



Generalized trapezoid inequality for multiplicatively convex functions

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ABSTRACT: In this paper, we first prove a new identity for multiplicative differentiable functions. Based on this identity, we establish a generalized trapezoid inequality for multiplicatively convex functions. Applications to special means are also given.

Key Words: Non-Newtonian calculus, generalized trapezoid inequality, multiplicatively convex functions.

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1. Introduction

Let I be an interval of real numbers

Definition 1.1 ([24]) *A function $f : I \rightarrow \mathbb{R}$ is said to be convex, if*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

Definition 1.2 ([24]) *A positive function $f : I \rightarrow \mathbb{R}$ is said to be logarithmically convex or multiplicatively convex, if*

$$f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$$

holds for all $x, y \in I$ and all $t \in [0, 1]$.

The fundamental inequality for convex functions is undoubtedly the Hermite-Hadamard inequality, which can be stated as follows: for every convex function f on the interval $[a, b]$ with $a < b$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2}. \quad (1.1)$$

If the function f is concave, then (1.1) holds in the reverse direction see [24].

The concept of convexity plays an important role in many areas and it is closely related to the development of the inequality theory, which is an important tool in the study of some properties of solutions of differential equations as well as in the error estimates of quadrature formulas.

In [8], Dragomir and Agarwal established the following inequalities connected with the inequality (1.1) known as trapezium inequalities

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$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|). \quad (1.2)$$

In [23], Pearce and Pečarić proved some midpoint type inequalities for functions whose absolute value of the first derivatives are convex

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|). \quad (1.3)$$

Kavurmaci et al. [14], discussed the following generalized trapezoid inequalities for convex functions

$$\begin{aligned} & \left| \frac{(b-x)f(b)+(x-a)f(a)}{\xi-\varsigma} - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(x-a)^2}{b-a} \left(\frac{|f'(x)|+2|f'(a)|}{6} \right) + \frac{(b-x)^2}{b-a} \left(\frac{|f'(x)|+2|f'(b)|}{6} \right). \end{aligned} \quad (1.4)$$

Concerning some papers dealing with inequality (1.1) see [13,15,16,17] and references therein.

In 1967, Grossman and Katz, created the first non-Newtonian calculation system, called geometric calculation. Over the next few years they had created an infinite family of non-Newtonian calculus, thus modifying the classical calculus introduced by Newton and Leibniz in the 17th century each of which differed markedly from the classical calculus of Newton and Leibniz known today as the non-Newtonian calculus or the multiplicative calculus, where the ordinary product and ratio are used respectively as sum and exponential difference over the domain of positive real numbers see [12]. This calculation is useful for dealing with exponentially varying functions.

The complete mathematical description of multiplicative calculus was given by Bashirov et al. [5]. Also in the literature, there remains a trace of a similar calculation proposed by the mathematical biologists Volterra and Hostinsky [26] in 1938 called the Volterra calculation which is identified as a particular case of multiplicative calculation. For a particularly useful reference, especially for those not fully familiar with multiplicative calculus, we refer readers to [25].

Recently, Ali et al. [1], gave the analogue of the Hermite-Hadamard inequality for multiplicatively convex functions.

Theorem 1.1 *Let f be a positive and multiplicatively convex function on interval $[a, b]$, then the following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \left(\int_a^b f(x)^{dx} \right)^{\frac{1}{b-a}} \leq \sqrt{f(a)f(b)}. \quad (1.5)$$

In the same paper, the authors proved the inequalities for the product and the quotient of two multiplicatively convex functions. In [2], Ali et al. studied the Hermite-Hadamard type inequalities for multiplicatively ϕ -convex and log- ϕ -convex functions. In [19] and [21], Özcan generalized the results obtained in [1] for s -convex and h -convex functions respectively. In [20], Özcan gave the analogue of Hermite-Hadamard type inequalities for multiplicatively preinvex functions. In [22], Özcan proposed the Hermite-Hadamard type inequalities for multiplicatively h -preinvex functions. In [3] Ali et al. investigate the Ostrowski and the Simpson type inequalities for multiplicatively convex functions. For more studies on multiplicative integral inequalities, we refer the reader to [4,6,9,10,18,27] and the references cited therein.

Motivated by the above papers, in this study we propose a new identity for multiplicatively differentiable functions. Based on that identity, we establish a generalized trapezoid inequality for multiplicatively convex functions. Applications to special means are also given.

2. Preliminaries

In this section, we begin by recalling some definitions, properties, and notions of derivation as well as multiplicative integration.

Definition 2.1 ([5]) *Let $f : \mathbb{R} \rightarrow \mathbb{R}^+$ be a positive function. The multiplicative derivative of the function f noted by f^* is defined as follows*

$$\frac{d^* f}{dt} = f^*(t) = \lim_{h \rightarrow 0} \left(\frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}}.$$

Remark 2.1 *If f has positive values and is differentiable at t , then f^* exists and the relation between f^* and ordinary derivative f' is as follows:*

$$f^*(t) = e^{(\ln f(t))'} = e^{\frac{f'(t)}{f(t)}}.$$

The multiplicative derivative admits the following properties:

Theorem 2.1 ([5]) *Let f and g be multiplicatively differentiable functions, and c is arbitrary constant. Then functions cf , fg , $f + g$, f/g and fg are $*$ differentiable and*

- $(cf)^*(t) = f^*(t);$
- $(fg)^*(t) = f^*(t) g^*(t);$
- $(f + g)^*(t) = f^*(t)^{\frac{f(t)}{f(t)+g(t)}} g^*(t)^{\frac{g(t)}{f(t)+g(t)}};$
- $\left(\frac{f}{g}\right)^*(t) = \frac{f^*(t)}{g^*(t)};$
- $(fg)^*(t) = f^*(t)^{g(t)} f(t)^{g'(t)}.$

In [5], Bashirov et al. introduced the concept of the $*$ integral called multiplicative integral which is noted $\int_a^b (f(t))^{dt}$. Which the sum of the terms of the product is used in the definition of a classical Riemann integral of f over $[a, b]$, while the product of the terms raised to the power is used in the definition of the multiplicative integral of f over $[a, b]$.

The relationship between the Riemann integral and the multiplicative integral is as follows:

Proposition 2.1 ([5]) *If f is Riemann integrable on $[a, b]$, then f is multiplicative integrable on $[a, b]$ and*

$$\int_a^b (f(t))^{dt} = \exp \left(\int_a^b \ln(f(t)) dt \right).$$

Moreover, Bashirov et al. show that multiplicative integral has the following results and properties:

Theorem 2.2 ([5]) *Let f be a positive and Riemann integrable on $[a, b]$, then f is multiplicative integrable on $[a, b]$ and*

- $\int_a^b ((f(t))^p)^{dt} = \left(\int_a^b (f(t))^{dt} \right)^p.$
- $\int_a^b (f(t) g(t))^{dt} = \int_a^b (f(t))^{dt} \int_a^b (g(t))^{dt}.$

- $\int_a^b \left(\frac{f(t)}{g(t)} \right)^{dt} = \frac{\int_a^b (f(t))^{dt}}{\int_a^b (g(t))^{dt}}.$
- $\int_a^b (f(t))^{dt} = \int_a^c (f(t))^{dt} \int_c^b (f(t))^{dt}, \quad a \leq c \leq b.$
- $\int_a^a (f(t))^{dt} = 1$ and $\int_a^b (f(t))^{dt} = \left(\int_b^a (f(t))^{dt} \right)^{-1}.$

Theorem 2.3 (Multiplicative Integration by Parts [5]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be $*$ -differentiable, let $g : [a, b] \rightarrow \mathbb{R}$ be differentiable so the function f^g is $*$ -integrable, and*

$$\int_a^b \left(f^*(t)^{g(t)} \right)^{dt} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \times \frac{1}{\int_a^b (f(t)^{g'(t)})^{dt}}.$$

Lemma 2.1 ([3]) *Let $f : [a, b] \rightarrow \mathbb{R}$ be multiplicative differentiable, let $g : [a, b] \rightarrow \mathbb{R}$ and let $h : J \subset \mathbb{R} \rightarrow \mathbb{R}$ be two differentiable functions. Then we have*

$$\int_a^b \left(f^*(h(t))^{h'(t)g(t)} \right)^{dt} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \times \frac{1}{\int_a^b (f(h(t))^{g'(t)})^{dt}}.$$

3. Main results

In order to prove our results, we need the following lemma

Lemma 3.1 *Let $f : [a, b] \rightarrow \mathbb{R}^+$ be a multiplicative differentiable mapping on $[a, b]$ with $a < b$. If f^* is multiplicative integrable on $[a, b]$, then we have the following identity for multiplicative integrals*

$$\begin{aligned} & (f(a))^{\frac{x-a}{b-a}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_a^b f(u)^{du} \right)^{\frac{1}{a-b}} \\ &= \left(\int_0^1 \left(f^*((1-t)a + tx)^{t-1} \right)^{dt} \right)^{\frac{(x-a)^2}{b-a}} \left(\int_0^1 \left(f^*((1-t)x + tb)^t \right)^{dt} \right)^{\frac{(b-x)^2}{b-a}}, \end{aligned}$$

where $x \in [a, b]$.

Proof: Let

$$I_1 = \left(\int_0^1 \left(f^*((1-t)a + tx)^{t-1} \right)^{dt} \right)^{\frac{(x-a)^2}{b-a}}$$

and

$$I_2 = \left(\int_0^1 \left(f^*((1-t)x + tb)^t \right)^{dt} \right)^{\frac{(b-x)^2}{b-a}}.$$

Using the integration by parts for multiplicative integrals, from I_1 we have

$$I_1 = \left(\int_0^1 \left(f^*((1-t)a + tx)^{t-1} \right)^{dt} \right)^{\frac{(x-a)^2}{b-a}}$$

$$\begin{aligned}
&= \int_0^1 \left(f^* \left((1-t)a + tx \right)^{\frac{(x-a)^2}{b-a}(t-1)} \right) dt \\
&= \int_0^1 \left(f^* \left((1-t)a + tx \right)^{(x-a) \left(\frac{x-a}{b-a}t - \frac{x-a}{b-a} \right)} \right) dt \\
&= \frac{1}{(f(a))^{\frac{x-a}{b-a}}} \cdot \frac{1}{\int_0^1 \left((f((1-t)a + tx))^{\frac{x-a}{b-a}} \right) dt} \\
&= (f(a))^{\frac{x-a}{b-a}} \frac{1}{\int_0^1 \left((f((1-t)a + tx))^{\frac{x-a}{b-a}} \right) dt} = (f(a))^{\frac{x-a}{b-a}} \left(\int_a^x f(u) du \right)^{\frac{1}{a-b}}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
I_2 &= \left(\int_0^1 \left(f^* \left((1-t)x + tb \right)^{\frac{(b-x)^2}{b-a}t} \right) dt \right)^{\frac{(b-x)^2}{b-a}} \\
&= \int_0^1 \left(f^* \left((1-t)x + tb \right)^{\frac{(b-x)^2}{b-a}t} \right) dt \\
&= \int_0^1 \left(f^* \left((1-t)x + tb \right)^{(b-x) \left(\frac{b-x}{b-a}t \right)} \right) dt \\
&= \frac{(f(b))^{\frac{b-x}{b-a}}}{1} \cdot \frac{1}{\int_0^1 \left((f((1-t)x + tb))^{\frac{b-x}{b-a}} \right) dt} \\
&= (f(b))^{\frac{b-x}{b-a}} \cdot \frac{1}{\int_0^1 \left((f((1-t)x + tb))^{\frac{b-x}{b-a}} \right) dt} = (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_x^b f(u) du \right)^{\frac{1}{a-b}}.
\end{aligned}$$

Multiplying above equalities we get

$$\begin{aligned}
I_1 \times I_2 &= (f(a))^{\frac{x-a}{b-a}} \left(\int_a^x f(u) du \right)^{\frac{1}{a-b}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_x^b f(u) du \right)^{\frac{1}{a-b}} \\
&= (f(a))^{\frac{x-a}{b-a}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_a^x f(u) du \int_x^b f(u) du \right)^{\frac{1}{a-b}} \\
&= (f(a))^{\frac{x-a}{b-a}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}},
\end{aligned}$$

which is the desired result. The proof is completed. \square

Theorem 3.1 Let $f : [a, b] \rightarrow \mathbb{R}^+$ be a multiplicative differentiable mapping on $[a, b]$ with $a < b$. If f^* is multiplicative convex on $[a, b]$, then we have

$$\left| (f(a))^{\frac{x-a}{b-a}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right|$$

$$\leq (f^*(a))^{\frac{(x-a)^2}{3(b-a)}} (f^*(x))^{\frac{(x-a)^2+(b-x)^2}{6(b-a)}} (f^*(b))^{\frac{(b-x)^2}{3(b-a)}}.$$

Proof: From Lemma 3.1, properties of multiplicative integral and the multiplicative convexity of f^* , we have

$$\begin{aligned} & \left| (f(a))^{\frac{x-a}{b-a}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right| \\ & \leq \left(\exp \frac{(x-a)^2}{b-a} \int_0^1 \left| \ln \left(f^*((1-t)a + tx)^{t-1} \right) \right| dt \right) \\ & \quad \times \left(\exp \frac{(b-x)^2}{b-a} \int_0^1 \left| \ln \left(f^*((1-t)x + tb)^t \right) \right| dt \right) \\ & = \left(\exp \frac{(x-a)^2}{b-a} \int_0^1 (1-t) \left| \ln \left(f^*((1-t)a + tx) \right) \right| dt \right) \\ & \quad \times \left(\exp \frac{(b-x)^2}{b-a} \int_0^1 t \left| \ln \left(f^*((1-t)x + tb) \right) \right| dt \right) \\ & \leq \left(\exp \frac{(x-a)^2}{b-a} \int_0^1 (1-t) \left| \ln \left(f^*(a)^{(1-t)} f^*(x)^t \right) \right| dt \right) \\ & \quad \times \left(\exp \frac{(b-x)^2}{b-a} \int_0^1 t \left| \ln \left(f^*(x)^{(1-t)} f^*(b)^t \right) \right| dt \right) \\ & = \left(\exp \frac{(x-a)^2}{b-a} \int_0^1 (1-t) \left| \ln \left(f^*(a)^{(1-t)} \right) + \ln \left(f^*(x)^t \right) \right| dt \right) \\ & \quad \times \left(\exp \frac{(b-x)^2}{b-a} \int_0^1 t \left| \ln \left(f^*(x)^{(1-t)} \right) + \ln \left(f^*(b)^t \right) \right| dt \right) \\ & = \left(\exp \frac{(x-a)^2}{b-a} \left[\ln(f^*(a)) \int_0^1 (1-t)^2 dt + \ln(f^*(x)) \int_0^1 t(1-t) dt \right] \right) \\ & \quad \times \left(\exp \frac{(b-x)^2}{b-a} \left[\ln(f^*(x)) \int_0^1 t(1-t) dt + \ln(f^*(b)) \int_0^1 t^2 dt \right] \right) \\ & = \left(\exp \frac{(x-a)^2}{b-a} \left[\frac{1}{3} \ln(f^*(a)) + \frac{1}{6} \ln(f^*(x)) \right] \right) \\ & \quad \times \left(\exp \frac{(b-x)^2}{b-a} \left[\frac{1}{6} \ln(f^*(x)) + \frac{1}{3} \ln(f^*(b)) \right] \right) \\ & = \left(\exp \left[\ln \left((f^*(a))^{\frac{(x-a)^2}{3(b-a)}} \right) + \ln \left((f^*(x))^{\frac{(x-a)^2}{6(b-a)}} \right) \right] \right) \\ & \quad \times \left(\exp \left[\ln \left((f^*(x))^{\frac{(b-x)^2}{6(b-a)}} \right) + \ln \left((f^*(b))^{\frac{(b-x)^2}{3(b-a)}} \right) \right] \right) \\ & = \left(\exp \left[\ln \left((f^*(a))^{\frac{(x-a)^2}{3(b-a)}} (f^*(x))^{\frac{(x-a)^2}{6(b-a)}} \right) \right] \right) \end{aligned}$$

$$\begin{aligned}
& \times \left(\exp \left[\ln \left((f^*(x))^{\frac{(b-x)^2}{6(b-a)}} (f^*(b))^{\frac{(b-x)^2}{3(b-a)}} \right) \right] \right) \\
&= \exp \left[\ln \left((f^*(a))^{\frac{(x-a)^2}{3(b-a)}} (f^*(x))^{\frac{(x-a)^2}{6(b-a)}} \right) + \ln \left((f^*(x))^{\frac{(b-x)^2}{6(b-a)}} (f^*(b))^{\frac{(b-x)^2}{3(b-a)}} \right) \right] \\
&= \exp \left[\ln \left((f^*(a))^{\frac{(x-a)^2}{3(b-a)}} (f^*(x))^{\frac{(x-a)^2 + (b-x)^2}{6(b-a)}} (f^*(b))^{\frac{(b-x)^2}{3(b-a)}} \right) \right] \\
&= (f^*(a))^{\frac{(x-a)^2}{3(b-a)}} (f^*(x))^{\frac{(x-a)^2 + (b-x)^2}{6(b-a)}} (f^*(b))^{\frac{(b-x)^2}{3(b-a)}}.
\end{aligned}$$

The proof is completed. \square

Corollary 3.1 *In Theorem 3.1, If we assume that $f^* \leq M$, we get*

$$\left| (f(a))^{\frac{x-a}{b-a}} (f(b))^{\frac{b-x}{b-a}} \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right| \leq M^{\frac{(x-a)^2 + (b-x)^2}{2(b-a)}}.$$

Corollary 3.2 *In Theorem 3.1, if we take $x = a$ we obtain*

$$\left| f(b) \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right| \leq (f^*(a))^{\frac{(b-a)}{6}} (f^*(b))^{\frac{(b-a)}{3}}.$$

Corollary 3.3 *In Theorem 3.1, if we take $x = b$ we obtain*

$$\left| f(a) \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right| \leq (f^*(a))^{\frac{(b-a)}{3}} (f^*(b))^{\frac{(b-a)}{6}}.$$

Corollary 3.4 *In Theorem 3.1, if we take $x = \frac{a+b}{2}$ we obtain*

$$\left| (f(a) f(b))^{\frac{1}{2}} \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right| \leq ((f^*(a)) (f^*(\frac{a+b}{2})) (f^*(b)))^{\frac{(b-a)}{12}}.$$

Moreover, if we use the multiplicative convexity of f^* i.e. $f^*(\frac{a+b}{2}) \leq \sqrt{f^*(a) f^*(b)}$, we get

$$\left| (f(a) f(b))^{\frac{1}{2}} \cdot \left(\int_a^b f(u) du \right)^{\frac{1}{a-b}} \right| \leq ((f^*(a)) (f^*(b)))^{\frac{(b-a)}{8}}. \quad (3.1)$$

4. Applications to special means

We shall consider the means for arbitrary real numbers a, b

The Arithmetic mean: $A(a, b) = \frac{a+b}{2}$.

The harmonic mean: $H(a, b) = \frac{2ab}{a+b}$, $a, b > 0$.

The logarithmic means: $L(a, b) = \frac{b-a}{\ln b - \ln a}$, $a, b > 0$ and $a \neq b$.

The p -Logarithmic mean: $L_p(a, b) = \left(\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)} \right)^{\frac{1}{p}}$, $a, b > 0$, $a \neq b$ and $p \in \mathbb{R} \setminus \{-1, 0\}$.

Proposition 4.1 *Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have*

$$e^{A(a^p, b^p) - L_p^p(a, b)} \leq \left(e^{a^{p-1} + b^{p-1}} \right)^{p \frac{(b-a)}{8}}.$$

Proof: The assertion follows from inequality (3.1) applied to the function $f(t) = e^{t^p}$ with $p \geq 2$ whose $f^*(t) = e^{pt^{p-1}}$ and $\left(\int_a^b f(u) du\right)^{\frac{1}{a-b}} = \exp(-L_p^p(a, b))$. \square

Proposition 4.2 Let $a, b \in \mathbb{R}$ with $0 < a < b$, then we have

$$e^{H^{-1}(a,b)-L^{-1}(a,b)} \leq M^{\frac{b-a}{8b^2}}.$$

Proof: The assertion follows from Corollary 3.1 with $x = \frac{a+b}{2}$ applied to the function $f(t) = e^{\frac{1}{t}}$ whose $f^*(t) = e^{-\frac{1}{t^2}}$, $M = e^{-\frac{1}{b^2}}$ and $\left(\int_a^b f(u) du\right)^{\frac{1}{a-b}} = \exp(-L^{-1}(a, b))$. \square

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Conflict of interest

There are no known conflicts of interest related to this work.

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