alpha_1+\alpha_2+1)(c+\alpha_3)\cdot d(m)$. If we assume that $\alpha_3\ge1$, then $d(m)=1$ and so $8(c+1)=(x+y+z+1)(\alpha_1+\alpha_2+1)(c+\alpha_3)$. When $\alpha_3=1$. That is, $8=(x+y+z+1)(\alpha_1+\alpha_2+1)$. The above equation has sense whenever $\alpha_1+\alpha_2=1$ and $x=y=z=1$, this is impossible. When $\alpha_3\ge2$, that is, $\alpha_1=\alpha_2=0$. We see that

$$c(7-(x+y+z))=(x+y+z+1)\alpha_3-8,$$

where $x+y+z\in\{4,5,6\}$. For $x=1$ and $y=z=2$ we have

$$n=2\cdot 3\cdot 5^{3\alpha_3-4}\cdot\left(2^2\cdot 5^{\alpha_3}+1\right). \tag{2.6}$$

For example, $\alpha_3=2$ we get $n=2\cdot 3\cdot 5^2\cdot\left(2^2\cdot 5^2+1\right)$. The proof is finished.

D. The case $n=2q_2q_3q_4^d$. Here, we will prove that there are only a finite number of solutions of this form. By applying (1.5), (1.7), $8(d+1)=(x+y+z+1)(\alpha_1+\alpha_2+1)(\alpha_3+1)\,d\cdot d(m)$. Note that $m=1$, so $8(d+1)=(x+y+z+1)(\alpha_1+\alpha_2+1)(\alpha_3+1)\,d$. The case $\alpha_1+\alpha_2\geq 1$ and $\alpha_3\geq 1$ is impossible. It remains the following possibilities:

Case 1. $\alpha_1+\alpha_2\geq 1$ and $\alpha_3=0$. This implies that $8(d+1)=(x+y+z+1)(\alpha_1+\alpha_2+1)\,d$, and so $d((x+y+z+1)(\alpha_1+\alpha_2+1)-8)=8$. Clearly, the last equation is not true whenever $\alpha_1+\alpha_2\geq 2$. If $\alpha_1+\alpha_2=1$, then $(x+y+z-3)\,d=4$. That is, d is either 2 or 4.

• For $d=2$ we have $x+y+z=5$. This implies $n=2\cdot 3\cdot 5\cdot 13^2$, which is the only solution for this case.

• For $d=4$ we have $x+y+z=4$. The only solution here is $n=2\cdot 3\cdot 5\cdot 7^4$.

Case 2. $\alpha_1=\alpha_2=0$ and $\alpha_3\geq 1$. This implies that $8(d+1)=(x+y+z+1)(\alpha_3+1)\,d$. If $\alpha_3\geq 2$, then we must have $\alpha_3=2$ and so the above equation has no solutions in positive integers.

Case 3. $\alpha_1=\alpha_2=\alpha_3=0$. This implies that $(x+y+z-7)\,d=8$. Since q_i $(1\leq i\leq 3)$ is a Fermat prime, we deduce that the above equation has the only solution $(x,y,z,d)=(1,2,8,2)$ and so $n=2\cdot 3\cdot 5\cdot 257^2$.

2.5. Proof of Part 5

Here, we will prove that our equation has solutions of the form $q_1^aq_2q_3q_4^d$, $q_1^aq_2q_3^cq_4$, $q_1^aq_2^bq_3q_4$ and $2^aq_2q_3^cq_4^d$.

A) Let $n=q_1^aq_2q_3q_4^d\in S_4$. By (1.4) and (1.6), we obtain

$$4(a+1)(d+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(\alpha_4+\alpha_5+1)(\alpha_6+1)\,d\cdot d(m).$$

We have $\alpha_4=\alpha_5=\alpha_6=0$ and $d(m)=1$. Implying that

$$4(a+1)(d+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)\,d.$$

Note that $\alpha_1+\alpha_2+\alpha_3$ cannot be 0 or 1 since $x+y+z+t\geq 8$ and so $9(a+1)\,d>4(a+1)(d+1)$ or $16ad>4(a+1)(d+1)$. If $\alpha_1+\alpha_2+\alpha_3\geq 2$, then d cannot be ≥ 4 and hence d is either 2 or 3. For the case $d=2$, we have $a(5-(x+y+z+t))=(x+y+z+t+1)(\alpha_1+\alpha_2+\alpha_3)-6$. This is impossible except for $(x,y,z,t)=(1,1,1,1)$. That is, $a=5(\alpha_1+\alpha_2+\alpha_3)-6$ from which it follows that

$$n=3^{5(\alpha_1+\alpha_2+\alpha_3)-6}\left(2\cdot 3^{\alpha_1}+1\right)\left(2\cdot 3^{\alpha_2}+1\right)\left(2\cdot 3^{\alpha_3}+1\right)^3. \tag{2.7}$$

For the case $d=3$, it is clear that $a=15(\alpha_1+\alpha_2+\alpha_3)-16$ and so

$$n=3^{15(\alpha_1+\alpha_2+\alpha_3)-16}\left(2\cdot 3^{\alpha_1}+1\right)\left(2\cdot 3^{\alpha_2}+1\right)\left(2\cdot 3^{\alpha_3}+1\right)^3.$$

For example, for $\alpha_1=1$, $\alpha_2=2$ and $\alpha_3=4$ we get $n=3^{89}\cdot 7\cdot 19\cdot 163^3$.

B) Let $n=q_1^aq_2q_3^cq_4\in S_4$. By (1.4) and (1.6), we obtain

$$4(a+1)(c+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(\alpha_4+\alpha_5+1)(c+\alpha_6)\cdot d(m).$$

Clearly, $\alpha_4=\alpha_5=0$ and $d(m)=1$. Therefore,

$$4(a+1)(c+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(c+\alpha_6).$$

There are only two possibilities: $\alpha_1+\alpha_2+\alpha_3\geq 1$ and $\alpha_6=0$ or $\alpha_1+\alpha_2+\alpha_3=0$ and $\alpha_6\geq 1$. When $\alpha_1+\alpha_2+\alpha_3\geq 1$ and $\alpha_6=0$. That is, $4(a+1)(c+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)\,c$. If $\alpha_1+\alpha_2+\alpha_3=1$, we obtain $c(x+y+z+t-3)=4$, and this is impossible since $x+y+z+t\geq 8$. If $\alpha_1+\alpha_2+\alpha_3\geq 2$, then c is either 2 or 3. When $c=2$, we obtain

$$6(a+1)=(x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3),$$

which gives $(x, y, z, t) = (1, 1, 1, 1)$ and so $a = 5(\alpha_1 + \alpha_2 + \alpha_3) - 6$ in which case

$$n = 3^{5(\alpha_1+\alpha_2+\alpha_3)-6} (2 \cdot 3^{\alpha_1} + 1)(2 \cdot 3^{\alpha_2} + 1)^2 (2 \cdot 3^{\alpha_3} + 1). \tag{2.8}$$

When $c = 3$, by the same argument we get $n = 3^{15(\alpha_1+\alpha_2+\alpha_3)-16} (2 \cdot 3^{\alpha_1} + 1)(2 \cdot 3^{\alpha_2} + 1)^3 (2 \cdot 3^{\alpha_3} + 1)$.

C) Let $n = q_1^a q_2^b q_3 q_4 \in S_4$. By (1.4) and (1.6), we obtain

$$4(a+1)(b+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5)(\alpha_6+1) \cdot d(m).$$

We remark that $\alpha_6 = 0$ and $d(m) = 1$. Hence,

$$4(a+1)(b+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)(b+\alpha_4+\alpha_5).$$

Also if $\alpha_1+\alpha_2+\alpha_3 \geq 1$ and $\alpha_4+\alpha_5 \geq 1$, then the above equation has no sense. Similarly, $\alpha_1+\alpha_2+\alpha_3 = 0$ and $\alpha_4 + \alpha_5 = 0$ give us a contradiction since $x + y + z + t \geq 15$. Then only two cases are possible:

Case 1. $\alpha_1 + \alpha_2 + \alpha_3 = 0$ and $\alpha_4 + \alpha_5 \geq 1$. This case implies that

$$4(a+1)(b+1) = (x+y+z+t+1)(b+\alpha_4+\alpha_5)a.$$

- If $\alpha_4 + \alpha_5 = 1$, then $a(x+y+z+t-3) = 4$. This is impossible since $x+y+z+t \geq 8$.
- If $\alpha_4+\alpha_5 \geq 2$, then a is either 2 or 3. For $a = 2$, we have $6(b+1) = (x+y+z+t+1)(b+\alpha_4+\alpha_5)$ and so $(x, y, z, t) = (1, 1, 1, 1)$, there by contradicting the fact that $q_1 < q_2$. For $a = 3$, we get the same contradiction as the case $a = 2$.

Case 2. $\alpha_1 + \alpha_2 + \alpha_3 \geq 1$ and $\alpha_4 + \alpha_5 = 0$. We consider the following two possibilities:

- If $\alpha_1 + \alpha_2 + \alpha_3 = 1$. This is impossible since there are three Fermat primes.
- If $\alpha_1 + \alpha_2 + \alpha_3 \geq 2$, then $4(a+1)(b+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)b$, and so b is either 2 or 3. For $b = 2$, we have $6(a+1) = (x+y+z+t+1)(a+\alpha_1+\alpha_2+\alpha_3)$, and therefore $(x, y, z, t) = (1, 1, 1, 1)$ in which case we get $n = 3^{5(\alpha_1+\alpha_2+\alpha_3)-6} (2 \cdot 3^{\alpha_1} + 1)^2 (2 \cdot 3^{\alpha_2} + 1)(2 \cdot 3^{\alpha_3} + 1)$. For $b = 3$, we may repeat the previous argument to obtain:

$$n = 3^{5(\alpha_1+\alpha_2+\alpha_3)-16} (2 \cdot 3^{\alpha_1} + 1)^3 (2 \cdot 3^{\alpha_2} + 1)(2 \cdot 3^{\alpha_3} + 1). \tag{2.9}$$

D) Assume that $n = 2^a q_2 q_3^c q_4^d \in S_4$. By (1.5) and (1.7), we have

$$2(a+1)(c+1)(d+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)d \cdot d(m). \tag{2.10}$$

Note that $d(m)$ cannot be ≥ 5. First, assume that $d(m) = 4$. In this case, the only possibilities are $(c, d) = (2, 2,), (3, 2)$ or $(2, 3)$. If $(c, d) = (2, 2)$, then $\alpha_1 = \alpha_2 = \alpha_3 = 0$, in which case the solution is given by $n = 2^{8(x+y+z)} (2^x m_1 + 1)(2^y m_2 + 1)^2 (2^z m_3 + 1)^2$, where $d(m_1 m_2 m_3) = 4$. For example, for $(x, y, z, m_1, m_2, m_3) = (1, 2, 3, 1, 7, 11)$ we have $n = n = 2^{39} \cdot 3 \cdot 5^2 \cdot 617^2$.

When $d(m) = 3$. We get $2(a+1)(c+1)(d+1) = 3(a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)d$. For example, when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ we obtain $2(a+1)(c+1)(d+1) = 3(a+x+y+z)cd$. If $(c, d) = (2, 2)$, then the solution is given by $n = 2^{3(x+y+z)-4} (2^x m_1 + 1)(2^y m_2 + 1)^3 (2^z m_3 + 1)^2$, where $d(m_1 m_2 m_3) = 3$. In particular, for $(x, y, z, m_1, m_2, m_3) = (1, 2, 2, 1, 1, 7^2)$ we have $n = 2^{11} \cdot 3 \cdot 5^3 \cdot 197^2$. When $d(m) = 2$. We have $\alpha_1 = \alpha_2 = 0$ or $\alpha_3 = 0$. In the case when $\alpha_1 = \alpha_2 = \alpha_3 = 0$ we obtain $(a+1)(c+1)(d+1) = (a+x+y+z)cd$. For example, for $(a, c, d, x, y, z) = (5, 2, 3, 1, 2, 4)$ we obtain $n = 2^5 \cdot 3 \cdot 17 \cdot 29^3$. When $d(m) = 1$. The solution is given by

$$n = 2^a (2^x + 1)(2^y q_1^{\alpha_1} + 1)^c (2^z q_1^{\alpha_2} q_2^{\alpha_3} + 1)^d. \tag{2.11}$$

For example, if $(x, y, z) = (1, 1, 4)$, $(\alpha_1, \alpha_2, \alpha_3) = (0, 1, 0)$ and $(a, c, d) = (3, 2, 2)$, then $n = 2^3 \cdot 3 \cdot 7^2 \cdot 17^2$.

2.6. Proof of Part 6

Let $n = 2q_2^b q_3^c q_4 \in S_4$. By (1.5) and (1.7), we can derive the following equation

$$4\left(b+1\right)\left(c+1\right) = \left(x+y+z+1\right)\left(b+\alpha_1+\alpha_2\right)\left(c+\alpha_3\right)\cdot d\left(m\right). \tag{2.12}$$

Note that (2.12) is not valid whenever $d\left(m\right) \geq 3$. Then we distinguish two cases:

Case 1. $d\left(m\right) = 2$. By (2.12), $2\left(b+1\right)\left(c+1\right) = \left(x+y+z+1\right)\left(b+\alpha_1+\alpha_2\right)\left(c+\alpha_3\right)$. Also the above equation is not valid for $\alpha_1+\alpha_2 \geq 1$ or $\alpha_3 \geq 1$, that is, $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore,

$$2\left(b+1\right)\left(c+1\right) = \left(x+y+z+1\right)bc. \tag{2.13}$$

Since there are only two Fermat primes; let, for example, q_2 and q_3, from (2.13) we deduce that

$$2\left(b+1\right)\left(c+1\right) \geq \left(4+z\right)cd \geq 5bc.$$

But this is impossible.

Case 2. $d\left(m\right) = 1$. The equation (2.12) can be rewritten as

$$4\left(b+1\right)\left(c+1\right) = \left(x+y+z+1\right)\left(b+\alpha_1+\alpha_2\right)\left(c+\alpha_3\right). \tag{2.14}$$

First, we notice that if $\alpha_1+\alpha_2 \geq 2$ and $\alpha_3 \geq 1$ or $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 \geq 2$, then the above equation has no sense. Thus, it remains the following possibilities:

a) $\alpha_1+\alpha_2 = 1$ and $\alpha_3 = 1$. By (2.14), we obtain $x = y = z = 1$ from which we get $q_2 = 3, q_3 = 7$ and $q_4 = 15$ or $q_2 = q_3 = 3$ and $q_4 = 7$. This is a contradiction with the fact that q_4 is prime and $q_2 < q_3$.

b) $\alpha_1+\alpha_2 = 0$ and $\alpha_3 \geq 0$. The equation (2.14) is equivalent to

$$4\left(b+1\right)\left(c+1\right) = \left(x+y+z+1\right)b\left(c+\alpha_3\right).$$

If $b \geq 5$, the above equation has no sense. Then $b \in \{2,3,4\}$.

b.1) $b = 4$ we obtain $20\left(c+1\right) = 4\left(x+y+z+1\right)\left(c+\alpha_3\right)$ and we deduce that

$$c\left(x+y+z-4\right) = 5-\alpha_3\left(x+y+z+1\right).$$

If $\alpha_3 = 0$ we obtain $c\left(x+y+z-4\right) = 5$, this implies that $c = 5$ and $x+y+z = 5$, impossible because q_2, q_3, q_4 are all Fermat primes. If $\alpha_3 = 1$, when $x+y+z = 4$; exactly $x = 1, y = 2$ and $z = 1$, we obtain $q_2 = 3, q_3 = 5$ and $q_4 = 11$. Then $n = 2\cdot 3^4\cdot 5^x\cdot 11$. However, when $x+y+z > 4$ or $x+y+z \leq 3$, there are no solutions. Similarly, if $\alpha_3 > 1$we have no solutions.

b.2) $b = 3$, we have $c\left(3\left(x+y+z+1\right)-16\right) = 16-3\alpha_3\left(x+y+z+1\right)$. If $\alpha_3 = 0$ we obtain $c\left(3\left(x+y+z+1\right)-16\right) = 16$. Then $\left(c, 3\left(x+y+z\right)-13\right)$ is either $(2,8)$, $(4,4)$, $(8,2)$ or $(16,1)$. For $c = 2$ and $3\left(x+y+z\right)-13 = 8$ we get $x+y+z = 7$; exactly $x = 1$, $y = 2$ and $z = 4$. Then $n = 2\cdot 3^3\cdot 5^2\cdot 17$. For $c = 4$ and $3\left(x+y+z\right)-13 = 4$ we also get $x+y+z = 17/3$ and this is impossible. For $c = 8$ and $3\left(x+y+z\right)-13 = 2$ implies that $x+y+z = 5$, this also is impossible. For $c = 16$ and $3\left(x+y+z\right)-13 = 1$ we have $x+y+z = 14/3$. A contradiction. If $\alpha_3 = 1$, we obtain

$$c\left(3\left(x+y+z\right)-13\right) = 16-3\left(x+y+z+1\right),$$

implying that $c = -1$, a contradiction with $c \geq 2$. If $\alpha_3 \geq 2$, then $c\left(3\left(x+y+z\right)-13\right) = 16-3\alpha_3\left(x+y+z+1\right)$. This is impossible. Therefore,

$$c\left(3\left(x+y+z\right)-13\right) = 16-3\alpha_3\left(x+y+z+1\right) = -8.$$

from which we obtain $x = y = z = 1$ and $q_2 = q_3 = 3$. This is a contradiction.

b.3) $b = 2$, we have $6\left(c+1\right) = \left(x+y+z+1\right)\left(c+\alpha_3\right)$, *i.e.,*

$$c\left(x+y+z-5\right) = 6-\alpha_3\left(x+y+z+1\right).$$

If $\alpha_3 = 0$, then we obtain $c(x+y+z-5) = 6$, gives $(c, x+y+z-5) = (2,3)$. This is impossible. Otherwise, $(c, x+y+z-5) = (3,2)$ implying that $x = 1, y = 2, z = 4$ and so $n = 2 \cdot 3^2 \cdot 5^3 \cdot 17$. Also if $(c, x+y+z-5) = (6,1)$, and this is impossible

c) $\alpha_3 = 0$ and $\alpha_1 + \alpha_2 \geq 1$. We have $4(b+1)(c+1) = (x+y+z+1)(\alpha_1+\alpha_2+b)c$. If $\alpha_1+\alpha_2 = 1$ and $\alpha_3 = 0$, we obtain $c(x+y+z-3) = 4$. Then $c = 2$ and $x+y+z = 5$ and this is impossible. Or $c = 4$ and $x+y+z = 4$, from which we obtain $(x,y,z) = (1,2,1)$. Hence, $q_2 = 3, q_3 = 5$ and $q_4 = 7$. Other cases are impossible. Then $n = 2 \cdot 3^x \cdot 5^4 \cdot 7$. If $\alpha_1 + \alpha_2 \geq 2$ and $\alpha_3 = 0$, we obtain

$$4(c+1) \geq (x+y+z+1)c.$$

If $x+y+z \geq 6$, then our equation has no solutions for all $c \geq 2$. Otherwise, we have the following observations:

i) $x+y+z = 5$, we obtain $4(b+1)(c+1) = 6(\alpha_1+\alpha_2+b)c$. For $c = 2$, implies that $12(b+1) = 12(\alpha_1+\alpha_2+b)$ this is impossible because $\alpha_1+\alpha_2 \geq 2$. For $c \geq 3$, we obtain $4(c+1) \geq 6c$, a contradiction.

ii) $x+y+z = 4$, we have $4(b+1)(c+1) = 5(\alpha_1+\alpha_2+b)c$. For $c = 2$, we have $12(b+1) = 10(\alpha_1+\alpha_2+b)$ implies $b = 5(\alpha_1+\alpha_2) - 6$. Thus, $(x,y,z) = (1,1,2)$ and so

$$n = 2 \cdot 3^{5(\alpha_1+\alpha_2)-6}\left(2 \cdot 3^{\alpha_1}+1\right)^2\left(2^2 \cdot 3^{\alpha_2}+1\right).$$

iii) $x+y+z = 3$, *i.e.*, $x = y = z = 1$ we have $4(b+1)(c+1) = 4(\alpha_1+\alpha_2+b)c$. Then $b = (\alpha_1+\alpha_2-1)c - 1$ and hence $n = 2 \cdot 3^{(\alpha_1+\alpha_2-1)c-1}\left(2 \cdot 3^{\alpha_1}+1\right)^c\left(2 \cdot 3^{\alpha_2}+1\right)$, where $2 \cdot 3^{\alpha_1}+1$ and $2 \cdot 3^{\alpha_2}+1$ are primes with $3^{\alpha_1} < 3^{\alpha_2}$. So there are infinitely many solutions of the desired form.

2.7. Proof of Part 7

Let $n = 2q_2^b q_3 q_4^d \in S_4$. As above, we have

$$4(b+1)(d+1) = (x+y+z+1)(b+\alpha_1+\alpha_2)(\alpha_3+1)d \cdot d(m). \tag{2.15}$$

We note that (2.15) has no solutions whenever $d(m) \geq 3$. Then we have two cases:

Case 1. $d(m) = 2$. It follows from (2.15) that

$$2(b+1)(d+1) = (x+y+z+1)(b+\alpha_1+\alpha_2)(\alpha_3+1)d.$$

Also the above equation is not valid for $\alpha_1+\alpha_2 \geq 1$ or $\alpha_3 \geq 1$, in which case $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore,

$$2(b+1)(d+1) = (x+y+z+1)bd. \tag{2.16}$$

Note that there are only two Fermat primes; let, for example, q_2 and q_3. The equation (2.16) becomes

$$2(b+1)(d+1) \geq (4+z)bd \geq 5bd.$$

But this is impossible.

Case 2. $d(m) = 1$. The equation (2.15) can be rewritten as

$$4(b+1)(d+1) = (x+y+z+1)(b+\alpha_1+\alpha_2)(\alpha_3+1)d.$$

First, we notice that if $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 \geq 1$, then the above equation has no solutions. Thus, it remains the following possibilities:

a) $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Here, q_2, q_3 and q_4 are Fermat primes. By (2.15), we obtain

$$4(b+1)(d+1) = \left(2^i + 2^j + 2^k + 1\right)bd,$$

where $0 \leq i < j < k \leq 2$; otherwise $4(b+1)(d+1) \geq \left(2^0+2^1+2^3+1\right)bd = 12bd$, which is impossible. That is $i = 0$, $j = 1$ and $k = 2$, from which it follows that $b = (d+1)/(d-1)$. This last equation has only two pairs of solutions $(b,d) = (3,2)$ or $(2,3)$. Observe that $q_2 = 3$, $q_3 = 5$ and $q_4 = 17$, in which case we get $n = 2 \cdot 3^3 \cdot 5 \cdot 17^2$ or $n = 2 \cdot 3^2 \cdot 5 \cdot 17^3$.

b) $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \geq 1$. Here, q_2 and q_3 are Fermat primes. From (2.15), we have

$$2\,(b+1)\,(d+1) \geq \left(2^i + 2^j + z + 1\right) bd,$$

where $0 \leq i < j$, and so $2\,(b+1)\,(d+1) \geq (4+z)\,bd \geq 5bd$. This is impossible.

c) $\alpha_1 + \alpha_2 \geq 1$ and $\alpha_3 = 0$. The equation (2.15) follows

$$4\,(b+1)\,(d+1) = (x+y+z+1)\,(\alpha_1+\alpha_2+b)\,d. \tag{2.17}$$

Here we see that $x+y+z$ cannot be ≥ 6; otherwise $4\,(d+1) \geq 7d$, which is impossible. Then consider three subcases:

c.1) Consider the case when $x+y+z=5$. By (2.17), we obtain $2b+2d+2 = 3\,(\alpha_1+\alpha_2)\,d + bd$. Note that d cannot be ≥ 3, i.e., $d=2$ and hence $\alpha_1 = 0$ and $\alpha_2 = 1$, in which case

$$n = 2\,(2^x+1\,)^b\,(2^y+1)\,(2^z\,(2^x+1)+1)^2,$$

where 2^x+1, 2^y+1 and $2^z\,(2^x+1)+1$ are primes with $x<y$. For example, for $x=1$ and $y=z=2$ we have $n = 2\cdot 3^b \cdot 5 \cdot 13^2$.

c.2) Consider the case when $x+y+z=4$. By (2.17), $4b+4d+4 = 5\,(\alpha_1+\alpha_2)\,d + bd$. Note that d cannot be ≥ 5, i.e., $d \in \{2,3,4\}$. Therefore,

$$n = 2\,(2^x+1\,)^{\frac{5(\alpha_1+\alpha_2)d-4d-4}{4-d}}\,(2^y\,(2^x+1)^{\alpha_1}+1)\,(2^z\,(2^x+1)^{\alpha_2}+1)^d.$$

For $d=2$, if $x=y=\alpha_1=1$, $z=2$ and $\alpha_2=3$ we have $n = 2\cdot 3^4\cdot 7\cdot 109^2$.

c.3) Consider the case when $x=y=z=1$. By (2.17), $b = (\alpha_1+\alpha_2-1)\,d-1$ and hence

$$n = 2\cdot 3^{(\alpha_1+\alpha_2-1)d-1}\,(2\cdot 3^{\alpha_1}+1)\,(2\cdot 3^{\alpha_2}+1)^d,$$

where $2\cdot 3^{\alpha_1}+1$ and $2\cdot 3^{\alpha_2}+1$ are distinct primes with $\alpha_1 < \alpha_2$. Here, we obtain infinitely many solutions.

2.8. Proof of Part 8

Assume that $n = 2q_2q_3^cq_4^d \in S_4$. We can obtain the following equation:

$$4\,(c+1)\,(d+1) = (x+y+z+1)\,(\alpha_1+\alpha_2+1)\,(c+\alpha_3)\,d\cdot d\,(m). \tag{2.18}$$

Note that (2.18) is not valid whenever $d\,(m) \geq 3$. Then we distinguish two cases:

Case 1. $d\,(m) = 2$. It follows from (2.18) that

$$2\,(c+1)\,(d+1) = (x+y+z+1)\,(\alpha_1+\alpha_2+1)\,(c+\alpha_3)\,d.$$

Also the above equation is not valid for $\alpha_1+\alpha_2 \geq 1$ or $\alpha_3 \geq 1$, in which case $\alpha_1=\alpha_2=\alpha_3=0$. The above equality implies

$$2\,(c+1)\,(d+1) = (x+y+z+1)\,cd, \tag{2.19}$$

where there are only two Fermat primes; let, for example, q_2 and q_3. The equation (2.19) becomes $2\,(c+1)\,(d+1) \geq (4+z)\,cd \geq 5cd$. But this is impossible.

Case 2. $d\,(m) = 1$. The equation (2.18) can be rewritten as

$$4\,(c+1)\,(d+1) = (x+y+z+1)\,(\alpha_1+\alpha_2+1)\,(c+\alpha_3)\,d. \tag{2.20}$$

First, we notice that if $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 \geq 1$, then the above equation has no sense. Thus, it remains the following observations:

a) $\alpha_1=\alpha_2=\alpha_3=0$. Here, q_2, q_3 and q_4 are Fermat primes. By (2.20), we obtain $4\,(c+1)\,(d+1) = \left(2^i+2^j+2^k+1\right) cd$, where $0 \leq i < j < k \leq 2$; otherwise $4\,(c+1)\,(d+1) \geq \left(2^0+2^1+2^3+1\right) cd = 12cd$, which is impossible. That is, $i=0$, $j=1$ and $k=2$, from which it follows that $c+d+1 = cd$, and hence

$c = (d+1)/(d-1)$. This last equation has only two pairs of solutions $(c, d) = (3, 2)$ or $(2, 3)$. Observe that $q_2 = 3$, $q_3 = 5$ and $q_4 = 17$, in which case we get $n = 2 \cdot 3 \cdot 5^3 \cdot 17^2$ or $n = 2 \cdot 3 \cdot 5^2 \cdot 17^3$.

b) $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 \geq 1$. Here, q_2 and q_3 are Fermat primes. From (2.20), we have $4(c+1)(d+1) = \left(2^i + 2^j + z + 1\right)(c + \alpha_3) d$, where $0 \leq i < j \leq 1$. Otherwise, $4(d+1) \geq \left(2^0 + 2^2 + z + 1\right) d \geq 7d$, and this is a contradiction. Hence,

$$4(c+1)(d+1) = (4+z)(c+\alpha_3) d. \tag{2.21}$$

We consider four possibilities:

• $z, \alpha_3 \geq 2$. We see that $4(c+1)(d+1) < (4+z)(\alpha_3 + c) d$, since $4(d+1) \leq (4+z) d$ and $c + 1 < c + \alpha_3$.

• $z \geq 2$ and $\alpha_3 = 1$. We have $4(d+1) = (4+z) d$, and therefore the only pair is $(d, z) = (2, 2)$. Hence $q_4 = 21$, contrary to our earlier observation.

• $z = 1$ and $\alpha_3 \geq 2$. From (2.21) we obtain that $4(c+1)(d+1) = 5(\alpha_3 + c) d$. This equation has no solution whenever $d \geq 4$. It remains to settle the possibilities $d = 2$ and $d = 3$. For $d = 2$, we have $c = 5\alpha_3 - 6$ and so $n = 2 \cdot 3 \cdot 5^{5\alpha_3 - 6} \cdot \left(2 \cdot 5^{\alpha_3} + 1\right)^2$. For $d = 3$, we also have $c = 15\alpha_3 - 16$ and so $n = 2 \cdot 3 \cdot 5^{15\alpha_3 - 16} \cdot \left(2 \cdot 5^{\alpha_3} + 1\right)^3$. Take for example $\alpha_3 = 3, 13$.

• $t = 1$ and $\alpha_3 = 1$. From (2.21) it is clear that $d = 4$ and consequently $n = 2 \cdot 3 \cdot 5^c \cdot 11^4$, where $c \geq 2$. This gives the infinite family of solutions above.

c) $\alpha_1 + \alpha_2 \geq 1$ and $\alpha_3 = 0$. The equation (2.20) is equivalent to

$$4(c+1)(d+1) = (x+y+z+1)(\alpha_1 + \alpha_2 + 1) cd. \tag{2.22}$$

Observe that if $\alpha_1 + \alpha_2 \geq 2$, then this equation has no solutions. This gives $\alpha_1 + \alpha_2 = 1$, and therefore also (2.22) becomes $2(c+1)(d+1) = (x+y+z+1) cd$. Here one can distinguish two possibilities:

Consider the case when $\alpha_1 = 0$ and $\alpha_2 = 1$. It follows that $2(c+1)(d+1) = \left(2^i + 2^j + z + 1\right) cd$, where $i = 0$ and $j = 1$. Thus,

$$c = \frac{2(1+d)}{(2+z) d - 2}. \tag{2.23}$$

This equation fails for $z \geq 3$, since $(2+z) d - 2 > 2(1+d)$. For $z = 2$, by (2.23) we get $c = 2$ and $d = 1$. This is impossible. For $z = 1$, by (2.23) we also get $c = 1$ and $d = 2$. This is impossible as well.

Consider the case when $\alpha_1 = 1$ and $\alpha_2 = 0$. Then $2(c+1)(d+1) = \left(2^i + z + 2^j + 1\right) cd$, where $i = 0$ and $j = 1$. Thus, $q_4 = 5$ which is impossible since q_4 is greater than q_3.

The proof Part 8 is finished.

2.9. Proof of Part 9

In this subsection, we will consider the forms stated in Part 9 and we prove that there are infinitely many solutions for each case.

A) Let $n = 2^a q_2^b q_3 q_4 \in S_4$. We first observe by (1.5) and (1.7) that

$$4(a+1)(b+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(\alpha_3+1) \cdot d(m). \tag{2.24}$$

We note that (2.24) has no solutions whenever $d(m) \geq 6$ since $(b+\alpha_1+\alpha_2)(\alpha_3+1) d(m) \geq 6b \geq 4(b+1)$ and $(a+x+y+z) > a+1$. Then we have five possibilities:

1. $d(m) = 5$. It follows from (2.24) that $4(a+1)(b+1) = 5(a+x+y+z)(b+\alpha_1+\alpha_2)(\alpha_3+1)$. The last equation is not valid whenever $b \geq 4$. It remains to settle the cases $b = 2$ and $b = 3$. When $b = 2$, we have $12(a+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+2)(\alpha_3+1)$. Here we must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$; otherwise, $12(a+1) \geq 5(a+3)(\alpha_1+\alpha_2+2)(\alpha_3+1) \geq 15(a+3)$. A contradiction. It follows that $a = 5(x+y+z) - 6$ and so $n = 2^{5(x+y+z)-6}\left(2^x m_1 + 1\right)^2\left(2^y m_2 + 1\right)\left(2^z m_3 + 1\right)$, where $m = m_1 m_2 m_3 = p^4$ for some odd prime p. For example, for $(x, y, z) = (1, 2, 4)$ and $p = 5$ we get $n = 2^{29} \cdot 3^2 \cdot 101 \cdot 401$.

When $b = 3$, we also have $16(a+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+3)(\alpha_3+1)$. Similarly, we must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$; otherwise, $16(a+1) \geq 5(a+3)(\alpha_1+\alpha_2+3)(\alpha_3+1) \geq 20(a+3)$. This is impossible. Therefore, $a = 15(x+y+z) - 16$ and hence

$$n = 2^{15(x+y+z)-16} \cdot \left(2^x m_1 + 1\right)^3\left(2^y m_2 + 1\right)\left(2^z m_3 + 1\right),$$

where $m = p^4$ for some odd prime p. For example, for $(x, y, z) = (1, 2, 4)$ and $p = 5$ we get $n = 2^{89} \cdot 3^3 \cdot 101 \cdot 401$.

2. $d(m) = 4$. It follows from (2.24) that $(a+1)(b+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(\alpha_3+1)$. We must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, $(a+1)(b+1) = (a+x+y+z)\,b$, and so $a = b(x+y+z-1)-1$. Hence, $n = 2^{b(x+y+z-1)-1}(2^x m_1+1)^b (2^y m_2+1)(2^z m_3+1)$, where m is either p^3 or pq for some odd primes p, q. For example, for $(x, y, z) = (1, 1, 2)$ and $m = 5 \cdot 7$ we get $n = 2^{3b-1} \cdot 3^b \cdot 11 \cdot 29$, $b \geq 2$.

3. $d(m) = 3$. It follows from (2.24) that $4(a+1)(b+1) = 3(a+x+y+z)(b+\alpha_1+\alpha_2)(\alpha_3+1)$. We must have $\alpha_3 = 0$. That is, $4(a+1)(b+1) = 3(a+x+y+z)(b+\alpha_1+\alpha_2)$. Some of the solutions are of the form $n = 2^a \cdot (2^x+1)^b \left(2^y(2^x+1)^{\alpha_1}+1\right)\left(2^z(2^x+1)^{\alpha_2}p^2+1\right)$. For example, for $a = 29$, $b = 3$, $x = y = z = \alpha_1 = \alpha_2 = 1$ and $m = 5^2$ we get $n = 2^{29} \cdot 3^3 \cdot 7 \cdot 151$.

4. $d(m) = 2$. It follows from (2.24) that $2(a+1)(b+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(\alpha_3+1)$. Here, $\alpha_3 = 0$ or $\alpha_1 = \alpha_2 = 0$. When $\alpha_3 = 0$, we get $2(a+1)(b+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)$. Some of the solutions are of the form $n = 2^a(2^x+1)^b\left(2^y(2^x+1)^{\alpha_1}+1\right)\left(2^z(2^x+1)^{\alpha_2}p+1\right)$. For example, for $a = 3$, $b = 2$, $x = y = z = \alpha_1 = \alpha_2 = 1$ and $m = 5$ we get $n = 2^3 \cdot 3^2 \cdot 7 \cdot 31$. When $\alpha_1 = \alpha_2 = 0$ we have $2(a+1)(b+1) = (a+x+y+z)(\alpha_3+1)\,b$. Some of the solutions are of the form $n = 2^a(2^x+1)^b(2^y+1)\left(2^z(2^y+1)^{\alpha_3}p+1\right)$. For example, for $x = z = \alpha_3 = 1$, $y = 2$ and $m = 7$ we get $n = 2^{3b-1} \cdot 3^b \cdot 5 \cdot 71$ $(b \geq 2)$.

5. $d(m) = 1$. It follows from (2.24) that $4(a+1)(b+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(\alpha_3+1)$. Thus, the solutions are of the form

$$n = 2^a \cdot (2^x+1)^b\left(2^y(2^x+1)^{\alpha_1}+1\right)\left(2^z(2^x+1)^{\alpha_2}\left(2^y(2^x+1)^{\alpha_1}+1\right)^{\alpha_3}+1\right).$$

For example, for $a = 3$, $b = 2$, $x = y = z = \alpha_1 = \alpha_2 = \alpha_3 = 1$ we obtain $n = 2^3 \cdot 3^2 \cdot 7 \cdot 43$.

B) Let $n = 2^a q_2 q_3^c q_4 \in S_4$. From (1.5) and (1.7), we obtain

$$4(a+1)(c+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3) \cdot d(m), \tag{2.25}$$

We note that (2.25) has no solutions whenever $d(m) \geq 6$ since $(\alpha_1+\alpha_2+1)(c+\alpha_3)\,d(m) \geq 6c \geq 4(c+1)$ and $(a+y+z+t) > a+1$. Then we have five cases:

1. $d(m) = 5$. It follows from (2.25) that $4(a+1)(c+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)$. The last equation is not valid whenever $c \geq 4$. It remains to settle the cases $c = 2$ and $c = 3$. When $c = 2$, the above equality implies $12(a+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+2)$. Here we must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$; otherwise, $12(a+1) \geq 5(a+3)(\alpha_1+\alpha_2+1)(\alpha_3+2) \geq 15(a+3)$. A contradiction. It follows that $a = 5(x+y+z)-6$ and so $n = 2^{5(x+y+z)-6}(2^x m_1+1)(2^y m_2+1)^2(2^z m_3+1)$, where $m = p^4$ for some odd prime p. For example, for $x = 1$, $y = 2$, $z = 4$ and $p = 5$ we get $n = 2^{29} \cdot 3 \cdot 101^2 \cdot 401$. When $c = 3$, we have $4(a+1)(c+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+3)$. Similarly, we must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$; otherwise, $16(a+1) \geq 5(a+3)(\alpha_1+\alpha_2+1)(\alpha_3+3) \geq 20(a+3)$. This is impossible. Therefore, $a = 15(x+y+z)-16$ and hence $n = 2^{15(x+y+z)-16}(2^x m_1+1)^3(2^y m_2+1)(2^z m_3+1)$, where $m = p^4$ for some odd prime p. For example, for $(x, y, z) = (1, 2, 4)$ and $p = 5$ we get $n = 2^{89} \cdot 3 \cdot 101^3 \cdot 401$.

2. $d(m) = 4$. It follows from (2.25) that $(a+1)(c+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)$. We must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, $(a+1)(c+1) = (a+x+y+z)\,c$, and so $a = c(x+y+z-1)-1$. Hence, $n = 2^{c(x+y+z-1)-1}(2^x m_1+1)(2^y m_2+1)^c(2^z m_3+1)$, where m is either p^3 or pq for some odd primes p, q. For example, for $(x, y, z) = (1, 1, 2)$ and $m = 5 \cdot 7$ we get $n = 2^{3c-1} \cdot 3 \cdot 11^c \cdot 29$, $c \geq 2$. This an infinite family of solutions.

3. $d(m) = 3$. It follows from (2.25) that $4(a+1)(c+1) = 3(a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)$. We must have $\alpha_1 = \alpha_2 = 0$. That is, $4(a+1)(c+1) = 3(a+x+y+z)(c+\alpha_3)$. Some of the solutions are of the form $n = 2^a(2^x+1)(2^y+1)^c\left(2^z(2^y+1)^{\alpha_3}p^2+1\right)$. For example, for $a = 8$, $c = 2$, $x = z = \alpha_3 = 1$, $y = 2$ and $m = 7^2$ we get $n = 2^8 \cdot 3 \cdot 5^2 \cdot 491$.

4. $d(m) = 2$. From (2.25),we obtain $2(a+1)(c+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)$. Thus, $\alpha_3 = 0$ or $\alpha_1 = \alpha_2 = 0$. When $\alpha_3 = 0$ we obtain $2(a+1)(c+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)\,c$, or, equivalently,

$$a = \frac{((x+y+z)(\alpha_1+\alpha_2+1)-2)\,c-2}{2+c\,(2-(\alpha_1+\alpha_2+1))}. \tag{2.26}$$

Here we must have $\alpha_1 + \alpha_2 \leq 1$. If $\alpha_1 + \alpha_2 = 1$, then some of the solutions are of the form

$$n = 2^{(x+y+z-1)c-1}\left(2^x+1\right)\left(2^y+1\right)^c\left(2^z\left(2^x+1\right)p+1\right).$$

For example, for $(x, y, z) = (1, 2, 2)$ and $m = 13$ we get $n = 2^{4c-1} \cdot 3 \cdot 5^c \cdot 157$, which is an infinite family of solutions. If $\alpha_1 = \alpha_2 = 0$, then by (2.26) we have $a = \dfrac{(x+y+z-2)c-2}{2+c}$, in which case the solutions are of the form $n = 2^{\frac{(x+y+z-2)c-2}{2+c}}\left(2^x m_1+1\right)\left(2^y m_2+1\right)^c\left(2^z m_3+1\right)$. For example, for $(x, y, z) = (2, 2, 4)$, $c = 5$ and $m = 7$ we get $n = 2^4 \cdot 5 \cdot 17^5 \cdot 29$. When $\alpha_1 = \alpha_2 = 0$ we have $2(a+1)(c+1) = (a+x+y+z)(\alpha_3+c)$. Some of the solutions are of the form $n = 2^a\left(2^x+1\right)\left(2^y+1\right)^c\left(2^z\left(2^y+1\right)^{\alpha_3}p+1\right)$. For example, for $x = z = \alpha_3 = 1$, $y = 2$ and $m = 7$ we get $n = 2^2 \cdot 3 \cdot 5^2 \cdot 71$.

5. $d(m) = 1$. It follows from (2.25) that $4(a+1)(c+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(c+\alpha_3)$. The solutions are of the form:

$$n = 2^a\left(2^x+1\right)\left(2^y\left(2^x+1\right)^{\alpha_1}+1\right)^c\left(2^z\left(2^x+1\right)^{\alpha_2}\left(2^y\left(2^x+1\right)^{\alpha_1}+1\right)^{\alpha_3}+1\right).$$

For example, for $(x, y, z) = (1, 1, 1)$, $(a, c) = (5, 2)$ and $\alpha_1 = \alpha_2 = \alpha_3 = 1$, we get $n = 2^5 \cdot 3 \cdot 7^2 \cdot 43$.

C) Let $n = 2^a q_2 q_3 q_4^d \in S_4$. As above, we have

$$4(a+1)(d+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)d \cdot d(m). \tag{2.27}$$

We note that (2.27) has no solutions whenever $d(m) \geq 6$ since $(\alpha_1+\alpha_2+1)(\alpha_3+1)d \cdot d(m) \geq 6d \geq 4(d+1)$ and $(a+x+y+z) > a+1$. Then we have five cases:

Case 1. $d(m) = 5$. It follows from (2.27) that

$$4(a+1)(d+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)d.$$

The last equation is not valid whenever $d \geq 4$. It remains to settle the cases $d = 2$ and $d = 3$. When $d = 2$, we have $6(a+1) = 5(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)$. Here we must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$; otherwise, $6(a+1) \geq 5(a+3)(\alpha_1+\alpha_2+1)(\alpha_3+1) \geq 10(a+3)$. A contradiction. It follows that $a = 5(x+y+z)-6$ and so $n = 2^{5(x+y+z)-6}\left(2^x m_1+1\right)\left(2^y m_2+1\right)\left(2^z m_3+1\right)^2$, where $m = p^4$ for some odd prime p. For example, for $x = 1$, $y = 2$, $z = 4$ and $p = 5$ we get $n = 2^{29} \cdot 3 \cdot 101 \cdot 401^2$. When $d = 3$, we also have

$$16(a+1) = 15(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1). \tag{2.28}$$

Similarly, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$; otherwise,

$$16(a+1) \geq 15(a+3)(\alpha_1+\alpha_2+1)(\alpha_3+1) \geq 30(a+3).$$

This is impossible. Therefore by (2.28), $a = 15(x+y+z)-16$ and hence

$$n = 2^{15(x+y+z)-16}\left(2^x m_1+1\right)^3\left(2^y m_2+1\right)\left(2^z m_3+1\right),$$

where $m = p^4$ for some odd prime p. For example, for $(x, y, z) = (1, 2, 4)$ and $p = 5$ we get $n = 2^{89} \cdot 3 \cdot 101 \cdot 401^3$.

Case 2. $d(m) = 4$. It follows from (2.27) that

$$(a+1)(d+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)d.$$

We must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Thus, $a = d(x+y+z-1)-1$ and so

$$n = 2^{d(x+y+z-1)-1}\left(2^x m_1+1\right)\left(2^y m_2+1\right)\left(2^z m_3+1\right)^d,$$

where $m = m_1 m_2 m_3$ is either p^3 or pq for some odd primes p, q. For example, for $(x, y, z) = (1, 1, 2)$ and $m = 5 \cdot 7$ we get $n = 2^{3d-1} \cdot 3 \cdot 11 \cdot 29^d$, $d \geq 2$. This an infinite family of solutions.

Case 3. $d(m) = 3$. It follows from (2.27) that

$$4(a+1)(d+1) = 3(a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)d.$$

We must have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. The above equality implies $4(a+1)(d+1) = 3(a+x+y+z)d$. Some of the solutions are of the form $n = 2^a(2^x+1)(2^y+1)\left(2^z p^2+1\right)^d$. For example, for $(x,y,z) = (1,2,2)$, $a = 3$, $d = 2$ and $m = 7^2$ we get $n = 2^3 \cdot 3 \cdot 5 \cdot 197^2$.

Case 4. $d(m) = 2$. By (2.27), we get $2(a+1)(d+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)d$. Here, we see that $\alpha_3 = 0$ or $\alpha_1 = \alpha_2 = 0$. When $\alpha_3 = 0$, we obtain

$$2(a+1)(d+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)d,$$

and therefore $a = \frac{((x+y+z)(\alpha_1+\alpha_2+1)-2)d-2}{2+d(2-(\alpha_1+\alpha_2+1))}$. Here we must have $\alpha_1+\alpha_2 \leq 1$. If $\alpha_1+\alpha_2 = 1$, then some of the solutions are of the form $n = 2^{(x+y+z-1)d-1}(2^x+1)(2^y+1)(2^z(2^x+1)p+1)^d$. For example, for $x = 1$, $y = z = 2$ and $m = 13$ we get $n = 2^{4d-1} \cdot 3 \cdot 5 \cdot 157^d$, $d \geq 2$. Here there infinitely many solutions of this kind.

Case 5. Assume that $d(m) = 1$, that is,

$$4(a+1)(d+1) = (a+x+y+z)(\alpha_1+\alpha_2+1)(\alpha_3+1)d. \tag{2.29}$$

First note that if $\alpha_1+\alpha_2 \geq 2$ and $\alpha_3 \geq 1$ or $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 \geq 2$, then both of which are contradictions. It remains to consider following three subcases:

Subcase 1. $\alpha_1+\alpha_2 = 0$ and $\alpha_3 \geq 0$. Clearly, $0 \leq \alpha_3 \leq 4$. When $\alpha_3 = 0$, we get by (2.29)

$$d = \frac{4(a+1)}{x+y+z-(4+3a)}.$$

For example, if $(x,y,z) = (1,8,16)$ then $(a,d) = (5,4)$, in which case $n = 2^5 \cdot 3 \cdot 257 \cdot 65\,537^4$. When $\alpha_3 = 1$, by (2.29) we also get $d = \frac{2(a+1)}{x+y+z-(a+2)}$. For example, if $(x,y,z) = (1,4,3)$ then $(a,d) = (4,5)$, and so $n = 2^4 \cdot 3 \cdot 17 \cdot 137^5$. When $\alpha_3 = 2$, the equality (2.29) gives $d = \frac{4(a+1)}{3(x+y+z)-(a+4)}$. For example, if $(x,y,z) = (1,4,10)$, then we get $(a,d) = (40,164)$. Hence, $n = 2^{40} \cdot 3 \cdot 17 \cdot 295\,937^{164}$. When $\alpha_3 = 3$, we obtain by (2.29) that $d = \frac{a+1}{x+y+z-1}$. For example, if $(x,y,z) = (1,4,5)$ then 9 divides $a+1$, in which case we get $n = 2^{9x-1} \cdot 3 \cdot 17 \cdot 157\,217^x$, where $x \geq 2$, which is an infinite family of solutions. When $\alpha_3 = 4$, by (2.29) we have $d = \frac{4(a+1)}{5(x+y+z)+a-4}$. For example, if $(x,y,z) = (1,4,4)$, then $a+41$ divides $4(a+1)$ from which we get (a,d) is either $(39,2)$ or $(119,3)$. Hence, the solutions are $n = 2^{39} \cdot 3 \cdot 17 \cdot 1336\,337^2$ and $n = 2^{119} \cdot 3 \cdot 17 \cdot 1336\,337^3$.

Subcase 2. $\alpha_1+\alpha_2 \geq 1$ and $\alpha_3 = 0$. It follows from (2.29) that

$$d = \frac{4(a+1)}{(\alpha_1+\alpha_2+1)(x+y+z)+(\alpha_1+\alpha_2+1)a-4(a+1)}.$$

Clearly, $1 \leq \alpha_1+\alpha_2 \leq 4$. For example, if $(x,y,z) = (1,2,4)$ and $(\alpha_1,\alpha_2) = (1,0)$ then (a,d) is either $(2,2)$, $(3,4)$ or $(4,10)$. Hence, $n = 2^2 \cdot 3 \cdot 13 \cdot 17^2$, $n = 2^3 \cdot 3 \cdot 13 \cdot 17^4$ and $n = 2^4 \cdot 3 \cdot 13 \cdot 17^{10}$.

Subcase 3. $\alpha_1+\alpha_2 = 1$ and $\alpha_3 = 1$. By (2.29), we have $d = \frac{a+1}{x+y+z-1}$. For example, if $(x,y,z) = (1,1,2)$ then we have 3 divides $a+1$ and $(\alpha_1,\alpha_2) = (1,0)$, in which case $n = 2^{3x-1} \cdot 3 \cdot 7 \cdot 29^x$, where $x \geq 2$. Hence there are infinitely many solutions of the desired form.

D) $n = 2^a q_2^b q_3^c q$. From (1.5) and (1.7), we get

$$2(a+1)(b+1)(c+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(c+\alpha_3) \cdot d(m).$$

We can easily show that $d(m)$ cannot be ≥ 3. Thus, we have only two cases:

Case 1. When $d(m) = 2$, we have $(a+1)(b+1)(c+1) = (a+x+y+z)(b+\alpha_1+\alpha_2)(c+\alpha_3)$. Note that $\alpha_1+\alpha_2 = 0$ or $\alpha_3 = 0$. If we assume $\alpha_1+\alpha_2 = 1$ and $\alpha_3 = 0$, then $(a+1)(c+1) =$

$(x+y+z+a)\,c$, which gives $a = (x+y+z-1)\,c - 1$. For $x = y = \alpha_1 = 1$, $z = 4$ and $m = 37$ we get $n = 2^{5c-1}\cdot 3^b \cdot 7^c \cdot 593$. The case $\alpha_1 + \alpha_2 \geq 2$ and $\alpha_3 = 0$ is impossible since

$$(a+x+y+z)\,(b+\alpha_1+\alpha_2)\,(c+\alpha_3) \geq (a+3)\,(2+b)\,c > (a+1)\,(b+1)\,(c+1)\,.$$

Case 2. When $d(m) = 1$, we obtain $2\,(a+1)\,(b+1)\,(c+1) = (x+y+z+a)\,(b+\alpha_1+\alpha_2)\,(c+\alpha_3)$. The solutions are of the form $n = 2^a\,(2^x+1)^b\,(2^y\,(2^x+1)^{\alpha_1}+1)^c\,\left(2^z\,(2^x+1)^{\alpha_2}\,(2^y\,(2^x+1)^{\alpha_1}+1)^{\alpha_3}+1\right)$. For example, if $x = \alpha_1 = \alpha_3 = 1$, $y = z = 1$, $a = 3$ and $\alpha_2 = 0$ we get $n = 2^3\cdot 3^b\cdot 13^c\cdot 53$. Thus we find an infinite family of solutions.

E) The proof of this case is similar to the proof of Case **D)**, where $n = 2^{3d-1}\cdot 3^b\cdot 5\cdot 7^d$ for $b, d \geq 2$.

F) Let $n = 2^a q_2^b q_3^c q_4^d \in S_4$, where $3 \leq q_2 < q_3 < q_4$ and $a, b, c, d \geq 2$. From (1.5),(1.7) we get

$$(a+1)\,(b+1)\,(c+1)\,(d+1) = (a+x+y+z)\,(b+\alpha_1+\alpha_2)\,(c+\alpha_3)\,d\cdot d\,(m)\,.$$

It is sufficient to assume that $\alpha_1+\alpha_2 = \alpha_3 = d(m) = 1$. Hence, $(a+1)\,(d+1) = (a+x+y+z)\,d$ from which if follows that $a = (x+y+z-1)\,d-1$. In particular, for $\alpha_1 = \alpha_3 = 1$, we obtain

$$n = 2^{(-1)d-1}\,(2^x+1)^b\,(2^y\,(2^x+1)+1)^c\,(2^z\,(2^y\,(2^x+1)+1)+1)^d\,.$$

For example, if $x = 2$, $y = 3$ and $z = 1$ then $n = 2^{5d-1}\cdot 5^b\cdot 41^c\cdot 83^d$, where $b, c, d \geq 2$. Thus our equation has infinitely many even solutions which have four powerful prime powers. This completes the proof of Part 9.

3. Corollary

As direct consequences of the proof of Parts 3, 4 and 5 we deduce the following corollary:

Corollary 3.1 *We have:*

1. *If there are infinitely many primes of the form $2\cdot 3^t+1$, then S_4 has infinitely many elements of the form $q_1^a q_2 q_3 q_4$, $2q_2^b q_3 q_4$, $q_1^a q_2 q_3 q_4^d$, $q_1^a q_2 q_3^c q_4$ and $q_1^a q_2^b q_3 q_4$.*
2. *If there are infinitely many primes of the form $2\cdot 5^t+1$, then S_4 has infinitely many elements of the form $q_1 q_2^b q_3 q_4$.*
3. *If there are infinitely many primes of the form $2^t\cdot p^3+1$, then S_4 has infinitely many elements of the form $2^a q_2 q_3 q_4$.*
4. *If there are infinitely many primes of the form $4\cdot 5^t+1$, then S_4 has infinitely many elements of the form $2q_2 q_3^c q_4$.*
5. *If there are infinitely many primes of the form $6\cdot 7^t+1$, then S_4 has infinitely many elements of the form $2^a q_1 q_3^c q_4^d$.*

Proof: We present the proof as follows:

1. Assume that there are infinitely many primes of the form $2\cdot 3^t+1$ (such primes[2] are related to Sophie Germain conjectures, see [10]).

a) In view of (2.2) with $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = t$, where 3 divides $5t+7$ and $2\cdot 3^t+1$ is prime, in which case $n = 3^{\frac{5t+7}{3}}\cdot 7\cdot 19\cdot (2\cdot 3^t+1)$.

b) In fact, by (2.5) with $x = y = z = \alpha_1 = 1$ and $\alpha_2 = t$, we get $n = 2\cdot 3^{t-2}\cdot 7\cdot (2\cdot 3^t+1)\,.$

c) This is by (2.7) with $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = t$ we have $n = 3^{5t+9}\cdot 7\cdot 19\cdot (2\cdot 3^t+1)^3\,.$

d) In view of (2.8), for $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = t$ we have $n = 3^{5t+9}\cdot 7\cdot 19^2\cdot (2\cdot 3^t+1)\,.$

e) In view of (2.9), for $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = t$ we have $n = 3^{5t-1}\cdot 7^3\cdot 19\cdot (2\cdot 3^t+1)\,.$

2. If there are infinitely many primes of the form $2\cdot 5^t+1$ with $t \equiv 0 \pmod 3$, then by (2.3), it suffices to choose $\alpha_4 = 1$ and $\alpha_5 = t$ in which case $n = 3\cdot 5^{\frac{5t-3}{3}}\cdot 11\cdot (2\cdot 5^t+1)$, where $2\cdot 5^t+1$ is prime (here we have $t \geq 24$).

[2] Note that the first values of t are 0, 1, 2, 4, 5, 6, 9, 16, 17, 30, 54, 57, 60, 65, 132, 180, 320, 696, 782, 822, 897, ...

3. If there are infinitely many primes of the form $2^t \cdot p^3 + 1$ with p prime, then in view of (2.4), for $(x, y) = (1, 2)$ and $z = t$ we have $n = 2^{t+1} \cdot 3 \cdot 5 \cdot \left(2^t \cdot p^3 + 1\right)$,where p and $2^t \cdot p^3 + 1$ must be prime.

4. If there are infinitely many primes of the form $4 \cdot 5^t + 1$, then by (2.6) with $\alpha_3 = t$, we have $n = 2 \cdot 3 \cdot 5^{3t-4} \cdot \left(2^2 \cdot 5^t + 1\right)$.

5. If there are infinitely many primes of the form $6 \cdot 7^t + 1$, then in view of (2.11), for $x = y = z = \alpha_1 = \alpha_2 = 1$ and $\alpha_3 = t$ we have $n = 2^a \cdot 3 \cdot 7^c \cdot \left(6 \cdot 7^t + 1\right)^d$, where by (2.10) the exponents a, c, d, t satisfy the equality $2(a+1)(c+1)(d+1) = 3(a+3)(c+t)d$ and $6 \cdot 7^t + 1$ is prime[3]. □

4. Conclusion

From Part 2, we deduce that any even square-free integer $n \in S_4$ is always formed by at least one odd Sophie Germain prime. So, if the Sophie Germain prime conjecture is true, then there are infinitely many square-free n such that $n \in S_4$. In addition, from Part 9, we proved that the elements of each of the following infinite families:

$$\begin{aligned}
\mathcal{F}_1 &= \left\{2^{3x-1} \cdot 3 \cdot 11 \cdot 29^x,\ x \geq 1\right\}, \\
\mathcal{F}_2 &= \left\{2^{3x-1} \cdot 3 \cdot 7 \cdot 29^x,\ x \geq 1\right\}, \\
\mathcal{F}_3 &= \left\{2 \cdot 3 \cdot 5^x \cdot 11^4,\ x \geq 1\right\}, \\
\mathcal{F}_4 &= \left\{2^{5x-1} \cdot 3^y \cdot 7^z \cdot 593,\ x, y, z \geq 1\ \right\}, \\
\mathcal{F}_5 &= \left\{2^3 \cdot 3^y \cdot 13 \cdot 17^4,\ y \geq 1\right\}, \\
\mathcal{F}_6 &= \left\{2^3 \cdot 3^y \cdot 13^z \cdot 53,\ y, z \geq 1\right\}, \\
\mathcal{F}_7 &= \left\{2^{5x-1} \cdot 5^y \cdot 41^z \cdot 83^x,\ x, y, z \geq 1\right\}, \\
\mathcal{F}_8 &= \left\{2^{9x-1} \cdot 3 \cdot 17 \cdot 157217^x,\ x \geq 1\right\}
\end{aligned}$$

are all in S_4. But, these sets contain only even solutions. However, as we see with odd solutions, we note that the existence of an infinite set of natural numbers that contains only odd numbers n such that $n \in S_4$ is related to some conjectures about the existence of an infinite number of primes with special forms, as well as the conjecture on the infinity of some generalized Sophie Germain primes. That is, primes of the form $2^x q^y + 1$, where q is odd prime and $x, y \geq 1$.

For further research, we close this paper by the following conjecture.

Conjecture 1 *S_4 has infinitely many odd numbers.*

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Bouhadjar Slimane and Bellaouar Djamel,
University 8 Mai 1945 Guelma,
Department of Mathematics,
B.P.401, Guelma, 24000,
Algeria.
E-mail address: bouhadjar.slimane@univ-guelma.dz, bellaouar.djamel@univ-guelma.dz