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Invariant, Anti Invariant and Slant Submanifolds of Locally Poly-Norden Manifolds

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ABSTRACT: In this paper, we study locally almost 3-poly-Norden manifolds and prove several properties of the curvature tensor and Ricci tensor of these manifolds. We investigate invariant, anti-invariant and slant submanifolds of almost 3-poly-Norden manifolds from various views. We characterize these submanifolds and give non-trivial examples.

Key Words: Locally 3-poly-Norden manifold, integrable structure, slant submanifold.

Contents

1. Introduction

As a generalization of invariant and anti-invariant submanifolds, we investigate slant submanifolds of almost 3-poly-Norden manifolds [12]. Many authors have studied special models of slant submanifolds in various structures, for example hemi-slant, invariant, anti-invariant, slant and semi-slant submanifolds of metallic manifolds have been reviewed in [1,3,6]. Furthermore, in [2,8] slant submanifolds of almost contact 3-structure manifolds, golden manifolds and locally conformal Kaehler manifolds have been investigated and the slant light-like submanifolds of indefinite cosymplectic manifold have been studied in [5].

At the present paper, we first define an almost 3-poly-Norden manifold and give an example for that in Section 2. In Section 3, we introduce ϕ-invariant and ϕ-anti-invariant submanifolds of almost 3-poly-Norden manifolds. Also, in Section 3, we investigate and characterize slant submanifolds and give an example of a slant submanifold of an almost 3-poly-Norden manifold.

2. Almost 3-Poly-Norden Manifolds

Definition 2.1 *Let $(\overline{M}, g)$ be a smooth Riemannian manifold. An almost poly-Norden structure on $\overline{M}$ is a (1,1)-tensor ϕ which satisfies*

$$\phi^2 = m\phi - I, \tag{2.1}$$

where I is identity operator on $\overline{M}$. In this case $(\overline{M}, \phi)$ is called an almost poly-Norden manifold [10].

We say that Riemannian metric $\overline{g}$ is ϕ-compatible if

$$\overline{g}(\phi X, \phi Y) = m\overline{g}(\phi X, Y) - \overline{g}(X, Y), \tag{2.2}$$

for any $X, Y \in \Gamma(T\overline{M})$. It follows ϕ is symmetric with respect to $\overline{g}$, that is

$$\overline{g}(\phi X, Y) = \overline{g}(X, \phi Y). \tag{2.3}$$

The Levi-Civita connection on $\overline{M}$ will be denoted by $\overline{\nabla}$.

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Definition 2.2 *An almost poly-Norden structure* $(\overline{M}, \phi)$ *is called integrable if its Nijenhuis tensor field* N_ϕ *vanishes, where* $N_\phi(X,Y) := \phi^2[X,Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$.

Note that $N_\phi = 0$ is equivalent to $\overline{\nabla}\phi = 0$ [10].

Definition 2.3 *An almost poly-Norden Riemannian manifold* $(\overline{M}, \phi, \overline{g})$ *is a locally almost poly-Norden manifold if* ϕ *is parallel with respect to the Levi-Civita connection associated to* $\overline{g}$.

Lemma 2.1 *Let* $(\overline{M}, \phi, \overline{g})$ *be a locally almost poly-Norden Riemannian manifold, then we have* $(\overline{\nabla}_X\phi)Y = -(\overline{\nabla}_Y\phi)X$ *for any* $X, Y \in \Gamma(T\overline{M})$.

Proof: In the account of the locally almost poly-Norden definition, we obtain for any $X \in \Gamma(T\overline{M})$

$$(\overline{\nabla}_X\phi)X = 0, \tag{2.4}$$

since by putting $Y := X$ in $(\overline{\nabla}_X\phi)Y = 0$ for any $X \in \Gamma(T\overline{M})$ in this case the above equation is obtained. So, we set $X + Y$ instead of X in the Equation (2.4), and then for any X,Y $\in \Gamma(T\overline{M})$ we conclude

$$\begin{aligned}
&(\overline{\nabla}_{X+Y}\phi)(X+Y) = 0,\\
&(\overline{\nabla}_X\phi)X + (\overline{\nabla}_Y\phi)X + (\overline{\nabla}_X\phi)Y + (\overline{\nabla}_Y\phi)Y = 0,\\
&(\overline{\nabla}_X\phi)Y = -(\overline{\nabla}_Y\phi)X.
\end{aligned}$$

□

Now as a generalization of almost poly-Norden manifolds, we introduce almost 3-poly-Norden manifolds. Let J_i, $i \in \{1,2,3\}$, be hyper-Hermitian structures on the Riemannian manifold $(\overline{M}, \overline{g})$ [9]. So, we have $J_i^2 = -1$ and $J_i \circ J_j = J_k$. By limiting the $m_i, m_j$and m_k to the interval $(0,2)$, we define

$$\begin{aligned}
\phi_i =& \frac{m_i}{2}I + \frac{\sqrt{4-{m_i}^2}}{2}J_i,\\
\phi_j =& \frac{m_j}{2}I + \frac{\sqrt{4-{m_j}^2}}{2}J_j,\\
\phi_k =& \frac{m_k}{2}I + \frac{\sqrt{4-{m_k}^2}}{2}J_k,
\end{aligned} \tag{2.5}$$

where I is the identity tensor field. Now, by using the condition $J_i \circ J_j = J_k$, we get the $\phi_i \circ \phi_j$ composition for any $X \in \Gamma(T\overline{M})$.

$$\begin{aligned}
\phi_i \circ \phi_j(X) &= \phi_i(\frac{m_j}{2}X + \frac{\sqrt{4-{m_j}^2}}{2}J_j(X)\\
&= \frac{m_j}{2}\phi_i(X) + \frac{\sqrt{4-{m_j}^2}}{2}\phi_i(J_j(X))\\
&= \frac{m_j}{2}(\frac{m_i}{2}X + \frac{\sqrt{4-{m_i}^2}}{2}J_i) + \frac{\sqrt{4-{m_j}^2}}{2}(\frac{m_i}{2}J_j(X)\\
&+ \frac{4-{m_i}^2}{2}J_i \circ J_j(X))\\
&= \frac{m_i m_j}{4}X + \frac{m_j\sqrt{4-{m_i}^2}}{4}J_i(X) + \frac{m_i\sqrt{4-{m_j}^2}}{4}J_j(X)\\
&+ \frac{\sqrt{(4-{m_j}^2)(4-{m_i}^2)}}{4}J_k(X)\\
&= \frac{m_i m_j}{4}X + \frac{m_j\sqrt{4-{m_i}^2}}{4}(\frac{-m_i}{\sqrt{4-{m_i}^2}}X + \frac{2}{\sqrt{4-{m_i}^2}}\phi_i(X))\\
&+ \frac{m_i\sqrt{4-{m_j}^2}}{4}(\frac{-m_j}{\sqrt{4-{m_j}^2}}X + \frac{2}{\sqrt{4-{m_j}^2}}\phi_j(X))\\
&+ \frac{\sqrt{(4-{m_j}^2)(4-{m_i}^2)}}{4}(\frac{-m_k}{\sqrt{4-{m_k}^2}}X + \frac{2}{\sqrt{4-{m_k}^2}}\phi_k(X))\\
&= -(\frac{m_k\sqrt{(4-{m_j}^2)(4-{m_i}^2)}}{4\sqrt{4-{m_k}^2}} + \frac{m_i m_j}{4})X + \frac{m_j}{2}\phi_i(X)\\
&+ \frac{m_i}{2}\phi_j(X) + \frac{\sqrt{(4-{m_i}^2)(4-{m_j}^2)}}{2\sqrt{4-{m_k}^2}}\phi_k(X).
\end{aligned}$$

So if we put $\alpha = \frac{m_k\sqrt{(4-{m_j}^2)(4-{m_i}^2)}}{4\sqrt{4-{m_k}^2}} + \frac{m_i m_j}{4}$ and $\beta = \frac{\sqrt{(4-{m_i}^2)(4-{m_j}^2)}}{2\sqrt{4-{m_k}^2}}$, in that case we drive

$$\phi_i \circ \phi_j = -\alpha I + \frac{m_i}{2}\phi_j + \frac{m_j}{2}\phi_i + \beta\phi_k. \tag{2.6}$$

Thus, we say the Riemannian structure $(\overline{M}, \overline{g}, \phi_i)_{i\in\{1,2,3\}}$ is an almost 3-poly-Norden manifold in which ϕ_i's are defined in the Equation (2.5).

It should be noted that, conversely if ϕ_i, ϕ_j and ϕ_k for $i, j, k \in \{1, 2, 3\}$, are almost poly-Norden structures on manifold $\overline{M}$, then each given almost poly-Norden structure induces complex structures as follows

$$J_i = \frac{-m_i}{\sqrt{4-{m_i}^2}}I + \frac{2}{\sqrt{4-{m_i}^2}}\phi_i,$$

$$J_j = \frac{-m_j}{\sqrt{4-{m_j}^2}}I + \frac{2}{\sqrt{4-{m_j}^2}}\phi_j,$$

$$J_k = \frac{-m_k}{\sqrt{4-{m_k}^2}}I + \frac{2}{\sqrt{4-{m_k}^2}}\phi_k.$$

Example 2.1 *Let $\overline{M} = \mathbb{R}^8$ and J_i's be the natural hyper-complex structures on $\mathbb{R}^8$. In the above equa-*

tions, put the indices i, j *and* k *respectively* $1, 2, 3$. *Now suppose* $m_1 = m_2 = m_3 = 1$ *then we have*

$$\phi_1 = \frac{1}{2}I + \frac{\sqrt{3}}{2}J_1 \quad , \quad J_1 = \frac{-1}{\sqrt{3}}I + \frac{2}{\sqrt{3}}\phi_1,$$
$$\phi_2 = \frac{1}{2}I + \frac{\sqrt{3}}{2}J_2 \quad , \quad J_2 = \frac{-1}{\sqrt{3}}I + \frac{2}{\sqrt{3}}\phi_2,$$
$$\phi_3 = \frac{1}{2}I + \frac{\sqrt{3}}{2}J_3 \quad , \quad J_3 = \frac{-1}{\sqrt{3}}I + \frac{2}{\sqrt{3}}\phi_3,$$
$$\phi_1 \circ \phi_2(X) = -(\frac{\sqrt{3}+3}{4\sqrt{3}})X + \frac{1}{2}\phi_1(X) + \frac{1}{2}\phi_2(X) + \frac{3}{2\sqrt{3}}\phi_3(X).$$

Example 2.2 *Suppose (1-1)-tensors* J_1, J_2, J_3 *are defined as follows,*

$$J_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad J_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad J_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix},$$

we put $m_1 = \frac{1}{2}, m_2 = \frac{2}{3}, m_3 = \frac{3}{2}$ *and by using equations (2.5) we get* $\phi_i, i = 1, 2, 3$.

$$\phi_1 = \frac{1}{4}I + \frac{\sqrt{15}}{4}J_1 = \begin{pmatrix} \frac{1}{4} & -\frac{\sqrt{15}}{4} & 0 & 0 \\ \frac{\sqrt{15}}{4} & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & -\frac{\sqrt{15}}{4} \\ 0 & 0 & \frac{\sqrt{15}}{4} & \frac{1}{4} \end{pmatrix},$$

$$\phi_2 = \frac{1}{3}I + \frac{2\sqrt{2}}{3}J_2 = \begin{pmatrix} \frac{1}{3} & 0 & -\frac{2\sqrt{2}}{3} & 0 \\ 0 & \frac{1}{3} & 0 & \frac{2\sqrt{2}}{3} \\ \frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} & 0 \\ 0 & -\frac{2\sqrt{2}}{3} & 0 & \frac{1}{3} \end{pmatrix},$$

$$\phi_3 = \frac{3}{4}I + \frac{\sqrt{7}}{4}J_3 = \begin{pmatrix} \frac{3}{4} & 0 & 0 & -\frac{\sqrt{7}}{4} \\ 0 & \frac{3}{4} & -\frac{\sqrt{7}}{4} & 0 \\ 0 & \frac{\sqrt{7}}{4} & \frac{3}{4} & 0 \\ \frac{\sqrt{7}}{4} & 0 & 0 & \frac{3}{4} \end{pmatrix}.$$

If there exist a Riemannian metric g on almost 3-poly-Norden manifold $\overline{M}$ such that for any $X, Y \in \Gamma(T\overline{M})$ and $i, j = 1, 2, 3$

$$g(\phi_i X, Y) = g(X, \phi_i Y), \tag{2.7}$$

then $\overline{M}$ has an almost 3-poly-Norden metric structure $(\phi_i, g), i = 1, 2, 3,$ and $(\overline{M}, \phi_i, g), i = 1, 2, 3,$ is called an almost 3-poly-Norden metric manifold.
This relation is equivalent to

$$g(\phi_i X, \phi_i Y) = m_i g(\phi_i X, Y) - g(X, Y), \tag{2.8}$$

and

$$g(\phi_i X, \phi_j Y) = -\alpha g(X, Y) + \frac{m_j}{2} g(X, \phi_i Y) + \frac{m_i}{2} g(X, \phi_j Y) + \beta g(X, \phi_k Y). \tag{2.9}$$

Definition 2.4 *An almost 3-poly-Norden manifold $(\overline{M}, \phi_i)$ is called integrable if its Nijenhuis tensor fields N_{ϕ_i} , $i = 1, 2, 3$ vanish, where*

$$N_{\phi_i}(X, Y) = {\phi_i}^2[X, Y] + [\phi_i X, \phi_i Y] - \phi_i[\phi_i X, Y] - \phi_i[X, \phi_i Y].$$

Note that $N_{\phi_i} = 0$ is equivalent to $\overline{\nabla}\phi_i = 0$, where $\overline{\nabla}$ is Levi-Civita connection on $\overline{M}$.

Lemma 2.2 *If $(\overline{M}, g, \phi_i)$, $i = 1, 2, 3$ is an integrable almost 3-poly-Norden manifold then we have $(\overline{\nabla}_X \phi_i)Y = -(\overline{\nabla}_Y \phi_i)X, \ \forall X, Y \in \Gamma(T\overline{M})$.*

Proof: Since $\overline{M}$ is an integrable manifold, thus $(\overline{\nabla}_X \phi_i)Y = 0, \ \forall X, Y \in \Gamma(T\overline{M})$. Set $Y := X$, in this case we get $(\overline{\nabla}_X \phi_i)X = 0, \ \forall X \in \Gamma(T\overline{M})$. Now put $X + Y$ instead of X,

$$\begin{aligned}
&(\overline{\nabla}_{X+Y}\phi_i)(X + Y) = 0,\\
&(\overline{\nabla}_X \phi_i)(X) + (\overline{\nabla}_X \phi_i)(Y) + (\overline{\nabla}_Y \phi_i)(X) + (\overline{\nabla}_Y \phi_i)(Y) = 0,
\end{aligned}$$

according to the above relations, the result is obtained. □

Lemma 2.3 *Let $(\overline{M}, \phi_i)$, $i = 1, 2, 3$ be an integrable almost 3-poly-Norden manifold, then $\forall X, Y \in \Gamma(T\overline{M})$,*

i) $(\overline{\nabla}_X \phi_i)\phi_i Y = -\phi_i(\overline{\nabla}_X \phi_i)Y,$

ii) $(\overline{\nabla}_X \phi_i)\phi_j Y = -\phi_j(\overline{\nabla}_X \phi_i)Y + \frac{m_j}{2}(\overline{\nabla}_X \phi_i)Y + \frac{m_i}{2}(\overline{\nabla}_X \phi_j)Y + \beta(\overline{\nabla}_X \phi_k)Y$
$-\phi_i \overline{\nabla}_X \phi_j Y + \phi_j \overline{\nabla}_X \phi_i Y.$

Proof: The proof of the first part is clear. We use the Equation (2.6) to prove the second part.

$$\begin{aligned}
(\overline{\nabla}_X \phi_i)\phi_j Y + \phi_j(\overline{\nabla}_X \phi_i)Y &= \overline{\nabla}_X \phi_i \circ \phi_j Y - \phi_i \overline{\nabla}_X \phi_j Y + \phi_j \overline{\nabla}_X \phi_i Y - \phi_j \circ \phi_i \overline{\nabla}_X Y\\
&= -\alpha \overline{\nabla}_X Y + \frac{m_j}{2} \overline{\nabla}_X \phi_i Y + \frac{m}{2} \overline{\nabla}_X \phi_j Y + \beta \overline{\nabla}_X \phi_k Y\\
&- \phi_i \overline{\nabla}_X \phi_j Y + \phi_j \overline{\nabla}_X \phi_i Y + \alpha \overline{\nabla}_X Y - \frac{m_i}{2} \phi_j \overline{\nabla}_X Y\\
&- \frac{m_j}{2} \phi_i \overline{\nabla}_X Y - \beta \phi_k \overline{\nabla}_X Y\\
&= \frac{m_j}{2}(\overline{\nabla}_X \phi_i)Y + \frac{m_i}{2}(\overline{\nabla}_X \phi_j)Y + \beta(\overline{\nabla}_X \phi_k)Y - \phi_i \overline{\nabla}_X \phi_j Y\\
&+ \phi_j \overline{\nabla}_X \phi_i Y.
\end{aligned}$$

□

From the previous results, the following lemma is clear.

Lemma 2.4 *If $(\overline{M}, \phi_i, \overline{g}), i = 1, 2, 3$ is an almost 3-poly-Norden Riemannian manifold, then*

$$\overline{g}((\overline{\nabla}_X \phi_i)Y, Z) = \overline{g}(Y, (\overline{\nabla}_X \phi_i)Z)$$

for any $X, Y, Z \in \Gamma(T\overline{M})$.

3. Submanifolds of an Almost 3-Poly-Norden Manifold

Let (M, g) be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, \overline{g}, \phi_i), i = 1, 2, 3$, where g is the induced metric on M. We use same symbol g for the induced metric g and the metric $\overline{g}$. The Gauss and Weingarten formulas are given by [7]

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{3.1}$$
$$\overline{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{3.2}$$

where h is the second fundamental form and A is the shape operator, which are related to each other as follows

$$g(h(X, Y), V) = g(A_V X, Y),$$

for any $X, Y \in \Gamma(TM), V \in \Gamma(TM^{\perp})$.

Lemma 3.1 *Let $(\overline{M}, g, \phi_i), i = 1, 2, 3$ be an integrable almost 3-poly-Norden Riemannian manifold, if σ_i is a (1-1)-tensor on submanifold (M, g) of $\overline{M}$ such that $\sigma_i = \phi_i \mid_M$, then (M, g, σ_i) is an integrable almost 3-poly-Norden submanifold if and only if $h(X, \sigma_i Y) = \sigma_i h(X, Y)$.*

Proof: First we show that (M, σ_i) is an almost 3-poly-Norden Riemannian submanifold and then by using Equation (3.1) we prove that it is integrable.

$$\sigma_i^2 X = \phi_i^2 \mid_M X = m\phi_i \mid_M X - X = m\sigma_i X - X$$

since $\overline{M}$ is integrable then $(\overline{\nabla}_X \phi_i \mid_M)Y = 0$ so we have

$$\begin{aligned}
&\overline{\nabla}_X \phi_i \mid_M Y - \phi_i \mid_M \overline{\nabla}_X Y = 0,\\
&\nabla_X \sigma_i Y + h(X, \sigma_i Y) - \sigma_i \nabla_X Y - \sigma_i h(X, Y) = 0,\\
&(\nabla_X \sigma_i)Y + h(X, \sigma_i Y) - \sigma_i h(X, Y) = 0.
\end{aligned}$$

So (M, σ_i) is an integrable submanifold that is $(\nabla_X \sigma_i)Y = 0$ if and only if $h(X, \sigma_i Y) = \sigma_i h(X, Y)$. □

Definition 3.1 *Let M be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, \phi_i), i = 1, 2, 3$. We say M is ϕ_i-invariant of $\overline{M}$ if $\phi_i(TM) \subset TM$ and M is ϕ_i-anti-invariant of $\overline{M}$ if $\phi_i(TM) \subset (TM^{\perp})$.*

For each $X \in \Gamma(TM), V \in \Gamma(TM^{\perp})$ and $i = 1, 2, 3$ we put

$$\phi_i X = T_i X + N_i X \tag{3.3}$$
$$\phi_i V = t_i V - n_i V \tag{3.4}$$

where

$$T_i : \Gamma(TM) \longrightarrow \Gamma(TM), \quad N_i : \Gamma(TM) \longrightarrow \Gamma(TM^{\perp}),$$
$$t_i : \Gamma(TM^{\perp}) \longrightarrow \Gamma(TM), \quad n_i : \Gamma(TM^{\perp}) \longrightarrow \Gamma(TM^{\perp}),$$

From (3.3) and (3.4) we can easily get the following equations

$$g(T_i X, Y) = g(X, T_i Y), \tag{3.5}$$
$$g(n_i U, V) = g(U, n_i V), \tag{3.6}$$
$$g(n_i X, V) = g(X, t_i V), \tag{3.7}$$

for any $X, Y \in \Gamma(TM)$ and $U, V \in \Gamma(TM^{\perp})$.

Lemma 3.2 *For any $X, Y \in \Gamma(TM), U, V \in \Gamma(TM^{\perp})$ the following relations are satisfied.*

$$T_i^2 = mT_i - I - t_iN_i, \qquad N_i = \frac{1}{m}(N_iT_i + n_iN_i), \tag{3.8}$$

$$t_i = \frac{1}{m}(T_it_i + t_in_i), \qquad n_i^2 = mn_i - I - N_it_i. \tag{3.9}$$

Theorem 3.1 *Let (M, g) be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, \phi_i), i = 1, 2, 3$. Then M is a ϕ_i-invariant submanifold if and only if the induced structure (T_i, g) and (n_i, g) of M are almost poly-Norden 3-structure.*

Proof: If M is ϕ_i-invariant submanifold in this case, according to (3.3), $N_i = 0$. So according to the first equality of the (3.8) and the second part of the (3.9), (T_i, g) and (n_i, g) are almost 3-poly-Norden structures and vice versa. □

The covariant derivative of T_i and $N_i(t_i$ and $n_i), i = 1, 2, 3$, respectively, are given by

$$(\nabla_X T_i)Y = \nabla_X T_iY - T_i\nabla_X Y, \qquad (\overline{\nabla}_X N_i)Y = \nabla^{\perp}_X N_iY - N_i\nabla_X Y, \tag{3.10}$$

$$(\nabla_X t_i)V = \nabla_X t_iV - t_i\nabla_X V, \qquad (\overline{\nabla}_X n_i)V = \nabla^{\perp}_X n_iV - n_i\nabla^{\perp}_X V. \tag{3.11}$$

Lemma 3.3 *Let M be a submanifold of an integrable almost 3-poly-Norden manifold $(\overline{M}, \phi_i), i = 1, 2, 3$, then for any $X, Y \in \Gamma(TM), V \in \Gamma(TM^{\perp})$ we get*

$$g((\overline{\nabla}_X N_i)Y, V) = g((\nabla_X t_i)V, Y).$$

Proof: Since the ϕ_i is integrable then for any $X, Y \in \Gamma(TM)$, we have $\overline{\nabla}_X\phi_iY = \phi_i\overline{\nabla}_X Y$. Now by using Gauss and Weingarten formulas and Equations (3.3) and (3.4) we conclude

$$\begin{aligned}
&\overline{\nabla}_X T_iY + \overline{\nabla}_X N_iY = \phi_i(\nabla_X Y + h(X, Y)),\\
&\nabla_X T_iY + h(X, T_iY) - A_{N_iY}X + \nabla^{\perp}_X N_iY = T_i\nabla_X Y + N_i\nabla_X Y + t_ih(X, Y)\\
&+ n_ih(X, Y).
\end{aligned}$$

Separating the tangential and normal components implies

$$(\nabla_X T_i)Y = A_{N_iY}X + t_ih(X, TY), \tag{3.12}$$

$$(\overline{\nabla}_X N_i)Y = n_ih(X, Y) - h(X, T_iY). \tag{3.13}$$

And for any $X \in \Gamma(TM)$ and $V \in \Gamma(TM^{\perp})$, we get $\overline{\nabla}_X\phi_iV = \phi_i\overline{\nabla}_X V$. In the similar way, we get

$$\begin{aligned}
&\overline{\nabla}_X t_iV + \overline{\nabla}_X n_iV = \phi_i(-A_V X + \nabla^{\perp}_X V),\\
&\nabla_X t_iV + h(X, t_iV) - A_{n_iV}X + \nabla^{\perp}_X n_iV = -T_iA_V X - N_iA_V X + t_i\nabla^{\perp}_X V\\
&+ n_i\nabla^{\perp}_X V.
\end{aligned}$$

So we get it by separating the tangential and normal component

$$(\nabla_X t_i)V = A_{n_iV}X - T_iA_V X, \tag{3.14}$$

$$(\overline{\nabla}_X n_i)V = -h(X, t_iV) - N_iA_V X. \tag{3.15}$$

And finally, by using the above equations and (3.5) and (3.6) we obtain

$$\begin{aligned}
g((\overline{\nabla}_X N_i)Y, V) &= g(h(X, Y), n_iV) - g(h(X, T_iY), V)\\
&= g(A_{n_iV}X - T_iA_V X, Y)\\
&= g((\nabla_X t_i)V, Y).
\end{aligned}$$

□

Lemma 3.4 *Let M be a submanifold of an integrable almost 3-poly-Norden manifold $(\overline{M}, \phi_i), i = 1, 2, 3$. Then $(\overline{\nabla}_X N_i)Y = 0$ and $(\nabla_X t_i)V = 0$, for any $X, Y \in \Gamma(TM), V \in \Gamma(TM^{\perp})$ if and only if the shape operator A satisfies*

$$A_{n_i V} X = T_i A_V X = T_i A_V X = A_V T_i X, \ \ i = 1, 2, 3.$$

Proof: By using (3.13)

$$g(h(X, Y), n_i V) - g(h(X, T_i Y), V) = 0.$$

According to the relation between h and A and Equation (3.3) we have

$$g(A_{n_i V} X, Y) - g(A_V X, T_i Y) = 0,$$
$$g(A_{n_i V} X, Y) - g(T_i A_V X, Y) = 0.$$

So equality is achieved on the left. On the other hand, since $(\nabla_X t_i)V = 0$, then by using (3.14) we get

$$g(A_{n_i V} X - T_i A_V X, Y) = 0.$$

The right-hand of equation can also be obtained by the following statement,

$$g(X, A_{n_i V} Y) - g(X, A_V T_i Y) = 0.$$

Conversely, all of the above steps are reversible. □

Theorem 3.2 *Let $\overline{R}$ is the curvature tensor of an integrable almost 3-poly-Norden $(\overline{M}, \phi_i, g), i = 1, 2, 3$. Then for any $X, Y, Z, W \in \Gamma(TM)$, following relations hold*

i) $\overline{R}(X, Y)\phi_i = -\phi_i \overline{R}(X, Y)$,

ii) $\overline{R}(X, \phi_i Y) = \overline{R}(\phi_i X, Y)$,

iii) $\overline{R}(\phi_i X, \phi_i Y) = m_i \overline{R}(\phi_i X, Y) - \overline{R}(X, Y)$,

iv) $g(\overline{R}(X, Y)\phi_i Z, \phi_i W) = m_i g(\overline{R}(X, Y)Z, \phi_i W) - g(\overline{R}(X, Y)Z, W)$,

v) $g(\overline{R}(X, Y)\phi_i Z, W) = g(\overline{R}(X, Y)Z, \phi_i W)$,

vi) $\overline{R}(\phi_i X, \phi_j Y)Z = -\alpha \overline{R}(X, Y)Z + \frac{m_j}{2}\overline{R}(X, \phi_i Y)Z + \frac{m_i}{2}\overline{R}(X, \phi_j Y)Z + \beta \overline{R}(X, \phi_k Y)Z$,

vii) $g(\overline{R}(X, Y)\phi_i Z, \phi_j W) = -\alpha \overline{g}(\overline{R}(X, Y)Z, W) + \frac{m_j}{2} g(\overline{R}(X, Y)Z, \phi_i W) + \frac{m_i}{2} g(\overline{R}(X, Y)Z, \phi_j W) + \beta g(\overline{R}(X, Y)Z, \phi_k W)$.

Proof: By using Equations (2.6), (2.7) part (i) and properties of curvature tensor we prove only (vi) and (vii) items.

$$\begin{aligned}
vi) \quad g(\overline{R}(\phi_i X, \phi_j Y)Z, W) &= \overline{R}(\phi_i X, \phi_j Y, Z, W) \\
&= \overline{R}(Z, W, \phi_i X, \phi_j Y) = g(\overline{R}(Z, W)\phi_i X, \phi_j Y) \\
&= g(\phi_i \overline{R}(Z, W)X, \phi_j Y) = g(\overline{R}(Z, W)X, \phi_i \circ \phi_j Y) \\
&= -\alpha g(\overline{R}(Z, W)X, Y) + \frac{m_j}{2} g(\overline{R}(Z, W)X, \phi_i Y) \\
&+ \frac{m_i}{2} g(\overline{R}(Z, W)X, \phi_j Y) + \beta g(\overline{R}(Z, W)X, \phi_k Y) \\
&= -\alpha \overline{R}(X, Y, Z, W) + \frac{m_j}{2} \overline{R}(X, \phi_i Y, Z, W) \\
&+ \frac{m_i}{2} \overline{R}(X, \phi_j Y, Z, W) + \beta \overline{R}(X, \phi_k Y, Z, W).
\end{aligned}$$

vii) $g(\overline{R}(X,Y)\phi_i Z, \phi_j W) = g(\phi_i \overline{R}(X,Y)Z, \phi_j W)$
$= g(\overline{R}(X,Y)Z, \phi_i \circ \phi_j W) = -\alpha \overline{g}(\overline{R}(X,Y)Z, W) + \frac{m_j}{2} g(\overline{R}(X,Y)Z, \phi_i W)$
$+ \frac{m_i}{2} g(\overline{R}(X,Y)Z, \phi_j W) + \beta g(\overline{R}(X,Y)Z, \phi_k W).$

□

So, the following results can be obtained directly by using the previous theorem.

Theorem 3.3 *Let $\overline{S}$ be the Ricci tensor of an integrable almost 3-poly-Norden manifold $(\overline{M}, \phi_i, g), i = 1,2,3$, then*

i) $\overline{S}(\phi_i^2 X, Y) = m_i \overline{S}(\phi_i X, Y) - \overline{S}(X,Y),$

ii) $\overline{S}(X, \phi_i^2 Y) = m_i \overline{S}(X, \phi_i Y) - \overline{S}(X,Y),$

iii) $\overline{S}(\phi_i X, Y) = \overline{S}(X, \phi_i Y),$

iv) $\overline{S}(\phi_i X, \phi_Y) = m_i \overline{S}(\phi_i X, Y) - \overline{S}(X,Y),$

v) $\overline{S}(\phi_i X, \phi_j Y) = -\alpha \overline{S}(X,Y) + \frac{m_j}{2} \overline{S}(X, \phi_i Y) + \frac{m_i}{2}(X, \phi_j) + \beta \overline{S}(X, \phi_k Y).$

Theorem 3.4 *Let $\overline{M}$ be an integrable almost 3-poly-Norden manifold then we get:*

i) $(\overline{\nabla}_W \overline{R})(X,Y)\phi_i Z = \phi_i (\overline{\nabla}_W \overline{R})(X,Y)Z$

ii) $(\overline{\nabla}_W \overline{S})(\phi_i X, Y) = (\overline{\nabla}_Z \overline{S})(X, \phi_i Z)$

4. Slant Submanifolds of Almost 3-Poly-Norden Manifolds

Definition 4.1 *Let (M,g) be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, g, \phi_i), i = 1,2,3$. Let X be a nonzero vector field tangent to M at p and $\theta(X)$ be the angle between T_pM and $\phi_i X$. If $\theta(X)$ is independent of the choice of $p \in M$ and $X \in T_pM$, then M is called a 3-slant submanifold and $\theta = \theta(X)$ is called slant angle.*

Thus, if

i) $\theta = 0$, M is ϕ_i-invariant submanifold.

ii) $\theta = \frac{\pi}{2}$, M is ϕ_i-anti-invariant submanifold.

iii) $0 < \theta < \frac{\pi}{2}$, M is called proper 3-slant submanifold.

According to the above definition we have:

$$cos\theta = \frac{g(\phi_i X, T_j X)}{\mid \phi_i X \mid\mid T_j X \mid}, \quad i,j \in \{1,2,3\}. \tag{4.1}$$

Lemma 4.1 *Let (M,g) be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, g, \phi_i)$. If M is a 3-slant submanifold with slant angle θ, then for any $X, Y \in \Gamma(TM)$ we have:*

i) $g(T_i X, T_i Y) = \cos^2\theta[m_i g(X, T_i Y) - g(X,Y)],$

ii) $g(T_iX, T_iY) = \cos^2[-\alpha g(X,Y) + \frac{m_j}{2}g(X,T_iY) + \frac{m_i}{2}g(X,T_jY) + \beta g(X,T_kY)]$,

iii) $g(N_iX, N_iY) = \sin^2\theta[m_i g(X,T_iY) - g(X,Y)]$,

iv) $g(N_iX, N_jY) = \sin^2\theta[\alpha g(X,Y) + \frac{m_j}{2}g(X,T_iY) + \frac{m_i}{2}g(X,T_jY) + \beta g(X,T_kY)]$.

Proof: By using Equation (4.1), we get

$$
\begin{aligned}
i)\ g(T_iX, T_iY) &= \cos^2\theta g(\phi_iX, \phi_iY) = \cos^2\theta g(\phi_i^2X, Y) = \cos^2\theta g((m_i\phi_i - I)X, Y) \\
&= \cos^2\theta[m_i g(\phi_iX, Y) - g(X,Y)] = \cos^2\theta[m_i g(X, T_iY) - g(X,Y)].
\end{aligned}
$$

$$
\begin{aligned}
ii)\ g(T_iX, T_jY) &= \cos^2\theta g(\phi_iX, \phi_jY) = \cos^2\theta g(X, \phi_i \circ \phi_jY) \\
&= \cos^2[-\alpha g(X,Y) + \frac{m_j}{2}g(X,\phi_iY) + \frac{m_i}{2}g(X,\phi_jY) + \beta g(X,\phi_kY)] \\
&= \cos^2[-\alpha g(X,Y) + \frac{m_j}{2}g(X,T_iY) + \frac{m_i}{2}g(X,T_jY) + \beta g(X,T_kY)].
\end{aligned}
$$

The proofs of the items (iii) and (iv) are similar to the proofs of (i) and (ii), respectively. □

Theorem 4.1 *Let (M, g) be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, g, \phi_i), i = 1, 2, 3$. If M is a 3-slant submanifold of $\overline{M}$, then there exist $\lambda \in [0, 1]$ such that*

i) $T_i^2 = \lambda(m_iT_i - I)$,

ii) $T_iT_j = \lambda(-\alpha I + \frac{m_j}{2}T_i + \frac{m_i}{2}T_j + \beta T_k)$.

Proof: Let M be a 3-slant submanifold of an almost 3-poly-Norden manifold $\overline{M}$, hence the angle θ is constant, so put $\lambda = cos^2\theta \in [0, 1]$. Now let use the previous lemma

$$
\begin{aligned}
i)\ g(T_i^2X, Y) &= g(T_iX, T_iY) = cos^2\theta[m_i g(T_iX, Y) - g(X,Y)] \\
&= cos^2\theta g(m_iT_iX - X, Y) = cos^2\theta g((m_iT_i - I)X, Y),
\end{aligned}
$$

for any $X, Y \in \Gamma(TM)$, therefore we have $T_i^2 = \lambda(m_iT_i - I)$.
Conversely, suppose there is a real number λ in the interval $[0, 1]$ such that $T_i^2 = \lambda(m_iT_i - I)$, put $\lambda = cos^2\theta \in [0, 1]$, hence λ is independent of the choice X.

$$
\begin{aligned}
ii)\ g(T_iT_jX, Y) &= g(T_iX, T_jY) \\
&= \cos^2[-\alpha g(X,Y) + \frac{m_j}{2}g(T_iX,Y) + \frac{m_i}{2}g(T_jX,Y) + \beta g(T_kX,Y)] \\
&= \cos^2[g((\alpha + \frac{m_j}{2}T_i + \frac{m_i}{2}T_j + \beta T - k)X, Y),
\end{aligned}
$$

so we have $T_iT_j = \lambda(-\alpha I + \frac{m_j}{2}T_i + \frac{m_i}{2}T_j + \beta T_k)$. □

Theorem 4.2 *Let (M, g) be a submanifold of an almost 3-poly-Norden manifold $(\overline{M}, g, \phi_i), i = 1, 2, 3$. If M is a 3-slant submanifold with slant angle θ, then*

i) $(\nabla_XT_i^2)Y = m_i\cos^2\theta(\nabla_XT_i)Y$, *ii)* $(\nabla_X T_i T_j)Y = \cos^2\theta[\frac{m_j}{2}(\nabla_X T_i)Y + \frac{m_i}{2}(\nabla_X T_j)Y + \beta(\nabla_X T_k)Y]$.

Proof: By using Lemma 4.1, we get

$$\begin{aligned} i)\ (\nabla_X T_i^2)Y &= \nabla_X T_i^2 Y - T_i^2 \nabla_X Y \\ &= \nabla_X cos^2\theta(m_i T_i Y - Y) - cos^2\theta(m_i T_i \nabla_X Y - \nabla_X Y) \\ &= m_i\, cos^2\theta \nabla_X T_i Y - cos^2\theta \nabla_X Y - m_i\, cos^2\theta\, T_i \nabla_X Y + cos^2\theta \nabla_X Y \\ &= m_i\, cos^2\theta(\nabla_X T_i)Y. \end{aligned}$$

$$\begin{aligned} ii)\ (\nabla_X T_i T_j)Y &= \nabla_X T_i T_j Y - T_i T_j \nabla_X Y \\ &= \nabla_X \cos^2\theta(-\alpha + \frac{m_j}{2}T_i + \frac{m_i}{2}T_j + \beta T_k)Y \\ &\quad - \cos^2\theta(-\alpha + \frac{m_j}{2}T_i + \frac{m_i}{2}T_j + \beta T_k)\nabla_X Y \\ &= (-\alpha\cos^2\theta + \alpha\cos^2\theta)\nabla_X Y + (\frac{m_j}{2}\cos^2\theta)(\nabla_X T_i)Y \\ &\quad + (\frac{m_i}{2}\cos^2\theta)(\nabla_X T_j)Y + \beta\cos^2\theta(\nabla_X T_k)Y \\ &= \cos^2\theta[\frac{m_j}{2}(\nabla_X T_i)Y + \frac{m_i}{2}(\nabla_X T_j)Y + \beta(\nabla_X T_k)Y]. \end{aligned}$$

□

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