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Common fixed points for generalized weakly contractive maps using simulation function

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ABSTRACT: In this paper, we shall introduce new notions of generalized $(\alpha_b - \psi_b)$ contractive mappings of type-I and type-II in generalized metric spaces. In addition to this, some fixed point results are also proved by making use of such types of contractions in the mentioned spaces.

Key Words: Generalized metric spaces, $(\alpha_b - \psi_b)$ contractions, fixed point.

Contents

1. Introduction

Fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. Moreover, being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive mapping. A point $x \in X$ is called a fixed point of T if $Tx = x$. The well-known Banach Contraction Principle ensures the existence and uniqueness of a fixed point of a contraction on a complete metric space. In 1977, Alber et al. [1] generalized Banach contraction principle by introducing the concept of weak contraction mappings in Hilbert spaces. Very recently, Samet et al. [2] suggested a very interesting class of mappings, known as $\alpha - \psi$ contractive mappings, to investigate the existence and uniqueness of a fixed point. Several fixed point results including the Banach contraction principle were concluded as a result of this paper. The techniques used in this paper have been improved by so many authors, [3,5,6,7,8,9].

2. Preliminaries

In the literature, notice that there are distinct notions that are called 'a generalized metric'. In the sequel, when we mention a'generalized metric' we mean that the metric introduced by Branciari [4] introduced the concept of generalized metric space. As such, any metric space is a generalized metric space but the converse is not true. He proved the Banach fixed point theorem in such a space. For more details, the readers can refer to [10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30].
$\mathbb{N}$ and $\mathbb{R}^+$ denote the set of positive integers and the set of nonnegative reals, respectively.
Let Ψ be the family of functions $\psi : [0, \infty) \to [0, \infty)$ satisfying the following conditions:

(i) ψ is upper semi-continuous;

(ii) $(\psi^n(t))_(n \in \mathbb{N})$ converges to 0 as $n \to \infty$, for all $t > 0$;

(iii) $\psi(t) < t$, for any $t > 0$.

In the following, we recall the notion of a generalized metric space introduced by Branciari [4].

Definition 2.1. [4] Let X be a non empty set and let $d : X \times X \to [0, \infty]$ satisfy the accompanying conditions, for all $x, y \in X$ and all particular $u, v \in X$ every one of which is different from x and y.

(GMS1) $d(x, y) = 0$ if and only if $x = y$;

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(GMS2) $d(x,y) = d(y,x)$;

(GMS3) $d(x,y) \leq d(x,u) + d(u,v) + d(v,y)$.

Then the map d is called a generalized metric. Here, the pair (X,d) is called a generalized metric space (GMS).

Definition 2.2. [4]

(i) A sequence$\{x_n\}$ in a GMS (X,d) is GMS convergent to a limit x if and only if $d(x_n,x) \to 0$ as $n \to \infty$;

(ii) A sequence $\{x_n\}$ in a GMS (X,d) is GMS Cauchy if and only if for every $\varepsilon > o$ there exists positive integer $N(\varepsilon)$ such that $d(x_n,x_m) < \varepsilon$, for all $n > m > N(\varepsilon)$;

(iii) A GMS (X,d) is called complete if every GMS Cauchy sequence in X is GMS convergent;

(iv) A mapping $T : (X,d) \to (X,d)$ is continuous if for any sequence $\{x_n\}$ in X such that $d(x_n,x) \to 0$ as $n \to \infty$, we have $d(Tx_n,Tx) \to 0$ as $n \to \infty$.

Recall that Samet et al. [2] introduced the following concepts:

Definition 2.3. [2] For a nonempty set X, let $T : X \to X$ and $\alpha : X \times X \to [0,\infty)$ be mappings. We say that T is α-permissible if, for all $x,y \in X$, we have

$$\alpha(x,y) \geq 1 \Rightarrow a(Tx,Ty) \geq 1. \tag{2.1}$$

Definition 2.4. [2] Let (X,d) be a metric space and $T : X \to X$ be a given mapping. We say that T is a $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and a specific ψ with the end goal that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)),\ for\ all\ x,y \in X. \tag{2.2}$$

Very recently, Karapinar [7] gave the analog of the notion of a $\alpha - \psi$ contractive mapping, with regards to generalized metric spaces as takes after.

Definition 2.5. [7] Let (X,d) be a generalized up metric space and $T : X \to X$ be a given mapping. We say that T is a $\alpha - \psi$ contractive mapping if there exist two functions $\alpha : X \times X \to [0,\infty)$ and a specific ψ such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(d(x,y)),\ for\ all\ x,y \in X. \tag{2.3}$$

Proposition 2.6. [31] Let $\{\bar{\gamma}_\mathfrak{n}\}$ is a convergent sequence in a GMS $(M,\tilde{d})$ with $lim_{m\to\infty}\tilde{d}(\bar{\gamma}_\mathfrak{n},\sqcap) = 0$, where $\sqcap \in X$. At that point $lim_{m\to\infty}\tilde{d}(\bar{\gamma}_\mathfrak{n},\delta) = \tilde{d}(\sqcap,\delta)$, for all $\delta \in M$. In Particular, $\{\bar{\gamma}_\mathfrak{n}\}$ series does not converge to δ if $\delta \neq \sqcap$.

Karapinar [7] additionally expressed the accompanying fixed point theorems.

Theorem 2.7. [7] Let (X,d) be a complete generalized metric space and $T : X \to X$ be a $\alpha - \psi$ contractive mapping. Assume that

(i) T is α-admissible;

(ii) there exists $x \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$;

(iii) T is continuous.

Then there exists a $u \in X$ such that $Tu = u$.

Theorem 2.8. Let (X,d) be a complete generalized metric space and $T : X \to X$ be a $\alpha - \psi$ contractive mapping. Assume that

(i) T is α-admissible;

(ii) there exists $x \in X$ such that $\alpha(x_0,Tx_0) \geq 1$ and $\alpha(x_0,T^2x_0) \geq 1$;

(iii) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$, for all n and $x_n \to x \in X$ as $n \to \infty$, at that point $\alpha(x_n, x) \geq 1$, for all n.
Then there exists a $u \in X$ such that $Tu = u$.

Let Z* be the set of simulation functions in the sense of Argoubi et al. [32].

Definition 2.9. [32] A simulation function is a mapping $\zeta : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfying the following conditions:

- (ζ_1) $\zeta(t,s) < s - t$ for all $t, s > 0$;
- (ζ_2) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $lim_{n\to\infty} t_n = lim_{n\to\infty} s_n = l \in (0,\infty)$, then

$$lim_{n\to\infty}\zeta(t_n, s_n) < 0.$$

Karapinar et al. [33] introduced some generalized (α, ψ)-contractive mappings.

Definition 2.10. [33] Let (X, d) be a generalized metric space be mappings and $T : X \to X$ be a given mapping. We say that T is a generalized α, ψ-contractive mapping of type-I if there exist two functions $\alpha : X \times X \to [0,\infty]$ and $\psi \in \Psi$, such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(M(x,y)),\ for\ all\ x,y \in X. \tag{2.4}$$

where

$$M(x,y) = max\{d(x,y), d(x,Tx), d(y,Ty)\}. \tag{2.5}$$

Definition 2.11. [33] Let (X, d) be a generalized metric space be mappings and $T : X \to X$ be a given mapping. We say that T is a generalized α, ψ-contractive mapping of type-II if there exist two functions $\alpha : X \times X \to [0,\infty]$ and $\psi \in \Psi$, such that

$$\alpha(x,y)d(Tx,Ty) \leq \psi(N(x,y)),\ \text{for all } x,y \in X, \tag{2.6}$$

where

$$N(x,y) = max\{d(x,y), \frac{d(x,Tx) + d(y,Ty)}{2}\}. \tag{2.7}$$

3. Main Results

We start the main section by introducing the new notions of generalized $\alpha_b - \psi_b$ contractive mappings of type-I and type-II with simulation function in generalized metric space.

Definition 3.1. Let $(\mathfrak{X}, \tilde{d})$ be a generalized metric space, $\hat{S} : \mathfrak{X} \times \mathfrak{X}$ be a map. We claim that $\hat{S}$ is a generalized $(\alpha_b - \psi_b)$ type-I contractive mapping regards ζ and $\zeta \in \mathbb{Z}$ if there are $\alpha_b :_X \times\mathfrak{X} \to [0,\infty)$ and $\psi_b \in \Psi_b$ s.t.

$$\zeta(\alpha_b(k,l)\tilde{d}(\hat{S}k,\hat{S}l), \psi_b(M_1(k,l))) \geq 0,$$
$$\alpha_b(k,l)\tilde{d}(\hat{S}k,\hat{S}l) \leq \psi_b(M_1(k,l)),\ \text{for all } k,l \in \mathfrak{X}, \tag{3.1}$$

where

$$M_1(k,l) = max\{\tilde{d}(k,l), \tilde{d}(k,\hat{S}k), \tilde{d}(l,\hat{S}l)\}. \tag{3.2}$$

Definition 3.2. Assume $(\mathfrak{X}, \tilde{d})$ be a generalized metric space and $\hat{S}$ be a mapping. We say that $\hat{S}$ is a generalized $(\alpha_b - \psi_b)$ type-II contractive mapping and $\zeta \in \mathbb{Z}$ if there are two functions α_b and $\psi_b \in \Psi_b$ s.t.

$$\zeta(\alpha_b(k,l)\tilde{d}(\hat{S}k,\hat{S}l), \psi_b(N_1(k,l))) \geq 0,$$
$$\alpha_b(k,l)\tilde{d}(\hat{S}k,\hat{S}l) \leq \psi_b(N_1(k,l)),\ \text{for all } k,l \in \mathfrak{X}, \tag{3.3}$$

where

$$N_1(k,l) = max\{\tilde{d}(k,l), \frac{\tilde{d}(k,\hat{T}k) + \tilde{d}(l,\hat{T}l)}{2}\}. \tag{3.4}$$

Theorem 3.3. Let the generalized metric space be $(\mathfrak{X}, \tilde{d})$, and $\hat{S} : \mathfrak{X} \times \mathfrak{X}$ be the mapping provided. We are claiming $\hat{S}$ is a $(\alpha_b - \psi_b)$ type-I contractive mapping generalised. Assume that the fact is

1. $\hat{S}$ is α_b-admissible;

2. there is $k_0 \in \mathfrak{X}$ s.t. $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2 k_0) \geq 1$;

3. $\hat{S}$ is constant.

Therefore, $v \in \mathfrak{X}$ occurs such that $\hat{S}v = v$.

Proof There is one point, by assumption(2), $k_0 \in \mathfrak{X}$ s.t. $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2 k_0) \geq 1$. We have a sequence specified as $\{k_t\}$ in $\mathfrak{X}$ by $k_{t+1} = \hat{S}k_t = \hat{S}^{t+1}k_0$, $\forall\ t \geq 0$. Expect that $k_{t_0} = k_{t_0+1}$ for some t_0. Since $v = k_{t_0} = k_{t_0+1} = \hat{S}k_{t_0} = \hat{S}v$. Therefore, all through the verification, we assume that

$$k_t \neq k_{t+1} \text{ for all } t. \tag{3.5}$$

Look out for this

$$\alpha_b(k_0, k_1) = \alpha_b(k_0, \hat{S}k_0) \geq 1 \Rightarrow \alpha_b(\hat{S}k_0, \hat{S}k_1) = \alpha_b(k_1, k_2) \geq 1,$$

Since $\hat{S}$ is α_b-admissible, we infer

$$\alpha_b(k_t, k_{t+1}) \geq 1, \text{ for all } t = 0, 1, 2, ... \tag{3.6}$$

By utilizing a similar method, we get

$$\alpha_b(k_0, k_2) = \alpha_b(k_0, \hat{S}^2 k_0) \geq 1 \Rightarrow \alpha_b(\hat{S}k_0, \hat{S}k_2) = \alpha_b(k_1, k_2) \geq 1,$$

The expression above yields

$$\alpha_b(k_t, k_{t+2}) \geq 1, \text{ for all } m = 0, 1, 2, ... \tag{3.7}$$

Step I: We claim that

$$lim_{t\to\infty} \tilde{d}(k_t, k_{t+1}) = 0. \tag{3.8}$$

Combining (3.1) and (3.6), we find that

$$\begin{aligned} 0 &\leq \zeta(\alpha_b(k_{t-1}, k_m)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_m)), \psi_b(M_1(k_{t-1}, k_t))) \\ &< \psi_b(M_1(k_{t-1}, k_t)) - \alpha_b(k_{t-1}, k_t)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \\ \alpha_b(k_{t-1}, k_t)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) &\leq \psi_b(M_1(k_{t-1}, k_t)) \end{aligned}$$

$$\tilde{d}(k_t, k_{t+1}) = \tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \leq \alpha_b(k_{t-1}, k_t)\tilde{d}(\hat{S}k_{t-1}, \hat{S}k_t) \leq \psi_b(M_1(k_{t-1}, k_t)), \tag{3.9}$$

for all $t \geq 1$, where

$$\begin{aligned} M_1(k_{t-1}, k_t) &= max\{\tilde{d}(k_{t-1}, k_t), \tilde{d}(k_{t-1}, \hat{S}k_{t-1}), \tilde{d}(k_t, \hat{S}k_t)\} \\ &= max\{\tilde{d}(k_{t-1}, k_t), \tilde{d}(k_{t-1}, k_t), \tilde{d}(k_t, k_{t+1})\} \\ &= max\{\tilde{d}(k_{t-1}, k_t), \tilde{d}(k_t, k_{t+1})\}. \end{aligned} \tag{3.10}$$

If for some t, $M_1(k_{t-1}, k_t) = \tilde{d}(k_t, k_{t+1}) (\neq 0)$, then the inequality (3.9) turns into

$$\tilde{d}(k_t, k_{t+1}) \leq \psi_b(M_1(k_{t-1}, k_t)) = \psi_b(\tilde{d}(k_t, k_{t+1}) < \tilde{d}(k_t, k_{t+1}),$$

a contradiction. Hence $M_1(k_{t-1}, k_t) = \tilde{d}(k_{t-1}, k_t)$, for all $t \in \mathbb{N}$, and (3.9) becomes

$$\begin{aligned} 0 &\leq \zeta(\tilde{d}(k_t, k_{t+1}), \psi_b(\tilde{d}(k_{t-1}, k_m))) \\ &< \psi_b(\tilde{d}(k_{t-1}, k_t)) - \tilde{d}(k_t, k_{t+1}) \\ \tilde{d}(k_t, k_{t+1}) &\leq \psi_b(\tilde{d}(k_{t-1}, k_t)), \text{ for all } t \in \mathbb{N}. \end{aligned} \tag{3.11}$$

This yields

$$\begin{aligned} 0 &\le \zeta(\tilde{d}(k_t,k_{t+1}),\tilde{d}(k_{t-1},k_t)) \\ &< \tilde{d}(k_t,k_{t+1}) - \tilde{d}(k_t,k_{t+1}) \\ \tilde{d}(k_t,k_{t+1}) &\le \tilde{d}(k_{t-1},k_t), \text{ for all } t\in\mathbb{N}. \end{aligned} \tag{3.12}$$

By (3.11), we have

$$\begin{aligned} 0 &\le \zeta(\tilde{d}(k_t,k_{t+1}),\psi_b^t(\tilde{d}(k_0,k_1))) \\ &< \psi_b^t(\tilde{d}(k_0,k_1)) - \tilde{d}(k_t,k_{t+1}) \\ \tilde{d}(k_t,k_{t+1}) &\le \psi_b^t(\tilde{d}(k_0,k_1)), \text{ for all } t\in\mathbb{N}. \end{aligned} \tag{3.13}$$

Through the ψ_b property, it is obvious that

$$lim_{m\to\infty}\tilde{d}(k_t,k_{t+1}) = 0.$$

Step II: We will show

$$lim_{t\to\infty}\tilde{d}(k_t,k_{t+2}) = 0. \tag{3.14}$$

By (3.1) and (3.7), we get

$$\begin{aligned} 0 &\le \zeta(\alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}),\psi_b(M_1(k_{t-1},k_{t+1}))) \\ &< \psi_b(M_1(k_{t-1},k_{t+1})) - \alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}) \\ \alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}) &\le \psi_b(M_1(k_{t-1},k_{t+1})). \end{aligned}$$

$$\begin{aligned} \tilde{d}(k_t,k_{t+2}) &= \tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}) \le \alpha_b(k_{t-1},k_{t+1})\tilde{d}(\hat{S}k_{t-1},\hat{S}k_{t+1}) \\ &\le \psi_b(M_1(k_{t-1},k_{t+1})), \end{aligned} \tag{3.15}$$

for all $t \ge 1$, where

$$\begin{aligned} M_1(k_{t-1},k_t) &= max\{\tilde{d}(k_{t-1},k_{t+1}),\tilde{d}(k_{t-1},\hat{S}k_{t-1}),\tilde{d}(k_{t+1},k_{t+2})\} \\ &= max\{\tilde{d}(k_{t-1},k_{t+1}),\tilde{d}(k_{t-1},k_t),\tilde{d}(k_{t+1},k_{t+2}). \end{aligned} \tag{3.16}$$

By (3.14), we have

$$M_1(k_{t-1},k_{t+1}) = max\{\tilde{d}(k_{t-1},k_{t+1}),\tilde{d}(k_{t-1},k_t)\}.$$

Thus, from (3.16)

$$b_t = \tilde{d}(k_t,k_{t+2}) \le \psi_b(M_1(k_{t-1},k_{t+1})) = \psi_b(max\{b_{t-1},c_{t-1}\}), \text{ for all } t\in\mathbb{N}. \tag{3.17}$$

Again, by (3.14)

$$c_t \le c_{t-1} \le max\{b_{t-1},c_{t-1}\}.$$

Therefore, the $max\{b_t,c_t\}$ sequence is non-increasing in monotony, and it converges to any $t \ge 0$. Suppose, $r > 0$. Now, by (3.8)

$$lim_{t\to\infty}b_t = lim_{t\to\infty}supmax\{b_t,c_t\} = lim_{t\to\infty}max\{b_t,c_t\} = r.$$

Putting $m \to \infty$ in (3.17), we get

$$\begin{aligned} z = lim_{t\to\infty}b_t &\le lim_{t\to\infty}sup\psi_b(max\{b_{t-1},c_{t-1}\}) \\ &\le \psi_b(lim_{t\to\infty}max\{b_{t-1},c_{t-1}\}) \\ &= \psi_b(r) < r, \end{aligned}$$

which appeared to be a contradiction.
Step III: We'll show

$$k_t \neq k_j, \text{ every } t \neq j. \tag{3.18}$$

For all of that $t, j \in \mathbb{N}$, presume $k_t = k_j$ with $t \neq j$. Since $\tilde{d}(k_s, k_{s+1}) > 0$, for each $s \in \mathbb{N}$. without loss of consensus, we may expect that $j > t + 1$.
Examine it next,

$$\begin{aligned} 0 &\leq \zeta(\alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j), \psi_b(M_1(k_{j-1}, k_j))) \\ &< \psi_b(M_1(k_{j-1}, k_j)) - \alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \end{aligned}$$
$$\alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \leq \psi_b(M_1(k_{j-1}, k_j))$$

$$\begin{aligned} \tilde{d}(k_t, k_{t+1}) = \tilde{d}(k_t, \hat{S}k_t) = \tilde{d}(k_j, \hat{S}k_j) &= \tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \leq \alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \\ &\leq \psi_b(M_1(k_{j-1}, k_j)). \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} M_1(k_{j-1}, k_j) &= max\{\tilde{d}(k_{j-1}, k_j), \tilde{d}(k_{j-1}, \hat{S}k_{j-1}), \tilde{d}(k_j, \hat{S}k_j)\} \\ &= max\{\tilde{d}(k_{j-1}, k_j), \tilde{d}(k_{j-1}, k_j), \tilde{d}(k_j, \hat{S}k_j)\} \\ &= max\{\tilde{d}(k_{j-1}, k_j), \tilde{d}(k_j, k_{j+1})\}. \end{aligned} \tag{3.20}$$

If $M_1(k_j, k_{j-1}) = \tilde{d}(k_{j-1}, k_j)$, then from (3.19), we get

$$\begin{aligned} \tilde{d}(k_t, k_{t+1}) &= \tilde{d}(k_t, \hat{S}k_t) = \tilde{d}(k_l, \hat{S}k_j) \\ &= \tilde{d}(k_j, k_{j+1}) \leq \alpha_b(k_j, k_{j+1})\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \\ &\leq \psi_b(M_1(k_{t+1}, k_t)) = \psi_b(\tilde{d}(k_{t+1}, k_t)) \\ &\leq \psi_b^{j-t}(\tilde{d}(k_t, k_{t+1})). \end{aligned} \tag{3.21}$$

If $M_1(k_{j-1}, k_j) = \tilde{d}(k_j, k_{j+1})$, (3.19) becomes

$$\begin{aligned} \tilde{d}(k_t, k_{t+1}) &= \tilde{d}(k_t, \hat{S}k_t) = \tilde{d}(k_j, \hat{S}k_j) \\ &= \tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \leq \alpha_b(k_{j-1}, k_j)\tilde{d}(\hat{S}k_{j-1}, \hat{S}k_j) \\ &\leq \psi_b(M_1(k_{j-1}, k_j)) = \psi_b(\tilde{d}(k_j, k_{j+1})) \\ &\leq \psi_b^{j-t+1}(\tilde{d}(k_t, k_{t+1})). \end{aligned} \tag{3.22}$$

Due to a property of ψ_b, (3.21) and (3.22) together yields

$$\tilde{d}(k_t, k_{t+1}) \leq \psi_b^{j-t}(\tilde{d}(k_t, k_{t+1})) < \tilde{d}(k_t, k_{t+1}) \tag{3.23}$$

and

$$\tilde{d}(k_t, k_{t+1}) \leq \psi_b^{j-t+1}(\tilde{d}(k_t, k_{t+1})) < \tilde{d}(k_t, k_{t+1}), \tag{3.24}$$

respectively. There is a contradiction in each case.
Step IV: We must show $\{k_t\}$ to be a cauchy sequence, that is,

$$lim_{t\to\infty}\tilde{d}(k_t, k_{t+h^*}) = 0, \text{ for all } h^* \in \mathbb{N}. \tag{3.25}$$

Two cases arise: $h^* = 1$ and $h^* = 2$ are proved by (3.8) and (3.14) respectively. Now, carry on the arbitrary $h^* \geq 3$. Two situations are plenty to look at.

Situation(I): Expect that $h^* = 2l + 1$, where $j \geq 1$. Next, along with Phase-III and Quadrilateral Inequality (3.13), we consider

$$\begin{aligned}
\tilde{d}(k_t, k_{t+h^*}) &= \tilde{d}(k_t, kt + 2j + 1) \leq \tilde{d}(k_t, k_{t+1}) + \tilde{d}(k_{t+1}, k_{t+2}) + ... + \tilde{d}(k_{t+2j}, k_{t+2j+1}) \\
&\leq \sum_{p=t+2}^{t+2j-1} \psi_b^p(\tilde{d}(k_0, k_1)) \\
&\leq \sum_{p=t}^{+\infty} \psi_b^p(\tilde{d}(k_0, k_1)) \to 0 \text{ as } t \to \infty. \qquad (3.26)
\end{aligned}$$

Case (II): Assume $h^* = 2j$, where $j \geq 2$ is. By the implementation of quadrilateral inequalities and step III along with (3.13), we consider again

$$\begin{aligned}
\tilde{d}(k_t, k_{t+h^*}) &= \tilde{d}(k_t, kt + 2j) \leq \tilde{d}(k_t, k_{t+1}) + \tilde{d}(k_{t+1}, k_{t+2}) + ... + \tilde{d}(k_{t+2j-1}, k_{t+2j}) \\
&\leq \tilde{d}(k_t, k_{t+2}) + \sum_{p=t}^{t+2j} \psi_b^p(\tilde{d}(k_0, k_1)) \\
&\leq \tilde{d}(k_t, k_{t+2}) + \sum_{p=t}^{+\infty} \psi_b^p(\tilde{d}(k_0, k_1)) \to 0 \text{ as } t \to \infty. \qquad (3.27)
\end{aligned}$$

Now, from these two expressions (3.26) and (3.27), we have

$$lim_{m\to\infty}\tilde{d}(k_j, k_{j+h^*}) = 0, \text{ for all } h^* \geq 3.$$

We conclude that a CS in $(\mathfrak{X}, \tilde{d})$ is $\{k_t\}$. Due to the completeness of $(\mathfrak{X}, \tilde{d})$, it occurs in such a way that $v \in \mathfrak{X}$ occurs

$$lim_{t\to\infty}\tilde{d}(k_t, v) = 0. \qquad (3.28)$$

Because $\hat{S}$ is continuous, we get that from (3.28)

$$lim_{t\to\infty}\tilde{d}(k_{t+1}, \hat{S}v) = lim_{t\to\infty}\tilde{d}(\hat{S}k_t, \hat{S}v) = 0, \qquad (3.29)$$

that is, $lim_{t\to\infty}k_{t+1} = \hat{S}v$.
Considering Proposition(2), we infer that $\hat{S}v = v$, i.e. v be fixed point of $\hat{S}$.
The below sentence is taken from the (3) Theorem due to the inequality of $N_1(k, l) \leq M_1(k, l)$.

Theorem 3.4. Let the generalized metric space be $(\mathfrak{X}, \tilde{d})$ and $\hat{S} : \mathfrak{X} \times \mathfrak{X}$ be the mapping provided. Expect that $\hat{S}v = v$ be fixed point of $\hat{S}$. We say that $\hat{S}$ is a generalized $(\alpha_b - \psi_b)$ type-II contractive mapping. Assume that

1. $\hat{S}$ is α_b-admissible;
2. there is $k_0 \in \hat{S}$ such that $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2k_0) \geq 1$;
3. $\hat{S}$ is constant.

There is then $v \in \mathfrak{X}$ such that $\hat{S}v = v$.

Theorem 3.5. If $\hat{S}$ is a generalized $(\alpha_b - \psi_b)$ type-II contractive mapping on generalized metric space $(\mathfrak{X}, \tilde{d})$. Assume that

1. $\hat{S}$ is α_b-admissible;
2. there is $k_0 \in \mathfrak{X}$ s.t. $\alpha_b(k_0, \hat{S}k_0) \geq 1$ and $\alpha_b(k_0, \hat{S}^2k_0) \geq 1$;
3. if $\{k_t\}$ is a $\mathfrak{X}$ series like $\alpha_b(k_t, k_{t+1}) \geq 1$, for all t and $k_t \to k \in \mathfrak{X}$ as $t \to \infty$, then there is a $\{k_t(h^*)\}$ subsequence of $\{k_t\}$, like $\alpha_b(k_t(h^*), x) \geq 1$, $\forall$ h^*.

So $v \in \mathfrak{X}$ exists, such that $\hat{S}v = v$.

Proof We know the $\{k_t\}$ series defined by $k_{t+1} = \hat{S}k_t \; \forall \; t \geq 0$ is a cauchy sequence and converges to some $v \in X$. Provided the Preposition(2),

$$lim_{h^* \to \infty} \tilde{d}(k_{t(h^*)+1}, \hat{S}v) = \tilde{d}(v, \hat{S}v). \tag{3.30}$$

Now, we 're going to know $\hat{S}v = v$. On the opposite, assume that $\hat{S}v \neq v$, so $\tilde{d}(\hat{S}v, v) > 0$. The subsequence $\{k_t(h^*)\}$ of $\{k_t\}$ occurs from (3.6) and (3) in such a way that $\alpha_b(k_t(h^*), v) \geq 1$, for all h^*. By applying (3.1), we get

$$\begin{aligned} 0 &\leq \zeta((\alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, v)), \psi_b(M_1(k_{t(h^*)}, v))) \\ &< \psi_b(M_1(k_{t(h^*)}, v)) - \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, v) \\ \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, v) &\leq \psi_b(M_1(k_{t(h^*)}, v)) \end{aligned}$$

$$d^*(k_{t(h^*)+1}, \hat{S}v) \leq \alpha_b(k_{t(h^*)}, v) d^*(\hat{S}k_{t(h^*)}, \hat{S}v) \leq \psi_b(M_1(k_{t(h^*)}, v)), \tag{3.31}$$

where

$$\begin{aligned} M_1(k_{t(h^*)}, v) &= max\{\tilde{d}(k_{t(h^*)}, v), \tilde{d}(k_{t(h^*)}, \hat{S}k_{t(h^*)}), \tilde{d}(v, \hat{S}v)\} \\ &= max\{\tilde{d}(k_{t(h^*)}, v), \tilde{d}(k_{t(h^*)}, k_{t(h^*)+1}), \tilde{d}(v, \hat{S}v)\}. \end{aligned} \tag{3.32}$$

By (3.8) and (3.30), we have

$$lim_{h^* \to \infty} M_1(k_{t(h^*)}, v) = \tilde{d}(v, \hat{S}v). \tag{3.33}$$

Making $h^* \to \infty$ in (3.31) and regarding that ψ_b is upper semi continuous

$$\tilde{d}(v, \hat{S}v) \leq \psi_b(\tilde{d}(v, \hat{S}v)) < \tilde{d}(v, \hat{S}v), \tag{3.34}$$

That's one contradiction. But we consider v to be a fixed point of $\hat{S}$, that is, $\hat{S}v = v$.
The upper semi-continuity hypothesis of ψ_b is not needed below. Similar to Theorem(3), we have the following for the generalized type-II $\alpha_b - \psi_b$ contractive mappings.

Theorem 3.5. If $\hat{S}$ is generalized $(\alpha_b - \psi_b)$ type-II contractive pair of mappings on generalized metric space $(\mathfrak{X}, \tilde{d})$,

1. $\hat{S}$ is α_b-admissible;
2. $k_0 \in \mathfrak{X}$ exists such that $\alpha(k_0, \hat{S}k_0) \geq 1$ and $\alpha(k_0, \hat{S}^2 k_0) \geq 1$ are available;
3. if $\{k_t\}$ is a series in $\mathcal{X}$ s.t. $\alpha_b(k_t, k_{t+1}) \geq 1$, for all t and $k_t \to \mathfrak{X} \in \mathfrak{X}$ as $t \to \infty$, then there exists a subsequence $\{k_t(h^*)\}$ of $\{k_t\}$ such that $\alpha_b(k_t(h^*), v) \geq 1$, for all h^*.

Then $\exists \; v \in \mathfrak{X}$ s.t. $\hat{S}v = v$.

Proof We know that the sequence $k_{m+1} = \hat{S}k_m$ for all $m \geq 0$ is cauchy and converges to some $v \in \mathfrak{X}$ after proof of this theorem is the same as the Theorem(3). Similarly, in Proposition(2), we obtain

$$lim_{h^* \to \infty} \tilde{d}(k_{t(h^*)+1}, \hat{S}v) = \tilde{d}(v, \hat{S}v). \tag{3.35}$$

We will show that $\hat{S}v = v$. Assume that $\hat{S}v \neq v$. From (3.6) and condition(3), there is a $\{k_t(h^*)\}$ subsequence to $\{k_t\}$ such that $\alpha_b(k_t(h^*), v) \geq 1$, for all h^*. By applying (3.3), for all h^*, we get

$$\begin{aligned} 0 &\leq \zeta(\alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, S^*v), \psi_b(N_1(k_{t(h^*)}, v))) \\ &< \psi_b(N_1(k_{t(h^*)}, v)) - \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) \\ \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) &\leq \psi_b(N_1(k_{t(h^*)}, v)) \end{aligned}$$

$$\tilde{d}(k_{t(h^*)+1}, \hat{S}v) \leq \alpha_b(k_{t(h^*)}, v)\tilde{d}(\hat{S}k_{t(h^*)}, \hat{S}v) \leq \psi_b(N_1(k_{t(h^*)}, v)), \tag{3.36}$$

where

$$N_1(k_{t(h^*)}, v) = max\{\tilde{d}(k_{t(h^*)}, v), \frac{\tilde{d}(k_{t(h^*)}, \hat{S}k_{t(h^*)}) + \tilde{d}(v, \hat{S}v)}{2}\}. \tag{3.37}$$

Letting $h^* \to \infty$ in (3.36), we have

$$lim_{h^* \to \infty} N_1(k_{t(h^*)}, v) = \frac{\tilde{d}(v, \hat{S}v)}{2}. \tag{3.38}$$

From(3.38), for a sufficiently large h^*, we have $N_1(k_{t(h^*)}, v) > 0$, which means

$$\begin{aligned} 0 &\leq \zeta(\psi_b(N_1(k_{t(h^*)}, v)), N_1(k_{t(h^*)}, v)) \\ &< N_1(k_{t(h^*)}, v) - \psi_b(N_1(k_{t(h^*)}, v)) \\ \psi_b(N_1(k_{t(h^*)}, v)) &\leq N_1(k_{t(h^*)}, v). \end{aligned}$$

We have h^* big enough from (3.38),

$$\psi_b(N_1(k_{t(h^*)}, v)) < N_1(k_{t(h^*)}, v).$$

Thus, from (3.36) and (3.38), we have

$$\tilde{d}(v, \hat{S}v) \leq \frac{\tilde{d}(v, \hat{S}v)}{2},$$

this's the fallacy.
We therefore consider v to be $\hat{S}$ as a fixed point. And that is, $\hat{S}v = v$.

References

1. Alber Ya. I. and Guerre-Delabriere S., Principle of weakly contractive maps in Hilbert spaces, New Results in Operator Theory and Its Applications, I. Gohberg and Y. Basel, Lyubich, Eds., vol. 98 of Operator Theory: Advances and Applications, pp. 7-12, Birkh¨auser, Switzerland, 1997.

2. Samet B, Vetro C, Vetro P: Fixed point theorems for $\alpha - \psi$-contractive type mappings. Nonlinear Anal. 2012, 75: 2154-2165. 10.1016/j.na.2011.10.014.

3. Ali MU, Kamran T:On (α^*, ψ)-contractive multi-valued mappings. Fixed Point Theory Appl. 2013. Article ID 137, 2013: Article ID 137.

4. Branciari A., A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Pub.Math.(Debr.),vol. 57, pp. 31-37, 2000.

5. Jleli M, Karapınar E, Samet B: Best proximity points for generalized $\alpha - \psi$-proximal contractive type mappings. J. Appl. Math. 2013. Article ID 534127, 2013: Article ID 534127.

6. Jleli M, Karapınar E, Samet B: Fixed point results for $\alpha - \psi_\alpha$-contractions on gauge spaces and applications. Abstr. Appl. Anal. 2013. Article ID 730825, 2013: Article ID 730825.

7. Karapınar E: Discussion on contractions on generalized metric spaces. Abstr. Appl. Anal. 2014. Article ID 962784, 2014: Article ID 962784.

8. Karapınar E, Samet B: Generalized $\alpha - \psi$-contractive type mappings and related fixed point theorems with applications. Abstr. Appl. Anal. 2012. Article ID 793486, 2012: Article ID 793486.

9. Mohammadi B, Rezapour S, Shahzad N: Some results on fixed points of $\alpha - \psi$-Ćirić generalized multifunctions. Fixed Point Theory Appl. 2013. Article ID 24, 2013: Article ID 24.

10. Aydi H, Karapınar E, Lakzian H: Fixed point results on the class of generalized metric spaces. Math. Sci. 2012. Article ID 46, 6: Article ID 46.

11. Azam A, Arshad M: Kannan fixed point theorems on generalized metric spaces. J. Nonlinear Sci. Appl. 2008, 1: 45-48.

12. Azam A, Arshad M, Beg I: Banach contraction principle on cone rectangular metric spaces. Appl. Anal. Discrete Math. 2009, 3: 236-241. 10.2298/AADM0902236A.

13. Bilgili N, Karapınar E:A note on 'Common fixed points for (ψ, α, β)-weakly contractive mappings in generalized metric spaces'. Fixed Point Theory Appl. 2013. Article ID 287, 2013: Article ID 287.

14. Das P: A fixed point theorem on a class of generalized metric spaces. Korean J. Math. Sci. 2002, 9: 29-33.

15. Das P: A fixed point theorem in generalized metric spaces. Soochow J. Math. 2007, 33: 33-39.

16. Das P, Lahiri BK: Fixed point of a Ljubomir Ćirić's quasi-contraction mapping in a generalized metric space. Publ. Math. (Debr.) 2002, 61: 589-594.

17. Das P, Lahiri BK: Fixed point of contractive mappings in generalized metric space. Math. Slovaca 2009, 59: 499-504. 10.2478/s12175-009-0143-2.

18. Erhan IM, Karapınar E, Sekulić T: Fixed points of (ψ, α) contractions on rectangular metric spaces. Fixed Point Theory Appl. 2012. Article ID 138, 2012: Article ID 138.

19. Jleli M, Samet B: The Kannan's fixed point theorem in a cone rectangular metric space. J. Nonlinear Sci. Appl. 2009,2(3):161-167.

20. Kikina L, Kikina K: A fixed point theorem in generalized metric space. Demonstr. Math. 2013, XLVI: 181-190.

21. Lakzian H, Samet B:Fixed points for (ψ, φ)-weakly contractive mapping in generalized metric spaces. Appl. Math. Lett. 2012, 25: 902-906. 10.1016/j.aml.2011.10.047.

22. Mihet D: On Kannan fixed point principle in generalized metric spaces. J. Nonlinear Sci. Appl. 2009,2(2):92-96.

23. Ming CM, Chen CH: Periodic points for the weak contraction mappings in complete generalized metric spaces. Fixed Point Theory Appl. 2012. Article ID 79, 2012: Article ID 79.

24. Samet B: Discussion on: a fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces by A. Branciari. Publ. Math. (Debr.) 2010,76(4):493-494.

25. Shatanawi W, Al-Rawashdeh A, Aydi H, Nashine HK: On a fixed point for generalized contractions in generalized metric spaces. Abstr. Appl. Anal. 2012. Article ID 246085, 2012: Article ID 246085.

26. Gupta V, Saini RK and Deep R: Some fixed point results in G-metric space involving generalized altering distances. International Journal of Applied Nonlinear Science. vol. 3, no. 1, 66-76, 2018.

27. Gupta V and Saini R:, Various fixed-point results of picard sequence in complete G-metric space. Journal of Physics: Conference Series. vol. 22, no. 67, Article 012006, 2022.

28. Kumar S and Garg SK: Expansion Mapping Theorem in Metric Spaces. Int. J. Contemp. Math. Sciences. Vol. 4, 2009, no. 36, 1749-1758.

29. Kumar M, Kumar P and Kumar S: Some common fixed point theorems in generalized metric spaces. Hindawi Publishing Corporation, Journal of Mathematics. vol. 2013, Article ID 719324, 7 pages.

30. Zhang Q and Song Y, Fixed point theory for generalized ϕ-weak contractions. Applied Mathematics Letters. 22 (2009), 75-78.

31. Kirk WA, Shahzad N: Generalized metrics and Caristi's theorem. Fixed Point Theory Appl. 2013. Article ID 129, 2013: Article ID 129.

32. Vetro C, Argoubi H and and Samet B: Nonlinear contractions involvingsimulation functions in a metric space with a partial order, Filomat 2015, 8, 1082–1094.

33. Ayadi H, Karapınar E, Samet B: Fixed points for generalized (α, ψ)-contractions on generalized metric spaces. Journal of Inequalities and Applications. 2014. Article Number 229(2014).

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