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An nonlinear elliptic problem involves two types of terms: degenerate coercivity and singular nonlinearity

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ABSTRACT: In this paper, we investigate the existence and regularity of solutions for nonlinear elliptic equations featuring degenerate coercivity and a singular right-hand side. The outcome of our study is contingent upon the summability of the function f within specific Lebesgue spaces, as well as the exponent of the singular terms.

Key Words: Singular term, degenerate elliptic equation, L^m data, fixed point theorem, weak solutions.

Contents

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^N$, where $N \geq 2$. We are interested in the following problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{a(x)|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right) = \dfrac{f}{|u|^\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is a non-negative function belonging to a suitable Lebsgue space $L^m(\Omega)$ $(m \geq 1)$, θ and γ are positive reals, $1 < p \leq 2$ and $a : \Omega \to \mathbb{R}$ is a measurable function such that $0 < \alpha \leq a(x) \leq \beta$ a.e. x in Ω, for two positive constants α and β.

We will now discuss some characteristics of problem (1.1) as well as the main difficulties we encounter. Firstly, let us observe that (1.1) could exhibit singularity in the following manner: the solution is required to be zero on the boundary of the domain; however, simultaneously, the right-hand side of (1.1) could become unbounded or "blow up." Another crucial feature is the absence of coercivity for positive θ. The operator $-\operatorname{div}\left(\frac{a(x)|\nabla u|^{p-2}\nabla u}{(1+|u|)^{\theta(p-1)}}\right)$ is well defined between $W_0^{1,p}(\Omega)$ and its dual space $W^{-1,p'}(\Omega)$, where $p' = \frac{p}{p-1}$; however, it is not coercive. Due to the absence of coercivity, the classical theory for elliptic operators that act between dual spaces cannot be applied, even when the data f exhibits sufficient regularity (as indicated in [18]).

As we will see, existence and summability of solutions to problem (1.1) depend on these features. We will overcome these two difficulties by approximation, truncating the degenerate coercivity of the operator term and the singularity of the right-hand side (see problems (2.1)). We will prove by Schauder's theorem that these problems admit a bounded finite energy solution u_n with the property that : for every subset $\omega \subset\subset \Omega$ there exists a positive constant $C_\omega > 0$ such that $u_n \geq C_\omega$ almost everywhere in ω for every $n \in \mathbb{N}$. This condition, combined with some a priori estimates on u_n obtained in Section 4, will allow us

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to pass to the limit in the approximating problems and get a solution to problem (1.1) in the sense of (2.1).

In the following, we provide a concise overview of several papers that have had a significant influence on our work. Problem (1.1), particularly in the coercive case where $\theta = 0$, was addressed in a study by Boccardo et al. [2]. They successfully demonstrated the existence and uniqueness of solutions to this problem.

$$\begin{cases} -\Delta u = \dfrac{f}{u^\gamma} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In [4,6], a nonlinear version of the above problem was studied, considering an operator as the p-Laplacian $-\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ instead of $-\Delta u$, the authors prove existence of and regularity results if f belong to $L^m(\Omega)$. For more and different aspects concerning singular problems we refer to [7,8,10].

When $\theta > 0$, $\gamma = 0$, and f is a measurable function with appropriate integrability properties, we refer to [1] for the existence of solutions. The result is obtained by means of approximation through suitable coercive problems. In particular, the authors show that if $f \in L^1(\Omega)$ a solution exists if $\theta < 1$. For $\theta = 1$ a bounded and finite energy solution to (1.1) is proven to exist if $f \in L^m(\Omega)$ with $m > \frac{N}{p}$. When $\theta > 1$ nonexistence of solutions is shown for datum f with norm large enough.

In [5], the author proved the existence and regularity of solution to the problem $p = 2$, $f \in L^m(\Omega)$ non-negative function and $a : \Omega \to \mathbb{R}$ is a measurable function such that $0 < \alpha \leq a(x) \leq \beta$ a.e. x in Ω. In [21], the author proved the existence and regularity of solution to the problem (1.1) with $\gamma > 0$ and $0 \leq \theta \leq 1$.

The existence and regularity results for weak solution of degenerate anisotropic elliptic equation with singularities data have been proved in [17]. The corresponding results for degenerate parabolic and elliptic equations with singularities have been developed in [15,9,11,12,13,14].

Our results will depend on the summability of f in some Lebesgue spaces, and on the values of γ : We distinguish the cases $(\theta - 1)(p - 1) < \gamma < \theta(p - 1) + 1$, $\gamma = \theta(p - 1) + 1$ and $\gamma > \theta(p - 1) + 1$.

2. Statements of Results

Definition 2.1 *Let $f \in L^m(\Omega)$, $m \geq 1$. A measurable function u is a solution of (1.1) in the sense of distributions if, for every $\omega \subset\subset \Omega$ there exists $c_\omega > 0$, such that $u \geq c_\omega > 0$ a.e. in ω and*

$$\int_\Omega a(x)\frac{|\nabla u|^{p-2}\nabla u.\nabla\varphi}{(1+u)^{\theta(p-1)}}dx = \int_\Omega \frac{f\varphi}{u^\gamma}dx, \quad \forall\varphi \in C_0^\infty(\Omega). \tag{2.1}$$

Theorem 2.1 *Let $(\theta - 1)(p - 1) < \gamma < \theta(p - 1) + 1$.*

(i) If $f \in L^m(\Omega)$, with $m > \frac{N}{p}$. Then, there exists a distributional solution u to problem (1.1) in the sense of (2.1), such that $u \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

(ii) If $f \in L^m(\Omega)$, with

$$m_1 = \frac{pN}{pN - (N-p)(\theta(p-1)+1-\gamma)} \leq m < \frac{N}{p}. \tag{2.2}$$

Then, there exists a distributional solution u to problem (1.1) in the sense of (2.1), such that $u \in W_0^{1,p}(\Omega) \cap L^s(\Omega)$, with

$$s = \frac{Nm[(p-1)(1-\theta)+\gamma]}{N-pm}. \tag{2.3}$$

(iii) if $f \in L^m(\Omega)$ with

$$\max\left\{1, \frac{N}{Np-(N-1)[\theta(p-1)+1-\gamma]}\right\} \leq m < m_1. \tag{2.4}$$

Then, there exists a distributional solution u to problem (1.1) in the sense of (2.1), such that $u \in W_0^{1,\eta}(\Omega)$, with

$$\eta = \frac{Nm[(p-1)(1-\theta)+\gamma]}{N-m[\theta(p-1)+1-\gamma]}. \tag{2.5}$$

Remark 2.1 *(i) The hypothesis (2.2) is meaningful, because*

$$N > p \ \text{ and } \ (\theta-1)(p-1) < \gamma < \theta(p-1)+1 \Leftrightarrow 1 < m_1 < \frac{N}{p},$$

(ii) Observe that the hypothesis (2.2) guarantee that $s \geq p^*$. If $(\theta-1)(p-1) < \gamma < \theta(p-1)+1$, we explicitly note that $m = m_1$ is equivalent $s = p^*$.

(iii) The hypothesis (2.4) implies $1 < \eta < p$.

Theorem 2.2 *Let $\gamma = \theta(p-1)+1$ and assume that $f \in L^1(\Omega)$. Then, there exists a distributional solution u to problem (1.1) in the sense of (2.1), such that $u \in W_0^{1,p}(\Omega)$.*

Theorem 2.3 *Let $\gamma > \theta(p-1)+1$ and assume that $f \in L^1(\Omega)$. Then, there exists a distributional solution $u \in W_{loc}^{1,p}(\Omega)$ to problem (1.1) in the sense of (2.1), such that $u^{\frac{\gamma+(p-1)(1-\theta)}{p}} \in W_0^{1,p}(\Omega)$.*

Remark 2.2 *If $p = 2$, the result of Theorems 2.1-2.3 coincides with regularity results for elliptic equations with coercivity and singularity (see [5]).*

3. Approximating problems

Let $n \in \mathbb{N}$, we approximate the problem (1.1) by the following non-degenerate and non-singular problem

$$\begin{cases} -\text{div}\left(\dfrac{a(x)|\nabla u_n|^{p-2}\nabla u_n}{(1+|T_n(u_n)|)^{\theta(p-1)}}\right) = \dfrac{T_n(f)}{(|u_n|+\frac{1}{n})^{\gamma}} & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

Lemma 3.1 *The problem (3.1) have a solution $u_n \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$ for every fixed $n \in \mathbb{N}$.*

Proof: This proofs derived from Schauder's fixed point argument in [19]. Let $n \in \mathbb{N}$ be fixed and $v \in L^p(\Omega)$ be fixed. We know that the following non-singular problem:

$$\begin{cases} -\text{div}\left(\dfrac{a(x)|\nabla w|^{p-2}\nabla w}{(1+|T_n(w)|)^{\theta(p-1)}}\right) = \dfrac{T_n(f)}{(|v|+\frac{1}{n})^{\gamma}} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{3.2}$$

has a unique solution $w \in W_0^{1,p}(\Omega)$ follows from the classical results [16,1,20]. In particular, it is well defined a map

$$S : L^p(\Omega) \to L^p(\Omega),$$

where $S(v) = w$. Again, thanks to regularity of the datum $\frac{T_n(f)}{(|v|+\frac{1}{n})^{\gamma}}$, we can take w as test function in (3.2), then

$$\alpha \int_\Omega \frac{|\nabla w|^p}{(1+n)^{\theta(p-1)}} dx \leq n^{\gamma+1} \int_\Omega |w| dx,$$

using the Poincaré inequality and Hölder's inequality on the right-hand side, we have

$$\begin{aligned} \int_\Omega |\nabla w|^p dx &\leq \frac{C_1}{\alpha} n^{\gamma+1}(1+n)^{\theta(p-1)} \int_\Omega |\nabla w| dx \\ &\leq C_2(n)|\Omega|^{\frac{1}{p'}} \left(\int_\Omega |\nabla w|^p dx\right)^{\frac{1}{p}}, \end{aligned}$$

thus,

$$\int_\Omega |\nabla w|^p dx \leq C_3(n, |\Omega|).$$

Using the Poincaré inequality on the left-hand side, we get

$$\|w\|_{L^p(\Omega)} \leq C_4(n, |\Omega|),$$

where $C_4(n, |\Omega|)$ is a positive constant independent form v and w. So that the ball B of $L^p(\Omega)$ of radius $C_4(n, |\Omega|)$ is invariant for the map S. Moreover it is easily seen that S is continuous and compact by the $W_0^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$ embedding. Ten by Schauder's fixed-point theorem there exist $u_n \in W_0^{1,p}(\Omega)$ such that $S(u_n) = u_n$, i.e., u_n solves (3.1). Observe that u_n is bounded by the results of [20]. Since $\dfrac{f_n}{(|u_n| + \frac{1}{n})^\gamma} \geq 0$, the maximum principle implies that $u_n \geq 0$. □

Lemma 3.2 *Let u_n be the solution to problem (3.1). Then, the sequence $\{u_n\}_n$ is increasing with respect to n.*

Proof: Due to $T_n(f) \leq T_{n+1}(f)$ and $\gamma > 0$,

$$-\text{div}\left(\frac{a(x)|\nabla u_n|^{p-2}\nabla u_n}{(1+T_n(u_n))^{\theta(p-1)}}\right) = \frac{T_n(f)}{(u_n + \frac{1}{n})^\gamma} \leq \frac{T_{n+1}(f)}{\left(u_n + \frac{1}{n+1}\right)^\gamma},$$

So that

$$\begin{aligned} &- \text{div}\left(\frac{a(x)|\nabla u_n|^{p-2}\nabla u_n}{(1+T_n(u_n))^{\theta(p-1)}} - \frac{a(x)|\nabla u_{n+1}|^{p-2}\nabla u_{n+1}}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}}\right) \\ &\leq T_{n+1}(f)\left[\frac{1}{\left(u_n + \frac{1}{n+1}\right)^\gamma} - \frac{1}{\left(u_{n+1} + \frac{1}{n+1}\right)^\gamma}\right] \leq 0. \end{aligned} \tag{3.3}$$

Let choose $T_k((u_n - u_{n+1})^+)$ (with $(u_n - u_{n+1})^+ = \max\{u_n - u_{n+1}, 0\}$) as test function in (3.3), by the hypotheses on a and the fact that

$$\begin{aligned} &\left|\frac{1}{(1+T_{n+1}(u_{n+1})^{\theta(p-1)}} - \frac{1}{(1+T_n(u_n)^{\theta(p-1)}}\right| \\ &\leq k\left[\frac{1}{(1+T_{n+1}(u_{n+1})^{\theta(p-1)}} + \frac{1}{(1+T_n(u_n)^{\theta(p-1)}}\right], \quad \text{in } \{0 \leq u_n - u_{n+1} \leq k\}, \end{aligned}$$

In the other hand, we have

$$\begin{aligned} \mathcal{I}_1 &= \alpha \int_\Omega \frac{1}{(1+T_n(u_n))^{\theta(p-1)}} \\ &\quad \times \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u_{n+1}|^{p-2}\nabla u_{n+1}\right]\nabla T_k((u_n - u_{n+1})^+)dx \\ &\leq \beta \int_\Omega |\nabla u_{n+1}|^{p-1}.\nabla T_k((u_n - u_{n+1})^+) \\ &\quad \times \left[\frac{1}{(1+T_{n+1}(u_{n+1})^{\theta(p-1)}} - \frac{1}{(1+T_n(u_n)^{\theta(p-1)}}\right] dx \\ &\leq \beta k \int_\Omega |\nabla u_{n+1}|^{p-1}\left|\nabla T_k((u_n - u_{n+1})^+)\right| \\ &\quad \times \left[\frac{1}{(1+T_{n+1}(u_{n+1})^{\theta(p-1)}} + \frac{1}{(1+T_n(u_n)^{\theta(p-1)}}\right] dx = \beta k \mathcal{I}_2. \end{aligned} \tag{3.4}$$

We recall the following well-known inequality that hold for any two real vectors ξ, η and a real $1 < p < 2$:

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right)(\xi - \eta) \geq (p-1)\frac{|\xi-\eta|^2}{(|\xi|+|\eta|)^{2-p}}. \tag{3.5}$$

Using the Hölder inequality, (3.5) and $u_n, u_{n+1} \in W_0^{1,p}(\Omega)$, we get

$$\begin{aligned}
\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^p}{(1+T_n(u_n))^{\theta(p-1)}}dx &= \int_\Omega \frac{|\nabla(u_n-u_{n+1}))|^p}{(1+T_n(u_n))^{\theta(p-1)\frac{p}{2}}(|\nabla u_n|+|\nabla u_{n+1}|)^{\frac{p(2-p)}{2}}}\frac{(|\nabla u_n|+|\nabla u_{n+1}|)^{\frac{p(2-p)}{2}}}{(1+T_n(u_n))^{\theta(p-1)\frac{2-p}{2}}}dx \\
&\leq \left(\int_\Omega \frac{|\nabla(u_n-u_{n+1}))|^2}{(1+T_n(u_n))^{\theta(p-1)}(|\nabla u_n|+|\nabla u_{n+1}|)^{2-p}}dx\right)^{\frac{p}{2}} \\
&\quad \times \left(\int_\Omega \frac{(|\nabla u_n|+|\nabla u_{n+1}|)^p}{(1+T_n(u_n))^{\theta(p-1)}}dx\right)^{\frac{2-p}{2}} \\
&\leq C_1\alpha^{-\frac{p}{2}}\mathcal{I}_1^{\frac{p}{2}}\left(\int_\Omega |\nabla u_n|^p + |\nabla u_{n+1}|^p dx\right)^{\frac{2-p}{2}} \\
&\leq C_1\alpha^{-\frac{p}{2}}\mathcal{I}_1^{\frac{p}{2}}\left[\|u_n\|_{W_0^{1,p}(\Omega)}^{\frac{p(2-p)}{2}} + \|u_{n+1}\|_{W_0^{1,p}(\Omega)}^{\frac{p(2-p)}{2}}\right] \\
&\leq C_2\mathcal{I}_1^{\frac{p}{2}}.
\end{aligned} \tag{3.6}$$

From (3.6), we obtain

$$\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^p}{(1+T_n(u_n))^{\theta(p-1)}}dx \leq C_3 k\mathcal{I}_2 \tag{3.7}$$

Now, for sufficiently small k, and for every $\mu > 0$, one has in $B_k = \{x \in \Omega : 0 \leq (u_n - u_{n+1})^+ \leq k\}$

$$\frac{1}{2^\mu}(1+T_n(u_n))^\mu \leq (1+T_{n+1}(u_{n+1}))^\mu \leq 2^\mu (1+T_n(u_n))^\mu.$$

This implies that

$$\begin{aligned}
\frac{1}{(1+T_{n+1}(u_{n+1})^{\theta(p-1)}} + \frac{1}{(1+T_n(u_n)^{\theta(p-1)}} &= \frac{(1+T_n(u_n))^{-\frac{\theta(p-1)^2}{p}}(1+T_{n+1}(u_{n+1}))^{\frac{\theta(p-1)^2}{p}}}{(1+T_n(u_n))^{\frac{\theta(p-1)}{p}}(1+T_{n+1}(u_{n+1}))^{\frac{\theta(p-1)}{p'}}} \\
&\quad + \frac{(1+T_{n+1}(u_{n+1}))^{\frac{-\theta(p-1)}{p}}(1+T_n(u_n))^{\frac{\theta(p-1)}{p}}}{(1+T_n(u_n))^{\frac{\theta(p-1)}{p}}(1+T_{n+1}(u_{n+1}))^{\frac{\theta(p-1)}{p'}}} \\
&\leq \frac{C_4}{(1+T_n(u_n))^{\frac{\theta(p-1)}{p}}(1+T_{n+1}(u_{n+1}))^{\frac{\theta(p-1)}{p'}}}.
\end{aligned}$$

From the previous estimate and using Hölder's inequality, we get

$$\begin{aligned}
\mathcal{I}_2 &\leq C_4 \int_{B_k} \frac{|\nabla u_{n+1}|^{p-1}}{(1+T_{n+1}(u_{n+1}))^{\frac{\theta(p-1)}{p'}}}\frac{|\nabla T_k((u_n-u_{n+1})^+)|}{(1+T_n(u_n))^{\frac{\theta(p-1)}{p}}}dx \\
&\leq C_4\left(\int_{B_k}\frac{|\nabla u_{n+1}|^p}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}}dx\right)^{\frac{1}{p'}}\left(\int_{B_k}\frac{|\nabla T_k((u_n-u_{n+1})^+)|^p}{(1+T_n(u_n))^{\theta(p-1)}}dx\right)^{\frac{1}{p}}.
\end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8) gives

$$\int_\Omega \frac{|\nabla T_k((u_n-u_{n+1})^+)|^p}{(1+T_n(u_n))^{\theta(p-1)}}dx \leq C_5 k^{p'}\int_{B_k}\frac{|\nabla u_{n+1}|^p}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}}dx \tag{3.9}$$

On the other hand, by Poincaré's inequality and (3.9), we have

$$\begin{aligned}
k^p \text{meas}\{B_k^c\} &= k^p \text{meas}\{x \in \Omega : (u_n - u_{n+1})^+ > k\} dx \\
&\le \int_\Omega |T_k((u_n - u_{n+1})^+)|^p dx \\
&\le C_6 \int_\Omega |\nabla T_k((u_n - u_{n+1})^+)|^p dx \\
&= C_6 \int_\Omega \frac{|\nabla T_k((u_n - u_{n+1})^+)|^p}{(1+T_n(u_n))^{\theta(p-1)}} (1+T_n(u_n))^{\theta(p-1)} dx \\
&\le C_6 \int_\Omega \frac{|\nabla T_k((u_n - u_{n+1})^+)|^p (1+n)^{\theta(p-1)}}{(1+T_n(u_n))^{\theta(p-1)}} dx \\
&\le C_7 k^{p'} (n+1)^{\theta(p-1)} \int_{B_k} \frac{|\nabla u_{n+1}|^p}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}} dx,
\end{aligned}$$

and so we get,

$$\text{meas}\{x \in \Omega : (u_n - u_{n+1})^+ > k\} \le C_{8,n} k^{p'-p} \int_{B_k} \frac{|\nabla u_{n+1}|^p}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}} dx.$$

Since, as k tends to zero, we have

$$B_k = \{x \in \Omega : 0 \le (u_n - u_{n+1})^+ \le k\} \to 0. \tag{3.10}$$

Using (3.10), $1 < p \le 2$ and the fact that $\frac{|\nabla u_{n+1}|^p}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}}$ belongs to $L^1(\Omega)$ (since $\nabla u_{n+1} \in L^p(\Omega)$ and $\frac{1}{(1+T_{n+1}(u_{n+1}))^{\theta(p-1)}} \le 1$), we deduce that

$$\text{meas}\{x \in \Omega : (u_n - u_{n+1})^+ \ge 0\} = 0.$$

This implies that $u_n \le u_{n+1}$ a.e. in Ω. □

Remark 3.1 *Ramark that, $1 < p \le 2 \Rightarrow p' - p > 0$. So, the condition $1 < p \le 2$ is necessary in order prove that $\{u_n\}$ is increasing. This result of monotonicity plays a crucial role in the proof of the following Lemma.*

Lemma 3.3 *Let u_n be the solution to problem (3.1). Then, for every $\omega \subset\subset \Omega$, there exists $c_\omega > 0$ such that $u_n \ge c_\omega$ a.e. in ω for every $n \in \mathbb{N}$.*

Proof: By lemma 3.1, one has u_1 is bounded, that is $\|u_1\|_{L^\infty(\Omega)} \le C_\infty$, for some positive constant C_∞. We set $h(s) = \frac{1}{(1+T_1(s))^{\theta(p-1)}}$. Then u_1 satisfies, distributionally,

$$-\text{div}(a(x)h(u_1)|\nabla u_1|^{p-2}\nabla u_1) \ge \frac{T_1(f)}{(1+\|u_1\|_{L^\infty(\Omega)})^\gamma} \ge \frac{T_1(f)}{(1+C_\infty)^\gamma},$$

that is,

$$-\text{div}(a(x)|\nabla u_1|^{p-2}\nabla u_1)h(u_1) - h'(u_1)a(x)|\nabla u_1|^p \ge C_9 T_1(f). \tag{3.11}$$

We define, for $s \ge 0$

$$\psi(s) = \int_0^s (h(t))^{\frac{\beta}{\alpha(p-1)}} dt. \tag{3.12}$$

We remark that (3.12) implies

$$\psi'(s) = (h(s))^{\frac{\beta}{\alpha(p-1)}}, \quad \text{and} \quad \frac{\psi''(s)}{\psi'(s)} = \frac{\beta}{\alpha(p-1)} \frac{h'(s)}{h(s)} \tag{3.13}$$

We define $v = \psi(u_1)$. Then

$$\begin{aligned}\operatorname{div}\big(a(x)|\nabla v|^{p-2}\nabla v\big) &= |\psi'(u_1)|^{p-2}\psi'(u_1)\operatorname{div}\big(a(x)|\nabla u_1|^{p-2}\nabla u_1\big)\\ &\quad + (p-1)a(x)|\nabla u_1|^p\psi''(u_1)|\psi'(u_1)|^{p-2},\end{aligned}$$

and therefore

$$\begin{aligned}-h(u_1)\operatorname{div}\big(a(x)|\nabla u_1|^{p-2}\nabla u_1\big) &= -h(u_1)\frac{\operatorname{div}\big(a(x)|\nabla v|^{p-2}\nabla v\big)}{|\psi'(u_1)|^{p-2}\psi'(u_1)}\\ &\quad + (p-1)a(x)h(u_1)\frac{\psi''(u_1)}{\psi'(u_1)}|\nabla u_1|^p. \end{aligned} \tag{3.14}$$

By inequality (3.11), (3.13) and (3.14), we have

$$-h(u_1)\frac{\operatorname{div}\big(a(x)|\nabla v|^{p-2}\nabla v\big)}{|\psi'(u_1)|^{p-2}\psi'(u_1)} + \frac{\beta}{\alpha}a(x)h'(u_1)|\nabla u_1|^p - a(x)h'(u_1)|\nabla u_1|^p \geq C_9T_1(f). \tag{3.15}$$

Using that $h'(s) \leq 0$, $\alpha \leq a(x) \leq \beta$, and (3.15), we obtain

$$-h(u_1)\frac{\operatorname{div}\big(a(x)|\nabla v|^{p-2}\nabla v\big)}{|\psi'(u_1)|^{p-2}\psi'(u_1)} \geq C_9T_1(f), \tag{3.16}$$

using the fact that

$$|\psi'(s)|^{p-2}\psi'(s) = (h(s))^{\frac{\beta}{\alpha}} \geq 0. \tag{3.17}$$

By (3.16),(3.17) and $h_1(u_1) \geq \frac{1}{(1+C_\infty)^{\theta(p-1)}}$, we get

$$-\operatorname{div}(a(x)|\nabla v|^{p-2}\nabla v) \geq C_9(h(u_1))^{\frac{\beta}{\alpha}-1}T_1(f) \geq \frac{C_9T_1(f)}{(C_\infty+1)^{\theta(p-1)(\frac{\beta}{\alpha}-1)}},$$

thus,

$$-\operatorname{div}(a(x)|\nabla v|^{p-2}\nabla v) \geq C_{10}T_1(f) \geq 0.$$

Let z be the $W_0^{1,p}(\Omega)$ solution to $-\operatorname{div}(a(x)|\nabla z|^{p-2}\nabla z) = C_{10}T_1(f)$. By the strong maximum principle, for every $\omega \subset\subset \Omega$ there exists a constant $\tilde{c}_\omega > 0$ such that $z \geq \tilde{c}_\omega$ a.e. in ω. By the comparison principle, we have $v \geq z \geq \tilde{c}_\omega$ a.e. in ω. Recalling that $v = \psi(u_1)$, one has $\psi(u_1) \geq \tilde{c}_\omega$ a.e. in ω. Since u_n is an increasing sequence and ψ is increasing, we deduce that $u_n \geq \psi^{-1}(\tilde{c}_\omega) = c_\omega > 0$ a.e. in ω for every $n \in \mathbb{N}$. □

4. Uniform estimates

In this section, C will denote a constant (not depending on n) that may change from line to line.

Lemma 4.1 *Assume that $(\theta-1)(p-1) \leq \gamma < \theta(p-1)+1$.*

(i) Let $f \in L^m(\Omega)$ with $m > \frac{N}{p}$ and let u_n be a solution of (3.1). Then, the sequence u_n is bounded in $W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$.

(ii) Let $f \in L^m(\Omega)$ with m satisfied (2.2) and let u_n be a solution of (3.1). Then, the sequence u_n is bounded in $W_0^{1,p}(\Omega) \cap L^s(\Omega)$, where s as in (2.3).

(iii) Let $f \in L^m(\Omega)$ with m satisfied (2.4) and let u_n be a solution of (3.1). Then, the sequence u_n is bounded in $W_0^{1,\eta}(\Omega)$, where η as in (2.5).

Proof:

(i) We define $H:\mathbb{R}^+\to\mathbb{R}$ as

$$H(s)=\int_0^s \frac{1}{(1+t)^{\theta-\frac{\gamma}{p-1}}}dt,$$

noting that $H(s)\to\infty$ as $s\to\infty$ if and only if $\gamma\geq(\theta-1)(p-1)$. Hence, the proof of (i) will be concluded once that is shown that $H(u_n)$ is bounded. Let us consider $\varphi(u_n)=G_k(H(u_n))(1+u_n)^\gamma, k\geq 1$, with $G_k(s)=s-T_k(s)$ for all $s\in\mathbb{R}$. Since $u_n\in W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$ be a solution of (3.1) and by definition of H and G_k, then φ it belongs to $W_0^{1,p}(\Omega)\cap L^\infty(\Omega)$. Testing (3.1) with $\varphi(u_n)$, using the hypotheses on a, dropping the non-negative terms on the left-hand side and the fact that

$$\nabla\varphi(u_n)=H'(u_n)(1+u_n)^\gamma\nabla u_n+\gamma(1+u_n)^{\gamma-1}G_k(H(u_n))\nabla u_n,$$

we obtain

$$\alpha\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{\frac{p[\theta(p-1)-\gamma]}{p-1}}}dx\leq\int_\Omega f_nG_k(H(u_n))\frac{(1+u_n)^\gamma}{u_n^\gamma}dx.$$

The previous inequality and the fact that

$$\forall a>0,\ \forall\mu>0,\ \exists\, C(\mu,a)>0,\ (1+t)^\mu\leq Ct^\mu,\quad \forall t\in[a,+\infty), \tag{4.1}$$

yielding

$$\begin{aligned}\int_\Omega|\nabla G_k(H(u_n))|^pdx&=\int_\Omega\frac{|\nabla u_n|^p}{(1+u_n)^{\frac{p[\theta(p-1)-\gamma]}{p-1}}}dx\\ &\leq C\int_\Omega fG_k(H(u_n))dx.\end{aligned} \tag{4.2}$$

the inequality (4.2) is exactly the starting point of Stampacchia's $L^\infty-$regularity proof (see [28]), so that there exists a constant C independent of n such that $0\leq H(u_n)\leq C$. Therefore, the strict monotonicity of H implies the boundedness of the sequence u_n in $L^\infty(\Omega)$. It is easy to get an estimation in $W_0^{1,p}(\Omega)$. Taking u_n as test function in formulation (3.1). Using the hypotheses on a, the boundedness of the sequence u_n in $L^\infty(\Omega)$ and Hölder inequality, we obtain

$$\alpha C\int_\Omega|\nabla u_n|^pdx\leq\left\|u_n^{1-\gamma}\right\|_{L^\infty(\Omega)}\int_\Omega fdx\leq C, \tag{4.3}$$

so that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. This finishes the proof of the case (i).

(ii) Let us use $\varphi=(1+u_n)^{\theta(p-1)+1}-1$ as test function in (3.1); the hypotheses on a and (4.1) imply that

$$\alpha[\theta(p-1)+1]\int_\Omega|\nabla u_n|^pdx\leq C\int_\Omega|f|u_n^{\theta(p-1)+1-\gamma}dx.$$

By Sobolev's inequality (with exponent $r=p^*$) on the left-hand side and Hölder's inequality (with exponent $m_1>1$) in the right one, we obtain

$$\left(\int_\Omega u_n^{p^*}dx\right)^{\frac{p}{p^*}}\leq C\|f\|_{L^{m_1}(\Omega)}\left(\int_\Omega u_n^{m_1'[\theta(p-1)+1-\gamma]}dx\right)^{\frac{1}{m_1'}}.$$

We remark that $p^*=m_1'[\theta(p-1)+1-\gamma]$. Moreover, $\frac{p}{p^*}\geq\frac{1}{m_1'}$. Then the above estimate implies that the sequence u_n is bounded in $L^{p^*}(\Omega)$ and in $W_0^{1,p}(\Omega)$.

Now, we are going to prove that u_n is bounded in $L^s(\Omega)$ where s as in (2.3). We take $\varphi=(1+u_n)^\lambda-1$ as test function in (3.1), by the hypotheses on a, one has

$$\alpha\lambda\int_\Omega\frac{|\nabla u_n|^p}{(1+u_n)^{\theta(p-1)-\lambda+1}}dx\leq\int_\Omega\frac{f_n}{(u_n+\frac{1}{n})^\gamma}[(1+u_n)^\lambda-1]dx,$$

thus,

$$\int_\Omega \left|\nabla\left[(1+u_n)^{\frac{-\theta(p-1)+\lambda+p-1}{p}}-1\right]\right|^p dx \le C\int_\Omega f(1+u_n)^{\lambda-\gamma}dx.$$

By Sobolev's inequality on the left-hand side and Hölder's inequality on the right one we have

$$\left(\int_\Omega \left|\left[(1+u_n)^{\frac{-\theta(p-1)+\lambda+p-1}{p}}-1\right]\right|^{p^*} dx\right)^{\frac{p}{p^*}} \le C\|f\|_{L^m(\Omega)}\left(\int_\Omega (1+u_n)^{m'(\lambda-\gamma)}dx\right)^{\frac{1}{m'}} \tag{4.4}$$

Let $\lambda>\gamma$ be such that

$$[-\theta(p-1)+\lambda+p-1]\frac{N}{N-p}=(\lambda-\gamma)\frac{m}{m-1},$$

the previous equality is equivalent to

$$\lambda=\frac{N(m-1)(1-\theta)(p-1)+\gamma m(N-p)}{N-pm},$$

since $m<\frac{N}{p}$ and $\gamma>(\theta-1)(p-1)$, we observe that

$$(\lambda-\gamma)\frac{m}{m-1}=\frac{Nm[(p-1)(1-\theta)+\gamma]}{N-pm}=s>1. \tag{4.5}$$

Hence, it follows from (4.4) and (4.5) that u_n is bounded in $L^s(\Omega)$.

(iii) Let us consider $\varphi=(1+u_n)^\tau-1$ as a test function in (3.1) with

$$\tau=\frac{N(m-1)(1-\theta)(p-1)+\gamma m(N-p)}{N-pm}.$$

With the same arguments as before, we have

$$\begin{aligned}\left(\int_\Omega \left|\left[(1+u_n)^{\frac{-\theta(p-1)+\tau+p-1}{p}}-1\right]\right|^{p^*} dx\right)^{\frac{p}{p^*}} &\le C\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{\theta(p-1)-\tau+1}}dx\\ &\le C\|f\|_{L^m(\Omega)}\left(\int_\Omega (1+u_n)^{m'(\tau-\gamma)}dx\right)^{\frac{1}{m'}}.\end{aligned} \tag{4.6}$$

As above, we conclude that u_n is bounded in $L^{\frac{N[(p-1)(1-\theta)+\tau]}{N-p}}(\Omega)$. We observe that $\gamma<\tau<\theta(p-1)+1$, by the assumption on m. Now, let $1<\eta<p$ By Hölder's inequality with exponent $\frac{p}{\eta}$ and (4.6), we obtain

$$\begin{aligned}\int_\Omega |\nabla u_n|^\eta dx &= \int_\Omega \frac{|\nabla u_n|^\eta}{(1+u_n)^{[\theta(p-1)-\tau+1]\frac{\eta}{p}}}(1+u_n)^{[\theta(p-1)-\tau+1]\frac{\eta}{p}}dx\\ &\le \left(\int_\Omega \frac{|\nabla u_n|^p}{(1+u_n)^{\theta(p-1)-\tau+1}}dx\right)^{\frac{\eta}{p}}\left(\int_\Omega (1+u_n)^{\frac{\eta[\theta(p-1)-\tau+1]}{p-\eta}}dx\right)^{\frac{p-\eta}{p}}\\ &\le C\left(\int_\Omega (1+u_n)^{\frac{\eta[\theta(p-1)-\tau+1]}{p-\eta}}dx\right)^{\frac{p-\eta}{p}}.\end{aligned} \tag{4.7}$$

The estimates (4.7) imply that the sequences u_n is bounded in $W_0^{1,\eta}(\Omega)$ if

$$\frac{\eta[\theta(p-1)-\tau+1]}{p-\eta}=\frac{N[(p-1)(1-\theta)+\tau]}{N-p},$$

that is

$$\eta=\frac{Nm[(p-1)(1-\theta)+\gamma]}{N-m[\theta(p-1)+1-\gamma]}.$$

This completes the proof of Lemma 4.1.

□

Lemma 4.2 *Suppose that the hypotheses of Theoreme 2.1 are satisfied. Then, the solution u_n to problem (3.1) are uniformly bounded in $W_0^{1,p}(\Omega)$.*

Proof: Let us choose $\varphi = (1+u_n)^{\theta(p-1)+1} - 1$ as a test function in (3.1). Using the hypotheses on a, we have

$$\alpha[\theta(p-1)+1]\int_\Omega |\nabla u_n|^p dx \leq C\int_\Omega |f|dx \leq C\|f\|_{L^1(\Omega)}.$$

The previous estimate implies that the sequence u_n is bounded in $W_0^{1,p}(\Omega)$. □

Lemma 4.3 *Suppose that the hypotheses of Theoreme 2.2 are satisfied. Then, the solution u_n to problem (3.1) are uniformly bounded in $W_{loc}^{1,p}(\Omega)$ and $u_n^{\frac{\gamma+(p-1)(1-\theta)}{p}}$ is uniformly bounded in $W_0^{1,p}(\Omega)$.*

Proof: Taking u_n^γ as test function in (3.1), since $\dfrac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1$, us the hypotheses on a, we obtain

$$\begin{aligned}\frac{p\alpha\gamma}{(\gamma+(p-1)(1-\theta))^p}\int_\Omega \left|\nabla\left(u_n^{\frac{\gamma+(p-1)(1-\theta)}{p}}\right)\right|^p dx &\leq \alpha\gamma\int_\Omega |\nabla u_n|^p u_n^{\gamma-1-\theta(p-1)}dx \\ &\leq \int_\Omega |f|dx.\end{aligned}$$

This proves that the sequence $u_n^{\frac{\gamma+(p-1)(1-\theta)}{p}}$ is bounded in $W_0^{1,p}(\Omega)$. Now by Sobolev's inequality applied in the left hand side, we deduce the boundedness of u_n in $L^s(\Omega)$ with $s = \frac{(\gamma+(p-1)(1-\theta))p^*}{p}$.

To prove the $W_{loc}^{1,p}(\Omega)$ bounded, let φ be a function in $C_0^\infty(\Omega)$, and let ω be the set $\{\varphi \neq 0\}$. Choosing $\left[(u_n+1)^{\theta(p-1)+1} - 1\right]\varphi^p$ as test function in (3.1), one has by the hypotheses on a and Lemma 3.3, we have

$$\begin{aligned}&\alpha(\theta(p-1)+1)\int_\Omega |\nabla u_n|^p\varphi^p dx + p\alpha\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\varphi^{p-1}u_n dx \\ &\leq \int_\Omega \frac{|f|}{(u_n+\frac{1}{n})^\gamma}\left[(u_n+1)^{\theta(p-1)+1} - 1\right]\varphi^p dx \\ &\leq c_\omega\int_\Omega |f|\varphi^p dx \\ &\leq c_\omega\|\varphi\|_{L^\infty(\Omega)}^p\int_\Omega |f|dx. \qquad (4.8)\end{aligned}$$

On other hand, using Young's inequality, we get

$$p\alpha\left|\int_\Omega |\nabla u_n|^{p-2}\nabla u_n\nabla\varphi\varphi^{p-1}u_n dx\right| \leq \varepsilon\int_\Omega |\nabla u_n|^p\varphi^p dx + C(\varepsilon)\int_\Omega |\nabla\varphi u_n|^p dx. \qquad (4.9)$$

By (4.8), (4.9) and the boundedness of u_n in $L^s(\Omega)$ (where $s \geq p$) yields

$$\begin{aligned}\int_\Omega |\nabla u_n|^p\varphi^p dx &\leq C\int_\Omega |\nabla\varphi|^p u_n^p dx + c_\omega\|\varphi\|_{L^\infty(\Omega)}^p\int_\Omega |f|dx \\ &\leq C\|\nabla\varphi\|_{L^\infty(\Omega)}^p\int_\Omega u_n^p dx + c_\omega\|\varphi\|_{L^\infty(\Omega)}^p\int_\Omega |f|dx \\ &\leq c_\omega,\end{aligned}$$

so that the sequence u_n is bounded in $W_{loc}^{1,p}(\Omega)$ as desired. □

5. Proof of Theorem 2.1-2.3

We only give the proof of Theorem 2.1 because Theorem 2.2 and 2.3 can be proved in a similar way. We restrict to the proof of point (1); the second and the third point are similar.

By Lemma 4.1, the sequence $(u_n)_n$ is bounded in $W_0^{1,p}(\Omega)$. Therefore, there exists a function $u \in W_0^{1,p}(\Omega)$ such that (up to a subsequence)

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,p}(\Omega), \\ u_n \to u & \text{a.e. in } \Omega. \end{cases} \tag{5.1}$$

Step 1 : Since u_n satisfies Lemma 3.3. For every $\varphi \in C_0^\infty(\Omega)$, we have

$$0 \le \left| \frac{f_n \varphi}{u_n + \frac{1}{n}} \right| \le \frac{\|\varphi\|_{L^\infty(\Omega)}}{c_\omega} f,$$

where ω is the set $\{\varphi \neq 0\}$. Therefore, by Lebesgue theorem, one has

$$\lim_{n\to+\infty} \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} dx = \int_\Omega \frac{f\varphi}{u^\gamma} dx, \tag{5.2}$$

Step 2 : Now, we are going to prove

$$\nabla u_n \to \nabla u \text{ a.e. in } \Omega.$$

Taking $u_n - u$ as a test function in (3.1), we have

$$\int_\Omega \frac{a(x)}{(1+T_n(u_n))^{\theta(p-1)}} |\nabla u_n|^{p-2} \nabla u_n \nabla(u_n - u) dx = \int_\Omega \frac{T_n(f)(u_n-u)}{(u_n + \frac{1}{n})^\gamma} dx.$$

So that

$$\begin{aligned} &\int_\Omega \frac{a(x)}{(1+T_n(u_n))^{\theta(p-1)}} \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right] \nabla(u_n-u) dx \\ &= \int_\Omega \frac{T_n(f)(u_n-u)}{(u_n+\frac{1}{n})^\gamma} dx + \int_\Omega \frac{a(x)}{(1+T_n(u_n))^{\theta(p-1)}} |\nabla u|^{p-2}\nabla u \nabla(u_n-u)dx. \end{aligned}$$

Passing to the limit as $n \to \infty$ in the previous inequality and using (5.1), (5.2), we can easily prove that

$$I_n = \int_\Omega \left[|\nabla u_n|^{p-2}\nabla u_n - |\nabla u|^{p-2}\nabla u \right] \nabla(u_n-u)dx \longrightarrow 0, \quad \text{as } n \to \infty. \tag{5.3}$$

Now, By Holder's inequality and (3.5) we get

$$\begin{aligned} \int_\Omega |\nabla u_n - \nabla u|^p \, dx &\le \int_\Omega \frac{|\nabla u_n - \nabla u|^p}{(|\nabla u_n| + |\nabla u|)^{\frac{p(2-p)}{2}}} (|\nabla u_n| + |\nabla u|)^{\frac{p(2-p)}{2}} \, dx \\ &\le \left(\int_\Omega \frac{|\nabla u_n - \nabla u|^2}{(|\nabla u_n| + |\nabla u|)^{2-p}} dx \right)^{\frac{p}{2}} \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^p dx \right)^{\frac{2-p}{p}} \\ &\le I_n \left(\int_\Omega (|\nabla u_n| + |\nabla u|)^p dx \right)^{\frac{2-p}{p}}. \end{aligned} \tag{5.4}$$

Since u_n is bounded in $W_0^{1,p}(\Omega)$ and $u \in W_0^{1,p}(\Omega)$, after letting $n \to +\infty$ in (5.4) we find

$$\lim_{n\to+\infty} \int_\Omega |\nabla u_n - \nabla u|^p \, dx = 0,$$

which implies,

$$\nabla u_n \to \nabla u \ \text{ strongly in } L^p(\Omega) \text{ and } \text{ a.e. in } \Omega. \tag{5.5}$$

Step 2 : We are going to prove (2.1) by passing to the limit in

$$\int_\Omega a(x)\frac{|\nabla u_n|^{p-2}\nabla u_n.\nabla\varphi}{(1+T_n(u_n))^{\theta(p-1)}}dx = \int_\Omega \frac{T_n(f)\varphi}{(u_n+\frac{1}{n})^\gamma}dx, \tag{5.6}$$

for every $\varphi \in C_0^\infty(\Omega)$. By (5.5) we have

$$|\nabla u_n|^{p-2}\nabla u_n \to |\nabla u|^{p-2}\nabla u \quad \text{weakly in } L^{p'}(\Omega). \tag{5.7}$$

Since

$$\frac{a(x)}{(1+T_n(u_n))^{\theta(p-1)}} \to \frac{a(x)}{(1+u)^{\theta(p-1)}} \text{in } L^r(\Omega), \text{for every } r \geq 1. \tag{5.8}$$

From (5.7), (5.8), we obtain

$$\int_\Omega \frac{a(x)|\nabla u_n|^{p-2}\nabla u_n\varphi}{(1+T_n(u_n))^{\theta(p-1)}}dx \longrightarrow \int_\Omega \frac{a(x)|\nabla u|^{p-2}\nabla u\varphi}{(1+u)^{\theta(p-1)}}dx, \tag{5.9}$$

for every $\varphi \in C_0^\infty(\Omega)$. Using (5.2) and (5.9), and passing to the limit in (5.6) we get (2.1).

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