Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–16.
ISSN-0037-8712
doi:10.5269/bspm.66261

Fixed Point for Family of Mappings with an Application in Dynamic Programming

Kavita * and Sanjay Kumar

ABSTRACT: In this paper, we investigate the existence of a unique common fixed point of families of weakly compatible mappings along with property$(E.A)$, common limit range property (CLR) and joint common limit range $(JCLR)$ property satisfying a generalized (ψ,ϕ)-weak contraction condition involving cubic terms of distance functions which generalize the result of Ćirić [6,7], Ćirić *et al.* [8], Chugh and Kumar [9], Jain and Kumar [11] , Jain *et al.* [13,15], Jungck [16], Kang *et al.* [19], Murthy and Prasad [24], Razani and Yazadi [25], Singh and Jain [28] and Zhang and Song [30]. As an application, we discuss the existence and uniqueness of a common solution of certain functional equations arising in dynamic programming.

Key Words: (ψ,ϕ)-weak contraction, weakly compatible mappings, property $(E.A)$, joint common limit range property $(JCLR)$, functional equations, dynamic programming.

Contents

1. Introduction and preliminaries

Banach contraction principle [3] is the basic result of fixed point theory which states that every contraction mapping T(say) defined on a complete metric space E(say) has a unique common fixed point. For the last ten decades, many researchers have been trying to generalize and extend this basic result in various directions. In 1976, Jungck [16] used the notion of commuting mappings for the generalization of the Banach contraction principle. In 1982, Sessa [27] relaxed the commutative condition of mapping to weak commutative mappings. Further, in 1986, Jungck [17] introduced the notion of compatible mappings to weaken the notion of commutativity/weak commutativity of mappings as follows:

Definition 1.1 [17] Two self mappings S and T of a metric space (E,d) are said to be compatible if and only if

$$\lim_{n\to\infty} d(STu_n, TSu_n) = 0,$$

whenever $\{u_n\}$ is a sequence in E such that $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z$, for some $z \in E$.

In 1996, the concept of weakly compatible mappings was introduced by Jungck [18] which may be considered as the minimal commutativity of mappings.

Definition 1.2 [18] Let S and T be two self mappings of a metric space (E,d). Then S and T are said to be weakly compatible mappings if the mappings commute at their coincidence points.

In 2002, Aamri and Moutawakil [1] introduced a generalization of noncompatible mappings in the form of property $(E.A)$.

Definition 1.3 [1] Two self mappings S and T defined on a metric space (E,d) are said to satisfy property $(E.A)$ if there exists a sequence $\{u_n\}$ in E such that

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z, \text{ for some } z \in E.$$

In 2005, Liu *et al.* [23] defined the notions of common property $(E.A)$ as follows.

* Corresponding author

Submitted December 12, 2022. Published February 09, 2023
2010 *Mathematics Subject Classification:* 47H10, 54H25, 90C39.

Definition 1.4 [23] Two pairs (f, S) and (g, T) of self mappings defined on a metric space (E, d) are said to satisfy common property $(E.A)$ if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in E such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} Tv_n = z, \text{ for some } z \in E.$$

In 2011, Sintunavarat and Kumam [29] presented the idea of common limit range property (CLR_T) as follows.

Definition 1.5 [29] A pair (S, T) of self mappings defined on a metric space (E, d) is said to satisfy the common limit range property (CLR_T) if there exists a sequence $\{u_n\}$ in E such that

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_n = z, \text{ for some } z \in T(E).$$

Remark 1.1 A pair (S, T) enjoying the property $(E.A)$ along with the closedness of the subspace $T(E)$ always satisfy the (CLR_T) property with respect to mapping T(see Examples 2.16-2.17 of [29])

In 2012, Imdad *et al.* [10] extended the notion of the common limit range property to two pairs of self mappings relaxing the requirements of closedness of subspaces under consideration.

Definition 1.6 [10] Let f, g, S and T be self mappings on a metric space (E, d). Two pairs (f, S) and (g, T) are said to satisfy the common limit range property with respect to mappings f and g (CLR_{fg}) if there exist two sequences $\{u_n\}$ and $\{v_n\}$ in E such that

$$\lim_{n\to\infty} fu_n = \lim_{n\to\infty} Su_n = \lim_{n\to\infty} gv_n = \lim_{n\to\infty} Tv_n = z, \text{ for some } z \in f(E) \cap g(E).$$

In 1971, Ćirić [6] investigated a class of self mappings on a metric space (E, d) satisfying the following condition.

$$d(fu, gv) \le k \max\{d(u, v), d(u, fu), d(v, fv), \frac{1}{2}[d(u, fv) + d(v, fu)]\}, \tag{1.1}$$

where $0 < k < 1$. In 1974, Ćirić [7] proved the common fixed point theorem for a family of mappings satisfying the condition (1.1) as follows.

Theorem 1.1 *[7] Let (E, d) be a complete metric space and $\{T_i\}_{i\in\Lambda}$ be a family of self mappings defined on E. If there exists a fixed $j \in \Lambda$ such that for each $i \in \Lambda$ and all $u, v \in E$*

$$d(T_i u, T_j v) \le \lambda \max\{d(u, v), d(u, T_i u), d(v, T_j v), \frac{1}{2}[d(u, T_j v) + d(v, T_i u)]\},$$

where $\lambda = \lambda(i) \in (0, 1)$, then all T_i have a unique common fixed point in E.

In 2005, Singh and Jain [28] proved the following fixed point theorem for commuting self mappings.

Theorem 1.2 *[28] Let (E, d) be a complete metric space and let A, B, P, Q, S and T be self mappings on E such that*

(H_1) *$P(E) \subset ST(E)$, $Q(E) \subset AB(E)$;*

(H_2) *$ST = TS$, $PB = BP$, $AB = BA$, $QT = TQ$;*

(H_3) *either AB or P is continuous;*

(H_4) *the pair (Q, ST) is weakly compatible and the pair (P, AB) is compatible;*

(H_5) *for all $u, v \in E$ and for some k , $0 < k < 1$,*

$$d(Pu, Qv) \le k \max\{d(Pu, ABv), d(Qv, STv), d(ABu, STv), \frac{1}{2}[d(Pu, STv) + d(Qv, ABu)]\}.$$

Then P, Q, S, T, A and B have a unique common fixed point.

In 2008, Ćirić *et al.* [8] proved common fixed point theorems for a family of mappings satisfying generalized non-linear contraction condition of type (1.1) in metric spaces and generalized the result of Singh and Jain [28].

In this paper, we prove some common fixed point theorems for a family of weakly compatible mappings along with the property$(E.A)$, the common limit range (CLR) property and the joint common limit range $(JCLR)$ property satisfying a generalized (ψ, ϕ)-weak contraction condition involving cubic terms of metric functions. Further, we apply our result to obtain common solution of system of certain functional equations arising in dynamic programming.

2. Main results

In 1969, Boyd and Wong [5] introduced ϕ contraction of the form $d(Tu, Tv) \leq \phi(d(u, v))$, for all $u, v \in E$, where T is a self mapping on a complete metric space E and $\phi : [0, \infty) \to [0, \infty)$ is an upper semi continuous function from right such that $0 \leq \phi(t) < t$, for all $t > 0$. In 1997, Alber and Guerre-Delabriere [2] generalized ϕ contraction to $\phi-$weak contraction in Hilbert spaces, which was further extended and proved by Rhoades [26] in complete metric spaces.

A self mapping T on a complete metric space is said to be a $\phi-$ weak contraction if for each $u, v \in E$, there exists a continuous non-decreasing function $\phi : [0, \infty) \to [0, \infty)$ satisfying $\phi(t) > 0$, for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$ such that

$$d(Tu, Tv) \leq d(u, v) - \phi(d(u, v)). \tag{2.1}$$

The function ϕ in the above inequality (2.1) is known as control function or altering distance function. The notion of control function was given by Khan *et al.* [21] : an altering distance is an increasing and continuous function $\phi : [0, \infty) \to [0, \infty)$ vanishing only at zero.

In 2009, Zhang and Song [30] gave the notion of generalized $\phi-$ weak contraction by generalizing the concept of $\phi-$weak contraction.

Definition 2.1 [30] Two self mappings S and T on a metric space (E, d) are said to be generalized $\phi-$weak contractions if there exists a mapping $\phi : [0, \infty) \to [0, \infty)$ with $\phi(t) > 0$ for all $t > 0$ and $\phi(0) = 0$ such that

$$d(Su, Tv) \leq M(u, v) - \phi(M(u, v)) \text{ for all } u, v \in E,$$

where $M(u, v) = \max\{d(u, v), d(u, Su), d(v, Tv), \frac{d(u,Tv)+d(v,Su)}{2}\}$.

In 2013, Murthy and Prasad [24] introduced a weak contraction that involves cubic terms of distance functions.

Theorem 2.1 *[24] Let T be a self mapping on a complete metric space E satisfying:*

$$\begin{aligned}[1 + pd(u, v)]d^2(Tu, Tv) \leq p \max \Big\{ &\frac{1}{2}[d^2(u, Tu)d(v, Tv) + d(u, Tu)d^2(v, Tv)], \\ &d(u, Tu)d(u, Tv)d(v, Tu), d(u, Tv)d(v, Tu)d(v, Tv) \Big\} \\ &+ m(u, v) - \phi(m(u, v)),\end{aligned}$$

where

$$\begin{aligned}m(u, v) = \max \Big\{ &d^2(u, v), d(u, Tu)d(v, Tv), d(u, Tv)d(v, Tu), \\ &\frac{1}{2}[d(u, Tu)d(u, Tv) + d(v, Tu)d(v, Tv)] \Big\},\end{aligned}$$

where $p \geq 0$ is a real number and $\phi : [0, \infty) \to [0, \infty)$ is a continuous function with $\phi(t) = 0$ if and only if $t = 0$ and $\phi(t) > 0$ for each $t > 0$. Then T has a unique fixed point in E.

Theorem 2.1 was extended and generalized for a variety of commuting self mappings on metric spaces [12,13,14,15,20,22]. In the present work, we shall discuss the existence and uniqueness of common fixed point for a family of weakly compatible mappings by using the control function $\psi \in \Psi$ and these results generalize and extend the results of Ćirić [7], Ćirić *et al.* [8], Chugh and Kumar [9], Jain and Kumar [11], Jain *et al.* [13,15], Kang *et al.* [19], Murthy and Prasad [24], Razani and Yazdi [25] and Singh and Jain [28] and Zhang and Song [30], where Ψ is a collection of all functions $\psi : [0,\infty)^4 \to [0,\infty)$ satisfying the following conditions:

(ψ_1) ψ is non decreasing and upper semi continuous in each coordinate variables,

(ψ_2) $\Delta(t) = \max\{\psi(t,t,0,0), \psi(0,0,0,t), \psi(0,0,t,0), \psi(t,t,t,t)\} \leq t$, for each $t > 0$.

Let Φ be a collection of all the functions $\phi : [0,\infty) \to [0,\infty)$ satisfying the following conditions:

(ϕ_1) ϕ is a continuous function,

(ϕ_2) $\phi(t) > 0$ for each $t > 0$ and $\phi(0) = 0$.

Throughout this section, we denote $A' = A_2A_4\cdots A_{2n}$ and $A'' = A_1A_3\cdots A_{2n-1}$, where A_i, $i = 1,2,\ldots,2n$ are as mentioned in the following theorems.

Theorem 2.2 *Let S, T and $\{A_i\}_{i=1}^{2n}$, be self mappings of a metric space (E,d) satisfying the following conditions:*

(C_1) *$T(E) \subset A'(E)$ and $S(E) \subset A''(E)$;*

(C_2) $A_1(A_3\cdots A_{2n-1}) = (A_3\cdots A_{2n-1})A_1$,
$A_1A_3(A_5\cdots A_{2n-1}) = (A_5\cdots A_{2n-1})A_1A_3$,
$\ldots$
$A_1A_3\cdots A_{2n-3}(A_{2n-1}) = (A_{2n-1})A_1A_3\cdots A_{2n-3}$;
$T(A_3\cdots A_{2n-1}) = (A_3\cdots A_{2n-1})T$,
$T(A_5\cdots A_{2n-1}) = (A_5\cdots A_{2n-1})T$,
$\ldots$
$TA_{2n-1} = A_{2n-1}T$;
$A_2(A_4\cdots A_{2n}) = (A_4\cdots A_{2n})A_2$,
$A_2A_4(A_6\cdots A_{2n}) = (A_6\cdots A_{2n})A_2A_4$,
$\ldots$
$A_2A_4\cdots A_{2n-2}(A_{2n}) = (A_{2n})A_2A_4\cdots A_{2n-2}$;
$S(A_4\cdots A_{2n}) = (A_4\cdots A_{2n})S$,
$S(A_6\cdots A_{2n}) = (A_6\cdots A_{2n})S$,
$\ldots$
$SA_{2n} = A_{2n}S$;

(C_3) *one of the subspaces $S(E)$ or $T(E)$ or $A'(E)$ or $A''(E)$ is complete;*

(C_4) *the pairs (T, A'') and (S, A') are weakly compatible;*

(C_5) *for all $u, v \in E$, there exists functions $\psi \in \Psi$ and $\phi \in \Phi$, a real number $p > 0$ such that*

$$[1 + pd(A'u, A''v)]d^2(Su, Tv) \leq p\psi\Big(d^2(A'u, Su)d(A''v, Tv), d(A'u, Su)d^2(A''v, Tv),$$
$$d(A'u, Su)d(A'u, Tv)d(A''v, Su), d(A'u, Tv)d(A''v, Su)d(A''v, Tv)\Big)$$
$$+ m(A'u, A''v) - \phi(m(A'u, A''v)),$$

where

$$m(A'u,A''v)=\max\Big\{d^2(A'u,A''v),d(A'u,Su)d(A''v,Tv),d(A'u,Tv)d(A''v,Su),\\ \frac{1}{2}[d(A'u,Su)d(A'u,Tv)+d(A''v,Su)d(A''v,Tv)]\Big\}.$$

Then S, T, A_1, A_2,..., A_{2n-1} and A_{2n} have a unique common fixed point in E.

Proof: Let $u_0 \in E$ be arbitrary, then by condition (C_1), there exists $u_1, u_2 \in E$ such that $Su_0 = A''u_1 = v_0$ and $Tu_1 = A'u_2 = v_1$. Continuing in this fashion, one can construct sequences $\{u_k\}$ and $\{v_k\}$ in E such that

$$v_{2k} = Su_{2k} = A''u_{2k+1} \quad \text{and} \quad v_{2k+1} = Tu_{2k+1} = A'u_{2k+2}, \tag{2.2}$$

for each $k = 0, 1, 2, 3....$ For simplicity, let us denote

$$\beta_k = d(v_k, v_{k+1)}, k = 0, 1, 2, 3, ... \tag{2.3}$$

Before proving main result, first we shall show that the sequence $\{\beta_k\}$ is non-increasing and $\lim_{k\to\infty} \beta_k = 0$. If k is even, i.e., $k = 2m, m = 0, 1, 2, \ldots$, taking $u = u_k = u_{2m}$ and $v = u_{k+1} = u_{2m+1}$ in (C_5) and using equation (2.2) and (2.3), we get

$$[1+p\beta_{2m-1}]\beta_{2m}^2 \le p\,\psi\big(\beta_{2m-1}^2\beta_{2m}, \beta_{2m-1}\beta_{2m}^2, 0, 0\big) + m(v_{2m-1}, v_{2m}) - \phi\big(m(v_{2m-1}, v_{2m})\big), \tag{2.4}$$

where $m(v_{2m-1}, v_{2m}) = \max\left\{\beta_{2m-1}^2, \beta_{2m-1}\beta_{2m}, 0, \frac{1}{2}\left[\beta_{2m-1}d(v_{2m-1}, v_{2n+1}) + 0\right]\right\}$.

Using triangular inequality, we get

$$d(v_{2m-1}, v_{2m+1}) \le d(v_{2m-1}, v_{2m}) + d(v_{2m}, v_{2m+1}) = \beta_{2m-1} + \beta_{2m}.$$

Hence,

$$m(v_{2m-1}, v_{2m}) \le \max\left\{\beta_{2m-1}^2, \beta_{2m-1}\beta_{2m}, 0, \frac{1}{2}\left[\beta_{2m-1}(\beta_{2m-1} + \beta_{2m})\right]\right\}. \tag{2.5}$$

Now, we claim that $\{\beta_k\}$, i.e., $\{\beta_{2m}\}$ is non-increasing. Suppose not, i.e., $\beta_{2m-1} < \beta_{2m}$, then by using the inequality (2.5) with the property of ϕ and ψ, inequality (2.4) becomes

$$[1+p\beta_{2m-1}]\beta_{2m}^2 \le p\beta_{2m-1}\beta_{2m}^2 + \beta_{2m-1}\beta_{2m} - \phi(\beta_{2m-1}\beta_{2m}),$$

i.e., $\beta_{2m}^2 < \beta_{2m}^2$, a contradiction. Therefore, $\beta_{2m} \le \beta_{2m-1}$.

Similarly, if k is odd, i.e., $k = 2m+1$,$m = 0, 1, 2 \ldots$, then one can obtain $\beta_{2m+1} \le \beta_{2m}$. It follows that the sequence $\{\beta_k\}$ is non-increasing for each k. Next, we claim that $\lim_{k\to\infty} \beta_k = 0$. Suppose not, i.e., for some $t > 0$

$$\lim_{k\to\infty} \beta_k = t. \tag{2.6}$$

Taking $k \to \infty$ and using inequality (2.5) and equation (2.6), inequality (2.4) reduces to

$$[1+pt]t^2 \le pt^3 + t^2 - \phi(t^2),$$

which implies that $\phi(t^2) \le 0$, a contradiction to the definition of ϕ. Therefore,

$$\lim_{k\to\infty} \beta_k = 0. \tag{2.7}$$

Now, we show that the sequence $\{v_k\}$ is a Cauchy sequence. Let us assume that $\{v_k\}$ is not a Cauchy sequence, so there exists an $\epsilon > 0$, for which, one can find two sequences of positive integers $\{m(k')\}$ and $\{n(k')\}$ such that $n(k') > m(k') > k'$ and

$$d(v_{m(k')}, v_{n(k')}) \geq \epsilon \quad \text{and} \quad d(v_{m(k')}, v_{n(k')-1}) < \epsilon, \tag{2.8}$$

for all positive integers k'. Now,

$$\epsilon \leq d(v_{m(k')}, v_{n(k')}) \leq d(v_{m(k')}, v_{n(k')-1}) + d(v_{n(k')-1}, v_{n(k')}).$$

Letting $k' \to \infty$, we get

$$\lim_{k' \to \infty} d(v_{m(k')}, v_{n(k')}) = \epsilon. \tag{2.9}$$

Using the triangular inequality, we have,

$$|d(v_{n(k')}, v_{m(k')+1}) - d(v_{m(k')}, v_{n(k')})| \leq d(v_{m(k')}, v_{m(k')+1}).$$

Taking limit as $k' \to \infty$ and using equations (2.7) and (2.9) in the above inequality, we have

$$\lim_{k' \to \infty} d(v_{n(k')}, v_{m(k')+1}) = \epsilon. \tag{2.10}$$

Again using the triangular inequality, we have

$$|d(v_{m(k')}, v_{n(k')+1}) - d(v_{m(k')}, v_{n(k')})| \leq d(v_{n(k')}, v_{n(k')+1})$$

Taking limit as $k' \to \infty$ and using equations (2.7) and (2.9), we have

$$\lim_{k' \to \infty} d(v_{m(k')}, v_{n(k')+1}) = \epsilon. \tag{2.11}$$

Similarly, using triangular inequality, we have

$$|d(v_{m(k')+1}, v_{n(k')+1}) - d(v_{m(k')}, v_{n(k')})| \leq d(v_{m(k')}, v_{m(k')+1}) + d(v_{n(k')}, v_{n(k')+1})$$

Taking limit as $k' \to \infty$ and using equations(2.7) and (2.9) in the above inequality, we have

$$\lim_{k' \to \infty} d(v_{n(k')+1}, v_{m(k')+1}) = \epsilon. \tag{2.12}$$

Taking $u = u_{m(k')}$ and $v = u_{n(k')}$ in (C_5) and using equations (2.2),(2.7)-(2.12) and then taking limit as $k \to \infty$, we get

$$[1 + p\epsilon]\epsilon^2 \leq p\psi(0, 0, 0, 0) + \epsilon^2 - \phi(\epsilon^2), \text{ i.e., } p\epsilon^3 + \phi(\epsilon^2) < 0,$$

which is a contradiction. Hence, the sequence $\{v_k\}$ is a Cauchy sequence in E. Suppose that $A'(E)$ is a complete subspace of E, therefore, there exists some $w \in E$ such that $v_{2k+1} = Tu_{2k+1} = A'u_{2k+2} \to w$, as $k \to \infty$. Therefore, one can find $z \in E$ such that $A'z = w$. A Cauchy sequence $\{v_k\}$ has a convergent subsequence $\{v_{2k+1}\}$, therefore the sequence $\{v_k\}$ converges and hence, we have $v_{2k} = Su_{2k} = A''u_{2k+1} \to w$, as $k \to \infty$.

(a) Now we claim that $Sz = w$. For this, putting $u = z$ and $v = u_{2k+1}$ in (C_5) and proceeding with $k \to \infty$, we have

$$[1 + pd(w, w)]d^2(Sz, w) \leq p\psi(0, 0, 0, 0) + m(w, w) - \phi(m(w, w)),$$

where

$$m(w, w) = \max\Big\{d^2(w, w), d(w, Sz)d(w, w), d(w, w)d(w, Sz), \frac{1}{2}[d(w, Sz)d(w, w) + d(w, Sz)d(w, w)]\Big\} = 0.$$

Simplifying the above inequality, we get $d^2(Sz, w) \leq 0$, which is true for $Sz = w$. Hence, $Sz = w = A'z =$, i.e., w is a coincidence point of S and A'.
(b) Since $S(E) \subset A''(E)$, there exists a point $x \in E$ such that $A''x = Sz = w$. Now, we claim that $Tx = w$, for this, substituting $u = u_{2k}$ and $v = x$ and taking limit $k \to \infty$ in (C_5), we get

$$[1 + pd(w, w)]d^2(w, Tx) \leq p\psi(0, 0, 0, 0) + m(w, w) - \phi(m(w, w)),$$

where

$$m(w, w) = \max \Big\{ d^2(w, w), d(w, w)d(w, Tx), d(w, Tx)d(w, w), \frac{1}{2}[d(w, w)d(w, Tx) + d(w, w)d(w, Tx)] \Big\} = 0.$$

After simplification, we get $d^2(w, Tx) = 0$, which implies that $Tx = w$. Hence, $Tx = w = A''x$, i.e., x is a coincidence point of T and A''.
(c) Since the pairs (T, A'') and (S, A') are weakly compatible, therefore, we have $A'w = A'Sz = SA'z = Sw$ and $A''w = A''Tx = TA''x = Tw$.
(d)Next, we show that $A'w = w$.Substituting $u = w$ and $v = v_{2k+1}$ in (C_5) and proceeding $k \to \infty$, we get

$$[1 + pd(A'w, w)]d^2(Sw, w) \leq p\psi(0, 0, 0, 0) + m(A'w, w) - \phi(m(A'w, w)),$$

where

$$m(A'w, w) = \max \Big\{ d^2(A'w, Sw), d(A'w, Sw)d(w, w), d(A'w, w)d(w, Sw), \frac{1}{2}[d(A'w, Sw)d(A'w, w) + d(w, Sw)d(w, w)] \Big\} = d^2(A'w, w).$$

Solving the above inequality, we get $A'w = w$. Hence, $Sw = A'w = w$.
(e) Now, we show that $A''w = w$, For this, taking $u = v = w$ in (C_5), we get

$$[1 + pd(w, A''w)]d^2(w, Tw) \leq p\psi(0, 0, 0, 0) + m(w, A''w) - \phi(m(w, A''w)),$$

where

$$m(w, A''w) = \max \Big\{ d^2(w, A''w), 0, d(w, Tw)d(A''w, w), 0 \Big\} = d^2(w, A''w).$$

After simplification, we conclude that $A''w = w$, hence, $A''w = Tw = w$.
(f) Now, taking $u = A_4 \cdots A_{2n}w$ and $v = w$ in (C_5) and applying condition (C_2), we have

$$[1 + pd(A_4 \cdots A_{2n}w, w)]d^2(A_4 \cdots A_{2n}w, w) \leq p\psi(0, 0, 0, 0) + m(A_4 \cdots A_{2n}w, w) - \phi(m(A_4 \cdots A_{2n}w, w)),$$

where

$$m(A_4 \cdots A_{2n}w, w) = \max\{d^2(A_4 \cdots A_{2n}w, w), 0, d^2(A_4 \cdots A_{2n}w, w), 0\} = d^2(A_4 \cdots A_{2n}w, w).$$

Solving, we get $p\, d^3(A_4 \cdots A_{2n}w, w) + \phi(d^2(A_4 \cdots A_{2n}w, w)) \leq 0$, which is possible only when $A_4 \cdots A_{2n}w = w$. Therefore, $w = A'w = A_2A_4 \cdots A_{2n}w = A_2w$. Continuing in this manner, we get $Sw = A_2w = A_4w = ... = A_{2n}w = w$.
(g) Now taking $u = w$ and $v = A_3 \cdots A_{2n-1}w$ in (C_5) and applying condition (C_2), we have

$$[1 + pd(w, A_3 \cdots A_{2n-1}w)]d^2(w, A_3 \cdots A_{2n-1}w) \leq p\psi(0, 0, 0, 0) + d^2(w, A_3 \cdots A_{2n-1}w) - \phi(d^2(w, A_3 \cdots A_{2n-1}w)).$$

After simplifying the above inequality, we get $pd^3(w, A_3 \cdots A_{2n-1}w) \le 0$, which is possible only if $d(w, A_3 \cdots A_{2n-1}w) = 0$. This implies that $A_3 \cdots A_{2n-1}w = w$. Hence, $Tw = A_1w = A_3w = ... = A_{2n-1}w = w$. Hence, $Sw = Tw = A_1w = A_2w = A_3w ... = A_{2n-1}w = A_{2n}w = w$. Similarly, the result holds if the subspace $S(E)$ or $T(E)$ or $A''(E)$ is assumed to be complete.

For the uniqueness, let $y \neq w$ be two common fixed points of the above mentioned mappings. Taking $u = w$ and $v = y$ in (C_5) , we get

$$[1 + pd(w, y)]d^2(w, y) \le p\psi(0, 0, 0, 0) + d^2(w, y) - \phi(d^2(w, y)).$$

Simplifying it, we get $pd^3(w, y) + \phi(d^2(w, y)) < 0$, which is a contradiction, hence, $y = w$. Thus, w is a unique common fixed point of S, T,A_1,A_2,..., A_{2n-1} and A_{2n}. □

Now, we prove a fixed point theorem for weakly compatible mappings relaxing the condition of completeness of the subspaces.

Theorem 2.3 *Let $A_i(i = 1, 2, \ldots, 2n)$, S and T be self mappings of a complete metric space (E, d) satisfying the conditions (C_1), (C_2), (C_4), (C_5) and*

(C_6) one of the subspaces $S(E)$ or $T(E)$ or $A'(E)$ or $A''(E)$ is closed;

Then S, T, A_1, A_2,..., A_{2n-1} and A_{2n} have a unique common fixed point in E.

Proof: It is well known that the subspaces of a complete metric space is complete if and only if it is closed. Hence, the conclusion follows easily from Theorem 2.2. □

Next, we establish the existence of a unique common fixed point for even number of weakly compatible mappings satisfying the common property $(E.A)$.

Theorem 2.4 *Let S,T, $A_i(i = 1, 2, \ldots, 2n)$ be self mappings of a metric space (E, d) satisfying the conditions (C_1), (C_2), (C_4), (C_5) and (C_6). If the pairs (T, A'') and (S, A') satisfy the common property $(E.A)$, then mappings A_1,A_2,..., A_{2n-1}, A_{2n}, S and T have a unique common fixed point in E.*

Proof: Since the pairs (S, A') and (T, A'') satisfy the common property $(E.A)$, there exist two sequences $\{u_k\}$ and $\{v_k\}$ in E such that

$$\lim_{k\to\infty} A'u_k = \lim_{k\to\infty} Su_k = \lim_{n\to\infty} A''v_k = \lim_{n\to\infty} Tv_k = w, \text{ for some } w \in E.$$

Assume that $A''(E)$ is a closed subset of E, then there exists $z \in E$ such that $w = A''z$. Now, we prove that $Tz = w$, for this taking $u = u_k$ and $v = z$ in (C_5) and taking limit $k \to \infty$, we get

$$[1 + pd(w, w)]d^2(w, Tz) \le p\psi(0, 0, 0, 0) + m(w, w) - \phi(m(w, w)),$$

where

$$m(w, w) = \max\Big\{d^2(w, w), d(w, w)d(w, Tz), d(w, Tz)d(w, w), \frac{1}{2}[d(w, w)d(w, Tz) + d(w, w)d(w, Tz)]\Big\} = 0.$$

After simplification, we get $d^2(w, Tz) = 0$, which implies that $Tz = w$. Since $T(E) \subset A'(E)$, there exists $x \in E$ such that $w = Tz = A'x$. Now, we claim that $Sx = w$. Substituting $u = x$ and $v = z$ in (C_5), we get

$$[1 + pd(w, w)]d^2(Sx, w) \le p\psi(0, 0, 0, 0) + m(w, w) - \phi(m(w, w)),$$

where

$$m(w, w) = \max\Big\{d^2(w, w), d(w, Sx)d(w, w), d(w, w)d(w, Sx), \frac{1}{2}[d(w, Sx)d(w, w) + d(w, Sx)d(w, w)]\Big\} = 0.$$

Simplifying the above inequality, we get $d^2(Sx, w) = 0$, which implies that $Sx = w$.Therefore, we have $Sx = A'x = Tz = A''z = w$. Since the pairs (S, A') and (T, A'') are weakly compatible mappings and x and z are coincidences points of each respectively, therefore, $A'w = A'Sx = SA'x = Sw$ and $A''w = A''Tz = TA''z = Tw$. Taking $u = x$ and $v = w$ in (C_5), we get

$$[1 + pd(w, A''w)]d^2(w, Tw) \leq p\psi(0, 0, 0, 0) + m(w, A''w) - \phi(m(w, A''w)),$$

where

$$m(w, A''w) = \max\Big\{d^2(w, A''w), 0, d(w, Tw)d(A''w, w), 0\Big\} = d^2(w, Tw).$$

After simplification, we have $pd^3(w, Tw) + \phi(d^2(w, Tw)) \leq 0$, which is true only for $Tw = w$, hence, $A''w = Tw = w$. Further, taking $u = v = w$ in (C_5), we get

$$[1 + pd(A'w, w)]d^2(Sw, w) \leq p\psi(0, 0, 0, 0) + m(A'w, w) - \phi(m(A'w, w)),$$

where

$$m(A'w, w) = \max\Big\{d^2(A'w, Sw), d(A'w, Sw)d(w, w), d(A'w, w)d(w, Sw), \frac{1}{2}[d(A'w, Sw)d(A'w, w) + d(w, Sw)d(w, w)]\Big\} = d^2(A'w, w).$$

Solving the above inequality, we get $A'w = w$. Hence, $Sw = A'w = w$. From the steps (f) and (g) of Theorem 2.2, it is clear that $Sw = A_2w = A_4w = ... = A_{2n}w = w$ and $Tw = A_1w = A_3w = ... = A_{2n-1}w = w$. Thus, w is a common fixed point of mappings S, T and $\{A_i\}_{i=1}^{2n}$. Similarly, one can complete the proof by considering $A'(E)$ or $T(E)$ or $S(E)$ a closed subspace of E. Uniqueness follows easily. This completes the proof. □

Now, we prove theorems for a family of mappings employing the common limit range property. Before proving the theorem, first, we prove the following lemma :

Lemma 2.1 *Let (E, d) be a metric space. Let S, T and $A_i (i = 1 \dots 2n)$ be self mappings defined on E satisfying the conditions (C_1), (C_5) and*

(C_7) *the pairs (S, A') and (T, A'') satisfy the property $(CLR_{A'})$ and the property $(CLR_{A''})$ respectively,*

(C_8) *one of the subspaces $A'(E)$ or $A''(E)$ is closed subset of E,*

(C_9) *$\{Su_k\}$ and $\{Tv_k\}$ converge for every sequences $\{u_k\}$ and $\{v_k\}$ in E whenever $\{A'u_k\}$ and $\{A''v_k\}$ converge,*

Then the pairs (S, A') and (T, A'') share the property $(CLR_{A'A''})$.

Proof: Suppose that the pair (A', S) satisfies the property $(CLR_{A'})$, so there exists a sequence $\{u_k\}$ in E such that $\lim_{k\to\infty} A'u_k = \lim_{k\to\infty} Su_k = z$, $z \in A'(E)$. Since $S(E) \subset A''(E)$, therefore, for each $\{u_k\}$ there corresponds a sequence $\{v_k\} \in E$ such that $Su_k = A''v_k$. Since $A''(E)$ is a closed subset, therefore $\lim_{k\to\infty} A''v_k = \lim_{k\to\infty} Su_k = z$, $z \in A''(E)$. Hence, $z \in A'(E) \cap A''(E)$. Thus, we have $A'u_k \to z$, $Su_k \to z$ and $A''v_k \to z$ as $k \to \infty$. By (C_9), $\{Tv_k\}$ converges. We claim that $\lim_{k\to\infty} Tv_k = z$. Suppose not, i.e., $\lim_{k\to\infty} Tv_k = t (\neq z)$. Taking $u = u_k$, $v = v_k$ in (C_5) and letting $k \to \infty$, we get

$$[1 + pd(z, z)]d^2(z, t) \leq p\psi\big(0, 0, 0, 0\big) + m(z, z) - \phi(m(z, z)),$$

where

$$m(z, z) = \max\Big\{d^2(z, z), d(z, z)d(z, t), d(z, t)d(z, z), \frac{1}{2}[d(z, z)d(z, t) + d(z, z)d(z, t)]\Big\} = 0.$$

Solving, we get $pd^3(z,t) \leq 0$. Since $z \neq t$, therefore, $pd^3(z,t) < 0$, but this is a contradiction, hence, $z = t$, i.e., $\lim_{k\to\infty} Tv_k = z$, which shows that the pairs (A', S) and (A'', T) share the property $(CLR_{A'A''})$. Hence the proof. □

Remark 2.1 Converse of the above Lemma 2.1 is not true in general,(see Example 3.5, [10]).

Theorem 2.5 *Let (E,d) be a metric space. Let $A_i(i = 1\ldots 2n)$, S and T be self mappings of E satisfying the conditions (C_2) and (C_5) of Theorem 2.2. If the pairs (S, A') and (T, A'') enjoy the property $(CLR_{A'A''})$, then the pairs (S, A') and (T, A'') have a coincidence point each. Moreover, the aforementioned mappings have a unique common fixed point in E, if the pairs (S, A') and (T, A'') are weakly compatible.*

Proof: Since the pairs (S, A') and (T, A'') enjoy the property $(CLR_{A'A''})$, there exists sequences $\{u_k\}$ and $\{v_k\}$ in E such that $\lim_{k\to\infty} A'u_k = \lim_{k\to\infty} Su_k = \lim_{n\to\infty} A''v_k = \lim_{k\to\infty} Tv_k = z$, $z \in A'(E) \cap A''(E)$. Also $z \in A'(E)$ implies the existence of a point $w \in E$ such that $A'w = z$. We claim that $Sw = A'w$. Putting $u = w$, $v = v_k$ in (C_5) and taking limit as $k \to \infty$, we get

$$[1 + pd(z,z)]d^2(Sw,z) \leq p\psi\big(0,0,0,0\big) + m(z,z) - \phi(m(z,z)),$$

where

$$m(z,z) = \max\Big\{d^2(z,z), d(z,Sw)d(z,z), d(z,z)d(z,Sw), \frac{1}{2}[d(z,Sw)d(z,z) + d(z,Sw)d(z,z)]\Big\} = 0.$$

Solving the above inequality, we have, $d^2(Sw,z) = 0$, implies that $Sw = z$. Therefore, w is a coincidence point of A' and S. As $z \in A''(E)$, so there exists a point $x \in E$ such that $A''x = z$. We claim that $A''x = Tx$, for this, replacing $u = w$, $v = x$ in (C_5), we get

$$[1 + pd(z,z)]d^2(z,Tx) \leq p\psi\big(0,0,0,0\big) + m(z,z) - \phi(m(z,z)),$$

where

$$m(z,z) = \max\Big\{d^2(z,z), d(z,z)d(z,Tx), d(z,Tx)d(z,z), \frac{1}{2}[d(z,z)d(z,Tx) + d(z,z)d(z,Tx)]\Big\} = 0.$$

Simplifying the above inequality, we have, $pd^3(z,Tx) \leq 0$, which holds only if $Tx = z$. So, $A''x = Tx = z$, i.e.,x is a coincidence point of A'' and T. Since the pairs (A', S) and (A'', T) are weakly compatible and w and x are coincidences point of each respectively, therefore, we have $A'z = A'Sw = SA'w = Sz$ and $A''z = A''Tx = TA''x = Tz$. Now, we claim that $A'z = z$. Taking $u = z$ and $v = x$ in (C_5), we have

$$[1 + pd(A'z,z)]d^2(Sz,z) \leq p\psi\big(0,0,0,0\big) + m(A'z,z) - \phi(m(A'z,z)),$$

where

$$m(A'z,z) = \max\Big\{d^2(A'z,z), d(A'z,Sz)d(z,z), d(A'z,z)d(z,Sz), \frac{1}{2}[d(A'z,Sz)d(A'z,z) + d(z,Sz)d(z,z)]\Big\} = d^2(A'z,z).$$

After simplification, we conclude that $d(A'z,z) = 0$, which implies that $A'z = z$. Hence, $Sz = A'z = z$. Next, we claim that $A''z = z$. For this, substituting $u = w, v = z$ in (C_5), we have

$$[1 + pd(z,A''z)]d^2(z,Tz) \leq p\psi\big(0,0,0,0\big) + m(z,A''z) - \phi(m(z,A''z)),$$

where

$$m(z, A''z) = \max\Big\{d^2(z, A''z), d(z,z)d(A''z, Tz), d(z, Tz)d(A''z, z),$$
$$\frac{1}{2}[d(z,z)d(z,Tz) + d(A''z, z)d(A''z, Tz)]\Big\} = d^2(z, A''z).$$

After simplification, we get $pd^3(z, A''z) + \phi(d^2(z, A''z)) \leq 0$, which holds only for $A''z = z$ and hence, $z = A''z = Tz$. Now, taking $u = A_4 \dots A_{2n}z$, $v = z$ in (C_5), applying condition (C_2) and following the step (f) of Theorem 2.2, we get $Sz = A_2z = A_4z = ... = A_{2n}z = z$. Next, taking $u = z$ and $v = A_3 \cdots A_{2n-1}z$ in (C_5) and applying condition (C_2) and following the step (g) of Theorem 2.2, we have, $Tz = A_1z = A_3z = ... = A_{2n-1}z = z$. Uniqueness follows easily. Thus, w is a unique common fixed point of S, T,A_1,A_2,..., A_{2n-1} and A_{2n}. □

Theorem 2.6 *Let S, T and $A_i(i = 1 \dots 2n)$ be self mappings of a metric space (E, d) satisfying the conditions (C_1), (C_2) and (C_5) of Theorem 2.2 and conditions $(C_7) - (C_9)$ of Lemma 2.1. If the pairs (S, A') and (T, A'') are weakly compatible, then the above-stated mappings have a unique common fixed point in E.*

Proof: It follows from Lemma 2.1 that the pairs (S, A') and (T, A'') satisfy the property $(CLR_{A'A''})$, hence all the conditions of the Theorem 2.5 are satisfied and both the pairs (S, A') and (T, A'') are weakly compatible, therefore, S, T,A_1,A_2,..., A_{2n-1} and A_{2n} have a unique common fixed point in E. □

Now, we present slight generalized form of the above stated Theorems.

Theorem 2.7 *Let (E, d) be a metric space and let $\{S_\lambda\}_{\lambda\in\Lambda}$ and $A_j(j = 1 \dots 2n)$ be two families of self mappings of E. Suppose there exists a fixed $\alpha \in \Lambda$ such that:*

(C_{10}) *$S_\alpha(E) \subset A''(E)$ for some $\alpha \in \Lambda$ and $S_\lambda(E) \subset A'(E)$ for each $\lambda \in \Lambda, \lambda \neq \alpha$;*

(C_{11}) $A_1(A_3 \cdots A_{2n-1}) = (A_3 \cdots A_{2n-1})A_1$,
$A_1A_3(A_5 \cdots A_{2n-1}) = (A_5 \cdots A_{2n-1})A_1A_3$,
...
$A_1A_3 \cdots A_{2n-3}(A_{2n-1}) = (A_{2n-1})A_1A_3 \cdots A_{2n-3}$;
$S_\lambda(A_3 \cdots A_{2n-1}) = (A_3 \cdots A_{2n-1})S_\lambda$,
$S_\lambda(A_5 \cdots A_{2n-1}) = (A_5 \cdots A_{2n-1})S_\lambda$,
...
$S_\lambda A_{2n-1} = A_{2n-1}S_\lambda$;
$A_2(A_4 \cdots A_{2n}) = (A_4 \cdots A_{2n})A_2$,
$A_2A_4(A_6 \cdots A_{2n}) = (A_6 \cdots A_{2n})A_2A_4$,
...
$A_2A_4 \cdots A_{2n-2}(A_{2n}) = (A_{2n})A_2A_4 \cdots A_{2n-2}$;
$S_\alpha(A_4 \cdots A_{2n}) = (A_4 \cdots A_{2n})S_\alpha$,
$S_\alpha(A_6 \cdots A_{2n}) = (A_6 \cdots A_{2n})S_\alpha$,
...
$S_\alpha A_{2n} = A_{2n}S_\alpha$;

(C_{12}) *one of the subspaces $S_\alpha(E)$ or $S_\lambda(E)$ or $A'(E)$ or $A''(E)$ is complete;*

(C_{13}) *the pairs (S_λ, A'') and (S_α, A') are weakly compatible;*

(C_{14}) *for* $\psi \in \Psi$, $\phi \in \Phi$, *real number* $p > 0$ *and for all* $u, v \in E$,

$$
\begin{aligned}
[1 + pd(A'u, A''v)]d^2(S_\alpha u, S_\lambda v) \leq p\psi\Big(& d^2(A'u, S_\alpha u)d(A''v, S_\lambda v), d(A'u, S_\alpha u)d^2(A''v, S_\lambda v), \\
& d(A'u, S_\alpha u)d(A'u, S_\lambda v)d(A''v, S_\alpha u), \\
& d(A'u, S_\lambda v)d(A''v, S_\alpha u)d(A''v, S_\lambda v)\Big) \\
& + m(A'u, A''v) - \phi(m(A'u, A''v)),
\end{aligned}
$$

where

$$
\begin{aligned}
m(A'u, A''v) = \max\Big\{ & d^2(A'u, A''v), d(A'u, S_\alpha u)d(A''v, S_\lambda v), d(A'u, S_\lambda v)d(A''v, S_\alpha u), \\
& \frac{1}{2}[d(A'u, S_\alpha u)d(A'u, S_\lambda v) + d(A''v, S_\alpha u)d(A''v, S_\lambda v)]\Big\}.
\end{aligned}
$$

Then all S_λ *and* A_j *have a unique common fixed point in* E.

Proof: Let $S_{\lambda_0} \in \{S_\lambda\}_{\lambda\in\Lambda}$ be fixed. By taking $S = S_\alpha$, $T = S_{\lambda_0}$ and applying Theorem 2.2, it follows that there exists some $z \in E$ such that $S_\alpha z = S_{\lambda_0} z = A_1 z = A_2 z = z = A_3 z = ... = A_{2n} z = z$.
Let $\lambda \in \Lambda$ be arbitrary. Then, by taking $u = v = z$ in (C_{14}), we get

$$[1 + pd(z, z)]d^2(z, S_\lambda z) \leq p\psi(0, 0, 0, 0) + 0 - \phi(0).$$

Simplifying it, we get $S_\lambda z = z$. Since λ was arbitrary, therefore $S_\lambda z = z$, for each $\lambda \in \Lambda$. Uniqueness follows easily. Thus, all S_λ and A_j have a unique common fixed point in E. □

Replacing the completeness of the above-mentioned subspaces in Theorem 2.7, the following results are obtained.

Theorem 2.8 *Let* $\{S_\lambda\}_{\lambda\in\Lambda}$ *and* $A_j(j = 1 \ldots 2n)$ *be two families of self mappings of a complete metric space* (E, d) *and* $\alpha \in \Lambda$ *be fixed such that conditions* $(C_{10}), (C_{11}), (C_{13})$ *and* (C_{14}) *of Theorem 2.7 are satisfied. If one of the subspaces* $S_\alpha(E)$ *or* $S_\lambda(E)$ *or* $A'(E)$ *or* $A''(E)$ *is closed, then all* S_λ *and* A_j *have a unique common fixed point.*

Theorem 2.9 *Let* $\{S_\lambda\}_{\lambda\in\Lambda}$ *and* $A_j(j = 1 \ldots 2n)$ *be two families of self mappings of a metric space* (E, d). *Let* $\alpha \in \Lambda$ *be fixed such that conditions* $(C_{11}), (C_{13})$ *and* (C_{14}) *of Theorem 2.7 are satisfied. If the pairs* (S_λ, A'') *and* (S_α, A') *enjoy the property* $(CLR_{A'A''})$, *then all* S_λ *and* A_j *have a unique common fixed point.*

Remark 2.2 Theorems 2.7 - 2.9 generalize the result of Ćirić *et al.* [7,8] and Razani and Yazadi [25] for family of mappings.

Remark 2.3 Taking $n = 2$, Theorems 2.2- 2.6 present a generalized version of Theorem 1.2 for six mappings.

Remark 2.4 Taking $n = 1$ in Theorems 2.2- 2.6, we get following extended and generalized versions of the results of Ćirić [6], Chugh and Kumar [9], Jain and Kumar [11] , Jain *et al.* [13,15], Jungck [16], Kang *et al.* [19], Murthy and Prasad [24] and Zhang and Song [30].

Theorem 2.10 *Let* (E, d) *be a complete metric space and* S, T, A_1 *and* A_2 *be four self mappings of* E *satisfying the following conditions*

(C_1*) $S(E) \subset A_1(E)$, $T(E) \subset A_2(E)$;

(C_2*) *$TA_1 = A_1T$, $SA_2 = A_2S$;*

(C_3*) *one of the subspace $S(E)$ or $T(E)$ or $A_1(E)$ or $A_2(E)$ is complete;*

(C_4*) *pairs (S, A_2) and (T, A_1) are weakly compatible;*

(C_5*) *for all $u, v \in E$, there exist functions $\phi \in \Phi$ and $\psi \in \Psi$ with a positive real number p such that*

$$[1 + pd(A_2u, A_1v)]d^2(Su, Tv) \leq p\psi\Big(d^2(A_2u, Su)d(A_1v, Tv), d(A_2u, Su)d^2(A_1v, Tv), d(A_2u, Su)d(A_2u, Tv)d(A_1v, Su), d(A_2u, Tv)d(A_1v, Su)d(A_1v, Tv)\Big) + m(A_2u, A_1v) - \phi(m(A_2u, A_1v)),$$

where

$$m(A_2u, A_1v) = \max\Big\{d^2(A_2u, A_1v), d(A_2u, Su)d(A_1v, Tv), d(A_2u, Tv)d(A_1v, Su), \frac{1}{2}[d(A_2u, Su)d(A_2u, Tv) + d(A_1v, Su)d(A_1v, Tv)]\Big\}.$$

Then S, T, A_1 and A_2 have a unique common fixed point in E.

Theorem 2.11 *Let S, T, A_1 and A_2 be four self mappings of a complete metric space (E, d) satisfying the conditions (C_1*), (C_2*), (C_4*), (C_5*) and*

(C_6*) *one of the subspace $S(E)$ or $T(E)$ or $A_1(E)$ or $A_2(E)$ is closed.*

Then S, T, A_1 and A_2 have a unique common fixed point in E.

Theorem 2.12 *Let S, T, A_1 and A_2 be four self mappings of a metric space (E, d) satisfying the conditions (C_1*), (C_2*), (C_4*), (C_5*) and (C_6*). If the pairs (S, A_2) and (T, A_1) satisfy the common property (E.A), then S, T, A_1 and A_2 have a unique common fixed point in E.*

3. Application

Throughout this section, we assume that U and V are Banach spaces, $\hat{S} \subseteq U$ and $D \subseteq V$ are state and decision spaces respectively. Let $\mathbb{R}$ denote the field of real numbers and $B(\hat{S})$ denotes the set of all bounded real valued functions on S.

Bellman and Lee [4] presented the basic form of functional equation of dynamic programming as follows:

$$h(u) = opt_v\, G(u, v, h(\tau(u, v))),$$

where u and v are the state and decision vectors respectively, τ is the transformation of the process and $h(u)$ is the optimal return with initial state u and *opt* denotes max or min.

As an application of Theorem 2.11, we investigate the existence and uniqueness of a common solution of the following functional equations arising in dynamic programming.

$$h_i(u) = \sup_{v \in D} G_i(u, v, h_i(\tau(u, v))), u \in \hat{S}, \tag{3.1}$$

$$k_i(u) = \sup_{v \in D} F_i(u, v, k_i(\tau(u, v))), u \in \hat{S}, \tag{3.2}$$

where $\tau : \hat{S} \times D \to S$ and $G_i, F_i : \hat{S} \times D \times \mathbb{R} \to \mathbb{R}, i = 1, 2$.

Define P_i and Q_i as follows

$$\begin{aligned} P_i f(u) &= \sup_{v \in D} F_i(u, v, f(\tau(u, v))), u \in \hat{S}, \\ Q_i g(u) &= \sup_{v \in D} G_i(u, v, g(\tau(u, v))), u \in \hat{S}, \end{aligned} \tag{3.3}$$

for all $u \in \hat{S}$; $f, g \in B(\hat{S})$, $i = 1, 2$.

Theorem 3.1 *Suppose that the following conditions hold:*

(D_1) *G_i and F_i are bounded for $i = 1, 2$.*

(D_2) *For sequences $\{f_n\}$, $\{g_n\} \subset B(\hat{S})$ and $f, g \in B(\hat{S})$ with*

$$\lim_{n\to\infty} \sup_{u\in\hat{S}} |f_n(u) - f(u)| = 0 \text{ and } \lim_{n\to\infty} \sup_{u\in\hat{S}} |g_n(u) - g(u)| = 0,$$

there exists $f_i, g_i \in B(\hat{S})$ such that $g = P_2 f_i$ and $f = P_1 g_i$, for $i = 1$ or 2.

(D_3) *For any $f \in B(\hat{S})$, there exists $g_1, g_2 \in B(\hat{S})$ such that $Q_1 f(u) = P_2 g_2(u)$ and $Q_2 f(u) = P_1 g_1(u)$, $u \in \hat{S}$.*

(D_4) *For any $f, g \in B(\hat{S})$, $P_1 f = Q_1 f$ implies that $Q_1 P_1 f = P_1 Q_1 f$ and $P_2 g = Q_2 g$ implies that $P_2 Q_2 g = Q_2 P_2 g$.*

(D_5) *For all $(u, v) \in \hat{S} \times D$, $f, g \in B(\hat{S}), t \in \hat{S}$ such that*

$$\begin{aligned}\big|G_1(u, v, f(t)) - G_2(u, v, g(t))\big|^2 \leq M^{-1}\bigg(p\,\psi\Big(&d^2(P_2 f, Q_2 f) d(P_1 g, Q_1 g),\\ &d(P_2 f, Q_2 f) d^2(P_1 g, Q_1 g), d(P_2 f, Q_2 f) d(P_2 f, Q_1 g) d(P_1 g, Q_2 f),\\ &d(P_2 f, Q_1 g) d(P_1 g, Q_2 f) d(P_1 g, Q_1 g)\Big) + m(P_2 f, P_1 g) - \phi(m(P_2 f, P_1 g))\bigg),\end{aligned}$$

where

$$\begin{aligned}m(P_2 f, P_1 g) = \max\big\{&d^2(P_2 f, P_1 g), d(P_2 f, Q_2 f) d(P_1 g, Q_1 g), d(P_2 f, Q_1 g) d(P_1 g, Q_2 f),\\ &\frac{1}{2}[d(P_2 f, Q_2 f) d(P_2 f, Q_1 g)] + d(P_1 g, Q_2 f) d(P_1 g, Q_1 g)\big\},\end{aligned}$$

$M = \big[1 + p \sup_{u\in\hat{S}} |P_2 f(u) - P_1 g(u)|\big]$, $\phi \in \Phi$, $\psi \in \Psi$, p is a positive real number and the mappings P_1, P_2, Q_1 and Q_2 are defined as in (3.3).

Then the system of functional equations given by (3.1) and (3.2) have a unique common solution in $B(\hat{S})$.

Proof: Let $d(h, k) = \sup_{u\in\hat{S}} |h(u) - k(u)|$, for any $h, k \in B(\hat{S})$. Obviously, $(B(\hat{S}), d)$ is a complete metric space. From conditions $(D_1) - (D_4)$, P_i, Q_i are self mappings of $B(\hat{S})$, for $i = 1, 2$, $Q_1(B(\hat{S})) \subset P_2(B(\hat{S}))$ and $Q_2(B(\hat{S})) \subset P_1(B(\hat{S}))$, $P_1(B(\hat{S}))$ or $P_2(B(\hat{S}))$ is closed subspace and the pairs (P_i, Q_i) are weakly compatible for $i = 1, 2$. For $\eta > 0$, $u \in \hat{S}$ and $g_1, g_2 \in B(\hat{S})$, there exists $v_1, v_2 \in D$ such that

$$Q_i g_i(u) < G_i(u, v_i, g_i(u_i)) + \eta, \tag{3.4}$$

where $u_i = \tau(u, v_i), i = 1, 2$. Also, we have

$$Q_1 g_1(u) \geq G_1(u, v_2, g_1(u_2)), \tag{3.5}$$

$$Q_2 g_2(u) \geq G_2(u, v_1, g_2(u_1)). \tag{3.6}$$

From (3.4),(3.6) and (D_5), we have

$$\begin{aligned}(Q_1 g_1(u) - Q_2 g_2(u))^2 <&(G_1(u, v_1, g_1(u_1)) - G_2(u, v_1, g_2(u_1))) + \eta)^2\\ =&(G_1(u, v_1, g_1(u_1)) - G_2(u, v_1, g_2(u_1)))^2 + \xi\\ &\leq M^{-1}\bigg(p\,\psi\Big(d^2(P_2 g_1, Q_2 g_1) d(P_1 g, Q_1 g_2), d(P_2 g_1, Q_2 g_1) d^2(P_1 g_2, Q_1 g_2),\\ &d(P_2 g_1, Q_2 g_1) d(P_2 g_1, Q_1 g_2) d(P_1 g_2, Q_2 g_1),\\ &d(P_2 g_1, Q_1 g_2) d(P_1 g_2, Q_2 g_1) d(P_1 g_2, Q_1 g_2)\Big) +\\ &m(P_2 g_1, P_1 g_2) - \phi(m(P_2 g_1, P_1 g_2))\bigg) + \xi,\end{aligned} \tag{3.7}$$

where $\xi = \eta^2 + 2\eta(G_1 - G2)$. From (3.4), (3.5) and (D_5), we have

$$\begin{aligned}(Q_1g_1(u) - Q_2g_2(u))^2 >& (G_1(u, v_2, g_1(u_2)) - G_2(u, v_2, g_2(u_2)) - \eta)^2 \\ &= (G_1(u, v_2, g_1(u_2)) - G_2(u, v_2, g_2(u_2))^2\xi_1 \\ &\geq -M^{-1}\Big(p\,\psi\Big(d^2(P_2g_1, Q_2g_1)d(P_1g, Q_1g_2), d(P_2g_1, Q_2g_1)d^2(P_1g_2, Q_1g_2), \\ &d(P_2g_1, Q_2g_1)d(P_2g_1, Q_1g_2)d(P_1g_2, Q_2g_1), \\ &d(P_2g_1, Q_1g_2)d(P_1g_2, Q_2g_1)d(P_1g_2, Q_1g_2)\Big)+ \\ &m(P_2g_1, P_1g_2) - \phi(m(P_2g_1, P_1g_2))\Big) - \xi,\end{aligned} \tag{3.8}$$

$\xi_1 = \eta^2 - 2\eta(G_1 - G2) < \xi$. From (3.7) and (3.8), we obtain

$$\begin{aligned}|Q_1g_1(u) - Q_2g_2(u)|^2 \leq M^{-1}\Big(p\,\psi\Big(&d^2(P_2g_1, Q_2g_1)d(P_1g, Q_1g_2), d(P_2g_1, Q_2g_1)d^2(P_1g_2, Q_1g_2), \\ &d(P_2g_1, Q_2g_1)d(P_2g_1, Q_1g_2)d(P_1g_2, Q_2g_1), d(P_2g_1, Q_1g_2)d(P_1g_2, Q_2g_1)d(P_1g_2, Q_1g_2)\Big)+ \\ &m(P_2g_1, P_1g_2) - \phi(m(P_2g_1, P_1g_2))\Big) + \xi,\end{aligned} \tag{3.9}$$

As $\eta > 0$ is arbitrary and (3.9) is true for all $u \in \hat{S}$, taking supremum, we get

$$\begin{aligned}[1 + pd(P_1g_1,P_2g_2)]d^2(Q_1g_1, Q_2g_2) \leq\ p\,\psi\Big(&d^2(P_2g_1, Q_2g_1)d(P_1g, Q_1g_2), d(P_2g_1, Q_2g_1)d^2(P_1g_2, Q_1g_2), \\ &d(P_2g_1, Q_2g_1)d(P_2g_1, Q_1g_2)d(P_1g_2, Q_2g_1), d(P_2g_1, Q_1g_2)d(P_1g_2, Q_2g_1)d(P_1g_2, Q_1g_2)\Big)+ \\ &m(P_2g_1, P_1g_2) - \phi(m(P_2g_1, P_1g_2))\Big).\end{aligned} \tag{3.10}$$

Therefore, Theorem 2.11 applies, where P_1, P_2, Q_1, Q_2 correspond to the mappings A_1, A_2, S, T respectively. So, P_1, P_2, Q_1 and Q_2 have a unique common fixed point $g^* \in B(\hat{S})$, i.e., $g^*(u)$ is a unique common solution of the system of functional equations (3.1) and (3.2). □

Acknowledgments

The authors thank the referees for their valuable suggestions and comments.

References

1. M. Aamri, D.EI Moutawakil, Some new common fixed point theorems under strict contractive conditions, Math. Anal. Appl., 270, 181-188, (2002).

2. Y. I. Alber, S. Guerre-Delabriere, Principle of weakly contractive maps in Hilbert spaces, New Results in Operator Theory Adv. Appl., 98, 7-22, (1997).

3. S. Banach, Surles operations dans les ensembles abstraites et leursapplications, Fundam. Math., 3, 133-181, (1922).

4. R. Bellman, B.S. Lee, Functional equations arising in dynamic programming, Aequationes Math., 17, 1-18, (1978).

5. D. W. Boyd, J. S. W. Wong, On nonlinear contractions, Proc. Amer. Math. Soc., 20(2), 458-464, (1969).

6. Lj. B. Ćirić, Generalized contractions and fixed point theorems, Publ. Inst. Math., 26, 19-26, (1971).

7. Lj. B. Ćirić, On a family of contractive maps and fixed points, Publ. Inst. Math., 31, 45-51, (1974).

8. Lj. B. Ćirić, A. Razani, S. Radenović and J.S. Ume, Common fixed point theorems for families of weakly compatible maps, Comput. Math. Appl., 55, 2533-2543, (2008).

9. R. Chugh, S. Kumar, Common fixed point for weakly compatible maps, Proc. Indian Acad. Sci.(Math. Sci.),111(2), 241-247, (2001).

10. M. Imdad, B.D. Pant, S. Chauhan, Fixed point theorems in Menger spaces using the CLR_{ST} property and applications, J. Nonlinear Anal. Optim., 3(2), 225-237, (2012).

11. D. Jain, S. Kumar, Weakly compatible mappings along with (JCLR) property satisfying a $\phi-$weak contraction, J. Nonlinear Anal. App., 2019(2), 23-28, (2019).

12. D. Jain, S. Kumar, Intimate mappings satisfying generalized $\phi-$ weak contraction involving cubic terms of metric function, J. Nonlinear Anal. Appl.,2018(2), 192-200.

13. D. Jain, S. Kumar, S. M. Kang, Weak contraction condition for mappings involving cubic terms of the metric function, Int. J. Pure Appl. Math., 116, 1115-1126, (2017).

14. D. Jain, S. Kumar, B. E. Rhoades, Common fixed points for $R-$weakly commuting mappings satisfying a weak contraction, Res. Fixed Point Theory Appl., 2019, Article ID 2019003, 18 pp, (2019).

15. D. Jain, S. Kumar, S. M. Kang and C. Jung, Weak contraction condition for compatible mappings involving cubic terms of the metric function, Far East J. Math. Sci., 103(4), 799-818, (2018).

16. G. Jungck, Commuting mappings and fixed points, Amer. Math. Monthly, 83, 261-263, (1976).

17. G. Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci., 9, 771-779, (1986).

18. G. Jungck, Common fixed points for non continuous non-self maps on non-metric spaces, Far East J. Math. Sci., 4(2), 199-215, (1996).

19. S. M. Kang, Y. J. Cho, G. Jungck, Common fixed points of compatible mappings, Int. J. Math. Math. Sci., 13(1), 61-66, (1990).

20. Kavita, Sanjay Kumar, Fixed points for Intimate mappings, J. Math. Comput. Sci., 12, Paper No. 48, (1990).

21. M. S. Khan, M. Swalek, S. Sessa, Fixed point theorems by altering distances between two points, Bull. Austral. Math. Soc., 30, 1-9, (1984).

22. R. Kumar, S. Kumar, Fixed points for weak contraction involving cubic terms of distance function, J. Math. Comput. Sci., 11, 1922-1954, (2021).

23. Y. Liu, J. Wu, Z. Li, Common fixed point of single valued and multivalued maps, Int. J. Math. Math. Sci., 19, 3045-3055, (2005).

24. P. P. Murthy, K. N. V. V. V. Prasad, Weak contraction condition involving cubic terms of $d(x, y)$ under the fixed point consideration, J.Math., Article ID 967045,5 pp, (2013). DOI 10.1155/2013/967045.

25. A. Razani, M. yazdi, Two common fixed point theorems for compatible mappings, Int. J. Nonlinear Anal. Appl., 2(2), 7-18, (2011).

26. B. E. Rhoades, Some theorems on weakly contractive maps, Nonlinear Anal., 47(4), 2683-2693, (2001).

27. S. Sessa, On a weak commutativity conditions of mappings in fixed point considerations, Publ. Inst. Math., 32(46), 149-153, (1982).

28. B. Singh, S. Jain, A fixed point theorem in Menger space through weak compatibility, J. Math. Anal. Appl., 301, 439-448, (2005).

29. W. Sintunavarat, P. Kumam, Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces, J. Appl. Math., 2011, 1–14, (2011).

30. Q. Zhang, Y. Song, Fixed point theory for generalized $\phi-$weak contractions, Appl. Math. Lett., 22, 75-78, (2009).

Kavita,
Department of Mathematics
Deenbandhu Chhotu Ram University of Science & Technology
Murthal, Sonepat-131039, Haryana(India).
E-mail address: kvtlather@gmail.com

and

Sanjay Kumar,
Department of Mathematics
Deenbandhu Chhotu Ram University of Science & Technology
Murthal, Sonepat-131039, Haryana(India).
E-mail address: sanjaymudgal2004@yahoo.com