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Decompositions of $P_m \odot P_n$ into Cycles, Paths and Claws

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ABSTRACT: In this article, we study and examine the decomposition of $P_m \odot P_n$ into C_n, P_m and $K_{1,3}$; where $P_m \odot P_n$ denotes the corona product graph of two paths P_m and P_n with $m + mn$ vertices and $m + m(n-1) + mn - 1$ edges. Specifically, we provide a thorough solution to the issue in the scenario when $m, n \geq 2$.

Key Words: Corona product of graph; decomposition of graph.

Contents

1. Introduction

A decomposition of a graph G is a collection of edge-disjoint subgraphs of $H_1, H_2, ..., H_r$, where each edge of G belongs to exactly one H_i. By placing conditions on the decomposition, multiple authors have explored various forms of decompositions and their accompanying properties. Every graph allows for a decomposition in which each subgraph H_i as a path, a cycle, a claw, and etc. It is clear that one of the obvious requirements for decomposition of G is that $\sum_{i=1}^{r} \alpha_i e(H_i) = e(G)$ exists. For ease of use, we refer to the equation $\sum_{i=1}^{r} \alpha_i e(H_i) = e(G)$ as a necessary sum condition.

In the past decade, the decomposition of graphs become an active area of research in graph theory. It is the most prominent area of research in graph theory and combinatorics and further, it has numerous applications in various fields such as networking, block designs, and bio-informatics, for instance, see [16,1,8,9] and the book [4] for a comprehensive overview of the G-decomposition of graphs. In the decomposition, researchers are particularly interested into k-cycles, for instance, see [5,6]. Recently, a good deal of interest has also been shown in decomposition into k-stars, for instance, see [7,11,18].

The corona product graph of two paths P_m and P_n with $m+mn$ vertices and $m+m(n-1)+mn-1$ edges is denoted by $P_m \odot P_n$, where m and n are any positive integers (cf [10,12]). A complete bipartite graph $K_{1,n}$ is known as an n-star, denoted by S_n. The tree is referred to as 'claw', and it serves as a representation of the complete bipartite graph $K_{1,3}$. A cycle of length n is referred to as an n-cycle and is symbolized by the symbol C_n. All the graphs we investigate here are finite and undirected, unless otherwise noted. The reader is directed to [3] for a glossary of common graph-theoretic terms, while [2,13,14,15] are references for studying the decomposition of graphs into paths, stars, and cycles. In this article, we study and determine the decomposition $D(P_m \odot P_n)$ of the corona product graph $P_m \odot P_n$ into cycles, paths and claws.

2. Decomposition of corona product graph $P_m \odot P_n$

Before, studying our results, we define the corona product of the graph as follows: The corona product graph of P_m and P_n are obtained by taking one copy of P_m and $|V(P_n)|$ copies of P_n and joining the $i-$th vertex of P_m to every vertex in the $i-$th copy of P_n, and the vertices are of the

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form $V(P_m \odot P_n) = \{u_1, u_2, u_3, ..., u_m, v_1, v_2........, v_n\}$ and the edges are of the form $E(P_m \odot P_n) = \{e_p = u_p u_{p+1}, e_q = u_1 v_q, e_q^{'} = u_2 v_q, e_q^{''} = u_3 v_q, ..., e_q^{m-1} = u_m v_i, e_r = v_r v_{r+1}\}$, where $q = \{1, 2, ..., n\}$, $p = \{1, 2, ..., m-1\}$, $r = \{1, 2, ..., n-1\}$. Now, we start with the following result.

Theorem 2.1 *Any corona product graph $P_m \odot P_n$, where $m \geq 2$, $n \geq 2$ can be decomposed into the following ways:*

$$D(P_m \odot P_n) = \begin{cases} P_m,\ \frac{mn}{2}C_3 \text{ and } (\frac{m-2}{2})nP_2, & \text{if } m,n \text{ is even,} \\ & m \geq 2, n \geq 2 \\ 3C_3 \text{ and } 4P_3, & \text{if } m = n = 3 \\ P_m,\ m\lfloor\frac{n}{2}\rfloor C_3,\ mP_3 \text{ and } m(\frac{n-3}{2})P_2, & \text{if } m,n \text{ is odd,} \\ & m > 3, n > 3 \end{cases}$$

Proof: Let $P_m \odot P_n$ be the corona product of P_m and P_n and $D(P_m \odot P_n)$ be the decompostion of $P_m \odot P_n$. Then the following cases complete the proof:

case 1. If m, n is even and $m \geq 2, n \geq 2$. Then we assume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_r\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_r\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_r\}$, where $q = \{1, 3, 5, ..., n-1\}, r = \{1, 3, 5, .., n-1\}$, $G_r = \{e_r\}$, $G_r^{'} = \{e_r\}$, ..., $G_r^{m-1} = \{e_r\}$, where $r = \{2, 4, 6, ..., n-2\}$. Following that the subgraph $< F >$ generates a single path P_m with length $m-1$, the subgraph $< H_q >$ generates a single cycle C_3 with length three, the subgraph $< H_q^{'} >$ generates a single cycle C_3 with length three, and it continued until the subgraph $< H_q^{m-1} >$ generates a single cycle C_3 with length three, the subgraph $< G_r >$ generates $\frac{n-2}{2}$ copies of paths P_2 with length one, the subgraph $< G_r^{'} >$ generates $\frac{n-2}{2}$ copies of paths P_2 with length one, and this process continues until the subgraph $< G_r^{m-1} >$ generates $\frac{n-2}{2}$ copies of paths P_2 with length one. Therefore, we conclude that $D(P_m \odot P_n)$ contains a single copy of P_m, $\dfrac{mn}{2}$ copies of C_3 and $\dfrac{m-2}{2}n$ copies of P_2.

case 2. If m, n is odd and $m = n = 3$. Then we assume that $F = \{e_p, e_{p+1}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_r\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_r\}$, where $q = \{1\}, r = \{1\}$, $E = \{e_q, e_r\}$, $E_1 = \{e_q^{'}, e_r\}$, $E_2 = \{e_q^{''}, e_r\}$, $E_3 = \{e_q^{'''}, e_r\}$, where $q = n, r = n-1$. Following that the subgraph $< F >$ generates a single path P_3 with length two, the subgraph $< H_q >$ generates a single cycle C_3 with length three, the subgraph $< H_q^{'} >$ generates a single cycle C_3 with length three, the subgraph $< H_q^{''} >$ generates a single cycle C_3 with length three, the subgraph $< E >$ generates a single path P_3 with length two, the subgraph $< E_1 >$ generates a single path P_3 with length two, the subgraph $< E_3 >$ generates a single path P_3 with length two. Therefore, we conclude that $D(P_m \odot P_n)$ contains 3 copies of C_3 and 4 copies of P_4.

case 3. If m, n is odd and $m > 3, n > 3$. Then we assume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_r\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_r\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_r\}$, where $q = r = \{1, 3, 5, ..., n-1\}$, $E = \{e_q, e_r\}$, $E_1 = \{e_q^{'}, e_r\}$,....,$H_{m-1} = \{e_q^{m-1}, e_r\}$, where $q = n, r = n-1$, $G_r = \{e_r\}$, $G_r^{'} = \{e_r\}$, ..., $G_r^{m-1} = \{e_r\}$, where $r = \{2, 4, 6..., n-3\}$. Following that the subgraph $< F >$ generates a single path P_m with length $m-1$, the subgraph $< H_q >$ generates $\lfloor\frac{n}{2}\rfloor$ copies of cycles C_3 with length three, the subgraph $< H_q^{'} >$ creates $\lfloor\frac{n}{2}\rfloor$ copies of cycles C_3 with length three, the subgraph $< H_q^{''} >$ generates $\lfloor\frac{n}{2}\rfloor$ copies of cycles C_3 of length three, and this process continues until the subgraph $< H_q^{m-1} >$ generates $\lfloor\frac{n}{2}\rfloor$ copies of cycles C_3 with length three, the subgraph $< E >$ generates a single path P_3 with length two, the subgraph $< E_1 >$ generates a single path P_3 with length two, the subgraph $< E_2 >$ generates a single path P_3 with length two, the subgraph $< E_3 >$ generates a single path P_3 with length two, and this process continues until the subgraph $< E_{m-1} >$ generates a single path P_3 with length two, the subgraph $< G_r >$ generates $\frac{n-3}{2}$ copies of paths P_2 with length one, the subgraph $< G_r^{'} >$ generates $\frac{n-3}{2}$ copies of paths P_2 with length one, and this process continues until the subgraph

$< G_r^{m-1} >$ generates $\frac{n-3}{2}$ copies of paths P_2 with length one. Therefore, we conclude that $D(P_m \odot P_n)$ contains one copy of P_m, $m\lfloor \frac{n}{2} \rfloor$ copies of C_3, m copies of P_3 and $m(\dfrac{n-3}{2})$ copies of P_2. □

Theorem 2.2 *Any corona product graph $P_m \odot P_n$, where $m \geq 2,\ n \geq 2$ can be decomposed into one copy of P_m, $m(n-1)$ copies of P_3 and m copies of P_2.*

Proof: Let $P_m \odot P_n$ be the corona product of P_m and P_n and $D(P_m \odot P_n)$ be the decompostion of $P_m \odot P_n$. Then the following cases complete the proof:

Case 1. If $m = n = 2b$, where $b = 1, 2, 3, ...$ Then we asuume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_r\}$, $H_q^{'} = \{e_q^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_r\}$,...,$H_q^{m-1} = \{e_q^{m-1}, e_r\}$, where $q = r = \{1, 2, 3, 4, ..., n-1\}$, $E_1 = \{e_q\}$, $E_2 = \{e_q^{'}\}$, $E_3 = \{e_q^{''}\}$,..., $E_m = \{e_q^{m-1}\}$, where $q = \{n\}$. Following that the subgraph $< F >$ generates a single path P_m, the subgraph $< H_i >$ generates $(n-1)$ copies of paths P_3, the subgraph $< H_i^{'} >$ generates $(n-1)$ copies of paths P_3, the subgraph $< H_i^{''} >$ generates $(n-1)$ copies of paths P_3, and this process continues until the subgraph $< H_i^{m-1} >$ generates $(n-1)$ copies of paths P_3, the subgraph $< E_1 >$ generates a single path P_2, the subgraph $< E_2 >$ generates a single path P_2, the subgraph $< E_3 >$ generates a single path P_2, and this process continues until the subgraph $< E_m >$ generates a single path P_2. Therefore, we conclude that $D(P_m \odot P_n)$ contains one copy of P_m, $m(n-1)$ copies of P_3, and m copies of P_2.

Case 2. If $m = n = 2b+1$, where $b = 1, 2, 3, ...$ Then we assume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_r\}$, $H_q^{'} = \{e_q^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_r\}$,...,$H_q^{m-1} = \{e_q^{m-1}, e_r\}$, where $q = r = \{1, 2, 3, 4, ..., n-1\}$, $E_1 = \{e_q\}$, $E_2 = \{e_q^{'}\}$, $E_3 = \{e_q^{''}\}$,..., $E_m = \{e_q^{m-1}\}$, where $q = \{n\}$. Following that the subgraph $< F >$ generates a single path P_m with length $m-1$, the subgraph $< H_i >$ generates $(n-1)$ copies of path P_3, the subgraph $< H_i^{'} >$ generates $(n-1)$ copies of cycles P_3, the subgraph $< H_i^{''} >$ generates $(n-1)$ copies of path P_3, and this process continues until the subgraph $< H_i^{m-1} >$ generates $(n-1)$ copies of paths P_3, the subgraph $< E_1 >$ generates a single path P_2, the subgraph $< E_2 >$ generates a single path P_2, the subgraph $< E_3 >$ generates a single path P_2, and this process continues until the subgraph $< E_m >$ generates a single path P_2. Therefore, we conclude that $D(P_m \odot P_n)$ contains one copy of P_m, $m(n-1)$ copies of P_3, and m copies of P_2. □

Theorem 2.3 *Any corona product graph $P_m \odot P_n$, where $m \geq 3,\ n \geq 3$ can be decomposed into following ways:*

$$D(P_m \odot P_n) = \begin{cases} P_m,\ \frac{mn}{3} K_{1,3} \text{ and } mP_n, & \text{if } m = n = 3b, \\ & \text{where } b = 1, 2, 3, ... \\ P_m,\ m(\frac{n-1}{3})K_{1,3},\ mP_n \text{ and } mP_2, & \text{if } m = n = 3b+1, \\ & \text{where } b = 1, 2, 3, ... \\ P_m,\ m(\frac{n-2}{3})K_{1,3},\ mP_n \text{ and } mP_3, & \text{if } m = n = 3b+2, \\ & \text{where } b = 1, 2, 3, ... \end{cases}$$

Proof: Let $P_m \odot P_n$ be the corona product of P_m and P_n and $D(P_m \odot P_n)$ be the decompostion of $P_m \odot P_n$. Then the following cases complete the proof:

Case 1. If $m = n = 3b$, where $b = 1, 2, 3, ...$ Then we assume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_{q+2}\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_{q+2}^{'}\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_{q+2}^{''}\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_{q+2}^{m-1}\}$, where $q = \{1, 4, ..., n-2\}$, $G_1 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, $G_2 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$,...., $G_m = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, where $r = \{1\}$. Following that the subgraph $< F >$ generates a single path P_m, the subgraph $< H_q >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{'} >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{''} >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, and this process continues until the subgraph $< H_q^{m-1} >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, the subgraph $< G_1 >$ generates a single path P_n, the subgraph $< G_2 >$ generates a single path P_n, and this process continues until the subgraph $< G_m >$ generates a single path P_n. Therefore, we conclude that $D(P_m \odot P_n)$ contains one copy of P_m, $\dfrac{mn}{3}$ copies of $K_{1,3}$ and m copies of P_n.

Case 2. If $m = n = 3b + 1$, where $b = 1, 2, 3, ...$ Then we assume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_{q+2}\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_{q+2}^{'}\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_{q+2}^{''}\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_{q+2}^{m-1}\}$, where $q = \{1, 4, ..., n-2\}$, $E_1 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, $E_2 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$,...., $E_m = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, where $r = \{1\}$, $G = \{e_q\}$, $G^{'} = \{e_i^{'}\}$,..., $G_q^{m-1} = \{e_q^{m-1}\}$, where $q = \{n\}$. Following that the subgraph $< F >$ generates a single path P_m with length $m-1$, the subgraph $< H_q >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{'} >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{''} >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, and this process continues until the subgraph $< H_q^{m-1} >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, the subgraph $< E_1 >$ generates a single path P_n with length $n-1$, the subgraph $< E_2 >$ generates a single path P_n with length $n-1$, and this process continues until the subgraph $< E_m >$ generates a single path P_n with length $n-1$, the subgraph $< G >$ generates a single path P_2 with length one, the subgraph $< G_1 >$ generates a single path P_2 with length one, the subgraph $< G_2 >$ generates a single path P_2 with length one, and this process continues until the subgraph $< G_{m-1} >$ generates a single path P_2 with length one. Therefore, we conclude that $D(P_m \odot P_n)$ contains one copy of P_m, $m(\dfrac{n-1}{3})$ copies of $K_{1,3}$, m copies of P_n and m copies of P_2.

Case 3. If $m = n = 3b + 2$, where $b = 1, 2, 3, ...$ Then we assume that $F = \{e_p, e_{p+1}, ..., e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_{q+2}\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_{q+2}^{'}\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_{q+2}^{''}\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_{q+2}^{m-1}\}$, where $q = \{1, 4, ..., n-2\}$, $E_1 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, $E_2 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$,...., $E_m = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, where $r = \{1\}$, $G = \{e_q, e_{q+1}\}$, $G^{'} = \{e_q^{'}, e_{q+1}^{'}\}$, $G^{''} = \{e_q^{''}, e_{q+1}^{''}\}$,...,$G = \{e_q^{m-1}, e_{q+1}^{m-1}\}$, where $q = \{n-1\}$. Following that, the subgraph $< F >$ generates a single path P_m with length $m-1$, the subgraph $< H_q >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{'} >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{''} >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$ and this process continues until the subgraph $< H_q^{m-1} >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$, the subgraph $< E_1 >$ generates a single path P_n with length $n-1$, the subgraph $< E_2 >$ generates a single path P_n with length $n-1$, and this process continues until the subgraph $< E_m >$ generates a single path P_n with length $n-1$, the subgraph $< G >$ generates a single path P_3 with length two, the subgraph $< G_1 >$ generates a single path P_3 with length two and this process continues until the subgraph $< G_{m-1} >$ generates a single path P_3 with length two. Therefore, we conclude that $D(P_m \odot P_n)$ contains one copy of P_m, $m(\dfrac{n-2}{2})$ copies of $K_{1,3}$, m copies of P_n and m copies of P_3. □

Theorem 2.4 *Any corona product graph $P_m \odot P_n$, where $m, n \geq 2$ can be decomposed into following ways:*

$$D(P_m \odot P_n) = \begin{cases} \frac{mn}{2}C_3 \textit{ and } \frac{mn-2}{2}P_2, & \textit{if } m, n \textit{ is even,} \\ & m, n \geq 2 \\ m\lfloor \frac{n}{2} \rfloor C_3,\ mP_3 \textit{ and } \frac{m(n-1)-2}{2}P_2, & \textit{if } m, n \textit{ is odd,} \\ & m, n \geq 3 \end{cases}$$

Proof: Let $P_m \odot P_n$ be the corona product of P_m and P_n and $D(P_m \odot P_n)$ be the decompostion of $P_m \odot P_n$. Then the following cases complete the proof:

case 1. If m, n is even and $m, n \geq 2$. Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_r\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_r\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_r\}$ where $q = \{1, 3, 5, ..., n-1\}, r = \{1, 3, 5, .., n-1\}$, $G_r = \{e_r\}$, $G_r^{'} = \{e_r\}$, ..., $G_r^{m-1} = \{e_r\}$, where $r = \{2, 4, 6..., n-2\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 with length one, the subgraph $< F_2 >$ generates a single path P_2 with length one, and this process continous until the subgraph $< F_{m-1} >$ generates a single path P_2 with length one, the subgraph $< H_q >$ generates a single cycle C_3 with length three, the subgraph $< H_q^{'} >$ generates a single cycle C_3 with length three, and this process continuous until the subgraph $< H_q^{m-1} >$ generates a single cycle C_3 with length three, the subgraph $< G_r >$ generates $\frac{n-2}{2}$ path P_2 with length one, the subgraph $< G_r^{'} >$ generates $\frac{n-2}{2}$ path P_2 with length one, and this process continuous until the subgraph $< G_r^{m-1} >$ generates $\frac{n-2}{2}$ path P_2 with length one. Therefore, $D(P_m \odot P_n)$ contains $\dfrac{mn}{2}$ copies of C_3 and $\dfrac{mn-2}{2}$ copies of P_2.

case 2. If n, m is even and $n \geq 3, m \geq 3$. Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_r\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_r\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_r\}$, where $q = r = \{1, 3, 5, ..., n-1\}$, $E = \{e_q, e_r\}$, $E_1 = \{e_q^{'}, e_r\}$,....,$H_{m-1} = \{e_q^{m-1}, e_r\}$, where $q = n, r = n-1$, $G_r = \{e_r\}$, $G_r^{'} = \{e_r\}$, ..., $G_r^{m-1} = \{e_r\}$, where $r = \{2, 4, 6..., n-3\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 with length one, the subgraph $< F_2 >$ generates a single path P_2 with length one, and this process continues until the subgraph $< F_{m-1} >$ generates a single path P_2 with length one, the subgraph $< H_q >$ generates $\lfloor \frac{n}{2} \rfloor$ copies of cycles C_3 with length three, the subgraph $< H_q^{'} >$ generates $\lfloor \frac{n}{2} \rfloor$ copies of cycles C_3 with length three, the subgraph $< H_q^{''} >$ generates $\lfloor \frac{n}{2} \rfloor$ copies of cycles C_3 of length three, and this process continues until the subgraph $< H_q^{m-1} >$ generates $\lfloor \frac{n}{2} \rfloor$ copies of cycles C_3 with length three, the subgraph $< E >$ generates a single path P_3 of length two, the subgraph $< E_1 >$ generates a single path P_3 of length two, the subgraph $< E_2 >$ generates a single path P_3 of length two, the subgraph $< E_3 >$ generates a single path P_3 of length two, and this process continues until the subgraph $< E_{m-1} >$ generates a single path P_3 of length two, the subgraph $< G_r >$ generates $\frac{n-3}{2}$ copies of paths P_2 with length one, the subgraph $< G_r^{'} >$ generates $\frac{n-3}{2}$ copies of paths P_2 with length one, and this process continues until the subgraph $< G_r^{m-1} >$ generates $\frac{n-3}{2}$ copies of paths P_2 with length one. Therefore, we conclude that $D(P_m \odot P_n)$ contains $m\lfloor \dfrac{n}{2} \rfloor$ copies of C_3, m copies of P_3 and $\dfrac{m(n-1)-2}{2}$ copies of P_2. $\square$

Theorem 2.5 *Any corona product graph $P_m \odot P_n$, where $m, n \geq 2$ can be decomposed into $2m-1$ copy of P_2 and $m(n-1)$ copies of P_3.*

Proof: Let $P_m \odot P_n$ be the corona product of P_m and P_n and $D(P_m \odot P_n)$ be the decompostion of $P_m \odot P_n$. Then the following cases complete the proof:

Case 1. If $m = n = 2b$, where $b = 1, 2, 3, ...$ Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_r\}$, $H_q^{'} = \{e_q^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_r\}$,...,$H_q^{m-1} = \{e_q^{m-1}, e_r\}$, where $q = r = \{1, 2, 3, 4, ..., n-1\}$, $E_1 = \{e_q\}$, $E_2 = \{e_q^{'}\}$, $E_3 = \{e_q^{''}\}$,..., $E_m = \{e_q^{m-1}\}$, where $q = \{n\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 with length one, the subgraph $< F_2 >$ generates a single path P_2 with length one, and this process continues until the subgraph $< F_{m-1} >$ generates a single path P_2 with length one, the subgraph $< H_i >$ generates $(n-1)$ copies of paths P_3 with length two, the subgraph $< H_i^{'} >$ generates $(n-1)$ copies of cycles P_3 with length two, the subgraph $< H_i^{''} >$ generates $(n-1)$ copies of paths P_3 with length two, and this process continues until the subgraph $< H_i^{m-1} >$ generates $(n-1)$ copies of paths P_3 with length two, the subgraph $< E_1 >$ generates a single path P_2 of length, the subgraph $< E_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< E_m >$ generates a single path P_2 of length one. Therefore, we conclude that $D(P_m \odot P_n)$ contains $2m-1$ copies of P_2 and $m(n-1)$ copies of P_3.

Case 2: If $m = m = 2b+1$, where $b = 1, 2, 3, ...$ Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_r\}$, $H_q^{'} = \{e_q^{'}, e_r\}$, $H_q^{''} = \{e_q^{''}, e_r\}$,...,$H_q^{m-1} = \{e_q^{m-1}, e_r\}$, where $q = r = \{1, 2, 3, 4, ..., n-1\}$, $E_1 = \{e_q\}$, $E_2 = \{e_q^{'}\}$, $E_3 = \{e_q^{''}\}$,..., $E_m = \{e_q^{m-1}\}$, where $q = \{n\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 of length one, the subgraph

$< F_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< F_{m-1} >$ generates a single path P_2 of length one, the subgraph $< H_i >$ generates $(n-1)$ copies of paths P_3 with length two, the subgraph $< H_i^{'} >$ generates $(n-1)$ copies of cycles P_3 with length two, the subgraph $< H_i^{''} >$ generates $(n-1)$ copies of paths P_3 with length two, and this process continues until the subgraph $< H_i^{m-1} >$ generates $(n-1)$ copies of paths P_3 with length two, the subgraph $< E_1 >$ generates a single path P_2 of length one, the subgraph $< E_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< E_m >$ generates a single path P_2 of length one. Therefore, we conclude that $D(P_m \odot P_n)$ contains $2m-1$ copies of P_2 and $m(n-1)$ copies of P_3. □

Theorem 2.6 *Any corona product graph $P_m \odot P_n$, where $m, n \geq 3$ can be decomposed into following ways:*

$$D(P_m \odot P_n) = \begin{cases} (m-1)P_2,\ \frac{mn}{3}K_{1,3} \ \textit{and} \ mP_n, & \textit{if } m = n = 3b, \\ & \textit{where } b = 1,2,3,... \\ m(\frac{n-1}{3})K_{1,3},\ mP_n \ \textit{and} \ (2m-1)P_2, & \textit{if } m = n = 3b+1, \\ & \textit{where } b = 1,2,3,... \\ (m-1)P_2,\ m(\frac{n-2}{3})K_{1,3},\ mP_n \ \textit{and} \ mP_3, & \textit{if } m = n = 3b+2, \\ & \textit{where } b = 1,2,3,... \end{cases}$$

Proof: Let $P_m \odot P_n$ be the corona product of P_m and P_n and $D(P_m \odot P_n)$ be the decompostion of $P_m \odot P_n$. Then the following cases complete the proof:

Case 1. If $m = n = 3b$, where $b = 1, 2, 3, ...$
Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_{q+2}\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_{q+2}^{'}\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_{q+2}^{''}\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_{q+2}^{m-1}\}$, where $q = \{1, 4, ..., n-2\}$, $G_1 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, $G_2 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$,...., $G_m = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, where $r = \{1\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 of length one, the subgraph $< F_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< F_{m-1} >$ generates a single path P_2 of length one, the subgraph $< H_q >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{'} >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{''} >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, and this process continues until the subgraph $< H_q^{m-1} >$ generates $\frac{n}{3}$ copies of claws $K_{1,3}$, the subgraph $< G_1 >$ generates a single path P_n of length $n-1$, the subgraph $< G_2 >$ generates a single path P_n of length $n-1$, and this process continues until the subgraph $< G_m >$ generates a single path P_n of length $n-1$. Therefore, we conclude that $D(P_m \odot P_n)$ contains $m-1$ copies of paths P_2, $\frac{mn}{3}$ copies of claw and m copies of paths P_n.

Case 2. If $m = n = 3b+1$, where $b = 1, 2, 3, ...$
Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_{q+2}\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_{q+2}^{'}\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_{q+2}^{''}\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_{q+2}^{m-1}\}$, where $q = \{1, 4, ..., n-2\}$, $E_1 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, $E_2 = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$,...., $E_m = \{e_r, e_{r+1}, ..., e_{r+(n-2)}\}$, where $r = \{1\}$, $G = \{e_q\}$, $G^{'} = \{e_i^{'}\}$,..., $G_q^{m-1} = \{e_q^{m-1}\}$, where $q = \{n\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 of length one, the subgraph $< F_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< F_{m-1} >$ generates a single path P_2 of length one, the subgraph $< H_q >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{'} >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{''} >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, and this process continues until the subgraph $< H_q^{m-1} >$ generates $\frac{n-1}{3}$ copies of claws $K_{1,3}$, the subgraph $< E_1 >$ generates a single path P_n of length $n-1$, the subgraph $< E_2 >$ generates a single path P_n of length $n-1$, and this process continues until the subgraph $< E_m >$ generates a single path P_n of length $n-1$, the subgraph $< G >$ generates a single path P_2 of length one, the subgraph $< G_1 >$ generates a single path P_2 of length one, the subgraph $< G_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< G_{m-1} >$ generates a single path P_2 of length one. Therefore,

we conclude that $D(P_m \odot P_n)$ contains $m(\dfrac{n-1}{3})$ copies of claw, m copies of P_n and $(2m-1)$ copies of P_2.

Case 3. If $n = m = 3b + 2$, where $b = 1, 2, 3, \ldots$.
Then we assume that $F_1 = \{e_p\}$, $F_2 = \{e_{p+1}\}$,..., $F_{m-1} = \{e_{p+m-2}\}$, where $p = \{1\}$, $H_q = \{e_q, e_{q+1}, e_{q+2}\}$, $H_q^{'} = \{e_q^{'}, e_{q+1}^{'}, e_{q+2}^{'}\}$, $H_q^{''} = \{e_q^{''}, e_{q+1}^{''}, e_{q+2}^{''}\}$,..., $H_q^{m-1} = \{e_q^{m-1}, e_{q+1}^{m-1}, e_{q+2}^{m-1}\}$, where $q = \{1, 4, \ldots, n-2\}$, $E_1 = \{e_r, e_{r+1}, \ldots, e_{r+(n-2)}\}$, $E_2 = \{e_r, e_{r+1}, \ldots, e_{r+(n-2)}\}$,...., $E_m = \{e_r, e_{r+1}, \ldots, e_{r+(n-2)}\}$, where $r = \{1\}$, $G = \{e_q, e_{q+1}\}$, $G^{'} = \{e_q^{'}, e_{q+1}^{'}\}$, $G^{''} = \{e_q^{''}, e_{q+1}^{''}\}$,...,$G = \{e_q^{m-1}, e_{q+1}^{m-1}\}$, where $q = \{n-1\}$. Following that, the subgraph $< F_1 >$ generates a single path P_2 of length one, the subgraph $< F_2 >$ generates a single path P_2 of length one, and this process continues until the subgraph $< F_{m-1} >$ generates a single path P_2 of length one, the subgraph $< H_q >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{'} >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$, the subgraph $< H_q^{''} >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$ and the subgraph $< H_q^{m-1} >$ generates $\frac{n-2}{3}$ copies of claws $K_{1,3}$, the subgraph $< E_1 >$ generates a single path P_n of length $n-1$, the subgraph $< E_2 >$ genereates a single path P_n of length $n-1$, and this process continues until the subgraph $< E_m >$ generates a single path P_n of length $n-1$, the subgraph $< G >$ generates a single path P_3 of length two, the subgraph $< G_1 >$ generates a single path P_3 of length two and this process continues until the subgraph $< G_{m-1} >$ generates a single path P_3 of length two. Therefore, we conclude that $D(P_m \odot P_n)$ contains $m-1$ copies of paths P_2, $m(\frac{n-2}{3})$ copies of claw, m copies of paths P_n and m copies of paths P_3. □

3. Conclusion

In this study, we defined the corona product graph $P_m \odot P_n$ of paths P_m and P_n, and we decomposed it into paths, cycles, and claws. Also, we decomposed it into $\frac{m(n-2)}{3}$ copies of $K_{1,3}$ and m copies of P_3 with length two, specifically for non-negative integers m, n, b and $m \geq 3, n \geq 3$, if $m = n = 3b + 2$. We believe to be natural for extending the results presented in this paper in near future.

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