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Exploration of Ring Structure From Multiset Context

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ABSTRACT: The primary aim of this research paper is to develop the concept of ring structures in the multiset framework and to explore some of their fundamental properties.

Key Words: Multiset, ring, subring.

Contents

1. Introduction

An unordered collection of elements in which elements may repeat is called a multiset (or m-set). A multiset is not a traditional (Cantorian) set because elements can appear more than once. Thus, a multiset differs from a set in that each element has a multiplicity, which is a natural number indicating how many times it occurs in the multiset. One of the most common and fundamental examples of a multiset is the multiset of prime factors of a positive integer n. For example, the number 10800 has the prime factorization $2^4 \times 3^3 \times 5^2$, which corresponds to the multiset $2, 2, 2, 2, 3, 3, 3, 5, 5$.

In 1988, the conception of the multiset was implicitly introduced by R. Dedekind. Later, Cerf et al. [2] developed the theory of multisets, treating it as a distinctive extension of set theory. The term "multiset," studied by Knuth [10], was originally proposed to him by N.G. de Bruijn in a private communication. Girish and John [4], [5], [6], [7] conducted research on relations, functions, and topological concepts in the context of multisets. In [1], [3], [9], [11], [14], and [15], various authors studied group theory in the context of multisets. In [8], the author studied multisets, introduced the notion of multiset spaces with new operations, and investigated their algebraic properties. Furthermore, in [12] and [13], the authors defined and examined some properties of the multiset ring and multi-ideal.

2. Preliminaries and Vital Characterizations

In this section, some important results and definitions from the existing literature are presented. These will be used in this article.

Definition 2.1. *A multiset A drawn from a set X is represented by a count function C_A defined as $C_A : X \to \mathbb{N}_0$, where $\mathbb{N}_0$ denotes the set of non-negative integers.*

Consider a multiset $A = \{m_1|x_1, m_2|x_2, \ldots, m_k|x_k\}$, drawn from the set $X = \{x_1, x_2, \ldots, x_k\}$. Here, $m_i|x_i$ means that the element x_i appears m_i times in A.

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In Definition 2.1, the number of occurrences of an element x in the multiset A is denoted by $C_A(x)$. Elements not contained in A are assigned a count of zero.

Definition 2.2. *The root set of a multiset A is an ordinary set, denoted by A^*. It is defined as $A^* = \{y \mid C_A(y) > 0\}$.*

An empty multiset A is one for which $C_A(y) = 0$ for all $y \in X$. The cardinality of a multiset A taken from a set X, denoted by $Card(A)$ or $|A|$, is defined as $Card(A) = \sum_{y \in X} C_A(y)$.

Definition 2.3. *Let A and B be two multisets drawn from a set X. A is a sub-multiset of B, denoted by $A \subseteq B$, if $C_A(y) \leq C_B(y)$ for all $y \in X$. If $A \subseteq B$ and $A \neq B$, then A is called a proper sub-multiset of B.*

It follows from the definition of multisets that $A = B$ if and only if $C_A(y) = C_B(y)$ for all $y \in X$. Also, $A = B$ implies $A^* = B^*$, but the converse need not hold. The relation $\subseteq$ is antisymmetric: $A \subseteq B$ and $B \subseteq A$ imply $A = B$. It defines a partial order on the class of multisets over a given set X.

Definition 2.4. *Let A and B be multisets drawn from a set X. The union of A and B is a multiset C, denoted by $C = A \cup B$, such that for all $y \in X$, $C_C(y) = \max\{C_A(y), C_B(y)\}$.*

Definition 2.5. *The intersection of two multisets A and B drawn from a set X is a multiset D, denoted by $D = A \cap B$, such that for all $y \in X$, $C_D(y) = \min\{C_A(y), C_B(y)\}$.*

Definition 2.6. *The addition of two multisets A and B drawn from a set X is a multiset E, denoted by $E = A \oplus B$, such that for all $y \in X$, $C_E(y) = C_A(y) + C_B(y)$.*

Definition 2.7. *The subtraction of two multisets A and B drawn from a set X is a multiset F, denoted by $F = A \ominus B$, such that for all $y \in X$, $C_F(y) = \max\{C_A(y) - C_B(y), 0\}$.*

Definition 2.8. *Let A and B be two multisets drawn from a set X. The Cartesian product of A and B is the multiset $A \times B = \{mn|(x, y) \mid x \in A, y \in B\}$, where the element $mn|(x, y)$ occurs mn times in $A \times B$, with $m = C_A(x)$ and $n = C_B(y)$.*

Definition 2.9. *A sub-multiset R of $A \times A$ is called a multiset relation on A if each element $mn|(x, y)$ in R has a count given by the product of $C_A(x)$ and $C_A(y)$. The relation between $m|x$ and $n|y$ is denoted by $(m|x)R(n|y)$.*

The domain and range of a multiset relation R on a multiset A can be defined as follows:

Domain of R:

$$Dom(R) = \{x \in^r A : \exists y \in^s A \text{ such that } (r|x)R(s|y)\}, \quad C_{Dom(R)}(x) = \sup\{C_R(x, y) : (r|x)R(s|y)\}.$$

Range of R:

$$Range(R) = \{y \in^s A : \exists x \in^r A \text{ such that } (r|x)R(s|y)\}, \quad C_{Range(R)}(y) = \sup\{C_R(x, y) : (r|x)R(s|y)\}.$$

Here, $C_{Dom(R)}(x)$ represents the count of occurrences of x in the domain of R, which is the maximum count of x among all pairs in R. Similarly, $C_{Range(R)}(y)$ represents the count of occurrences of y in the range of R, which is the maximum count of y among all pairs in R. The counts r and s represent the occurrences of x and y, respectively, in the multiset relation R.

Example 2.1. Let $A = \{4|a, 8|b, 7|c\}$ be a multiset.

$$R = \{3 \times 4|(a, b),\ 2 \times 5|(a, b),\ 4 \times 2|(a, a),\ 3 \times 3|(b, c),\ 4 \times 4|(b, b)\}$$

is a multiset relation defined on A. Then, $Dom(R) = \{4|a, 4|b\}$ and $Range(R) = \{5|b, 2|a, 3|c\}$.

Throughout this paper, we assume that if A is a multiset with maximum multiplicity m, then for any element $k|x \in A$ (or $x \in^k A$), we have $k \leq m$, unless otherwise stated.

3. Main Results

We introduce the following definitions:

Definition 3.1. *Let A be a non-empty multiset with maximum multiplicity m, and let A^* denote the root set of A. Let $n_1|y_1, n_2|y_2 \in A$. We define addition and multiplication on A as follows:*

Addition:

$$n_1|y_1 \oplus n_2|y_2 = (n_1 \oplus_1 n_2) \,|\, (y_1 \oplus_2 y_2),$$

where

(i) $\oplus_1$ is a binary operation on $\mathbb{N}$ such that $n_1 \oplus_1 n_2 \leq m$,

(ii) $\oplus_2$ is a binary operation on A^*.

Multiplication:

$$n_1|y_1 \otimes n_2|y_2 = (n_1 \otimes_1 n_2) \,|\, (y_1 \otimes_2 y_2),$$

where

(iii) $\otimes_1$ is a binary operation on $\mathbb{N}$ such that $n_1 \otimes_1 n_2 \leq m$,

(iv) $\otimes_2$ is a binary operation on A^*.

Then $(A, \oplus, \otimes)$ is called a multi-ring of order m if it satisfies the following properties:

1. $n_1|y_1 \oplus n_2|y_2 = n_2|y_2 \oplus n_1|y_1$, *(Commutative law for addition)*
2. $n_1|y_1 \oplus (n_2|y_2 \oplus n_3|y_3) = (n_1|y_1 \oplus n_2|y_2) \oplus n_3|y_3$, *(Associative law for addition)*
3. There exists an element $m|0 \in A$ such that $n|x \oplus m|0 = n|x$ for all $n|x \in A$, *(Additive identity)*
4. For each $n|y \in A$, there exists $l|z \in A$ such that $n|y \oplus l|z = m|0$. Then $l|z$ is called the additive inverse of $n|y$, denoted by $\ominus n|y$,
5. $n_1|y_1 \otimes (n_2|y_2 \otimes n_3|y_3) = (n_1|y_1 \otimes n_2|y_2) \otimes n_3|y_3$, *(Associative law for multiplication)*
6. $n_1|y_1 \otimes (n_2|y_2 \oplus n_3|y_3) = n_1|y_1 \otimes n_2|y_2 \oplus n_1|y_1 \otimes n_3|y_3$ *and* $(n_1|y_1 \oplus n_2|y_2) \otimes n_3|y_3 = n_1|y_1 \otimes n_3|y_3 \oplus n_2|y_2 \otimes n_3|y_3$, *(Distributive laws)*

Definition 3.2. *A multi-ring $(A, \oplus, \otimes)$ of order m is called commutative if*

$$n_1|y_1 \otimes n_2|y_2 = n_2|y_2 \otimes n_1|y_1, \quad \textit{for all } n_1|y_1, n_2|y_2 \in A,$$

otherwise it is called non-commutative.

Definition 3.3. *A multi-ring $(A, \oplus, \otimes)$ of order m is said to have an identity of multiplicity m (or unity of multiplicity m) if there exists an element $m|1 \in A$ such that*

$$n|y \otimes m|1 = n|y = m|1 \otimes n|y, \quad \text{for all } n|y \in A.$$

Example 3.1. Let
$$A = (Z_n)^n = \{n|[0], n|[1], n|[2], \dots, n|[n-1]\},$$
where $Z_n = \{[0], [1], [2], \dots, [n-1]\}$ is the set of integers modulo n. Define addition $\oplus$ and multiplication $\otimes$ on A as follows:

$$n_1|[y_1] \oplus n_2|[y_2] = \ll n_1 + n_2 \gg_m \ |[y_1 + y_2], \quad \text{where } \ll n_1 + n_2 \gg_m = \begin{cases} n_1 + n_2, & \text{if } n_1 + n_2 \le m, \\ n_1 + n_2 - m, & \text{if } n_1 + n_2 > m \end{cases}$$

Similarly,
$$n_1|[y_1] \otimes n_2|[y_2] = \ll n_1 n_2 \gg_m \ |[y_1 \cdot y_2], \quad \text{where } \ll n_1 n_2 \gg_m = \frac{n_1 n_2}{m}.$$
Then, A is a commutative multi-ring of order m with unity.

Example 3.2. Consider
$$R = (M_2(Z_2))^{16},$$
the collection of all 2×2 matrices over Z_2, where each element has multiplicity 16. Under the following operations, R forms a non-commutative multi-ring of order 16 with unity:

$$n_1|A \oplus n_2|B = \ll n_1 + n_2 \gg_m \ |(A + B), \quad \ll n_1 + n_2 \gg_m = \begin{cases} n_1 + n_2, & n_1 + n_2 \le m, \\ n_1 + n_2 - m, & n_1 + n_2 > m \end{cases}$$

Similarly,
$$n_1|A \otimes n_2|B = \ll n_1 n_2 \gg_m \ |(A \cdot B), \quad \ll n_1 n_2 \gg_m = \frac{n_1 n_2}{m}.$$

Example 3.3. Let
$$R = \{1|0, n|n : n \in \mathbb{Z} \setminus \{0\}\},$$
then R is an example of a multi-ring of infinite order with the following operations:

$$n_1|y_1 \oplus n_2|y_2 = (n_1 + n_2)|(y_1 + y_2), \quad n_1|y_1 \otimes n_2|y_2 = (n_1 n_2)|(y_1 \cdot y_2).$$

Example 3.4. Let
$$R = (Z_3)^3 = \{[0], [1], [1], [2], [2], [2]\},$$
where $Z_3 = \{[0], [1], [2]\}$ is the set of integers modulo 3. Define addition $\oplus$ and multiplication $\otimes$ on R as follows:

$$n_1|[y_1] \oplus n_2|[y_2] = \ll n_1 + n_2 \gg_3 \ |[y_1 + y_2], \quad \ll n_1 + n_2 \gg_3 = \begin{cases} n_1 + n_2 - 1, & n_1 + n_2 \le 4, \\ n_1 + n_2 - 4, & n_1 + n_2 > 4 \end{cases}$$

$$n_1|[y_1] \otimes n_2|[y_2] = \ll n_1 n_2 \gg_3 \ |[y_1 \cdot y_2], \quad \ll n_1 n_2 \gg_3 = \begin{cases} n_1 n_2/2, & \text{if } n_1 n_2 \text{ is even}, \\ \lfloor n_1 n_2/2 \rfloor + 1, & \text{if } n_1 n_2 \text{ is odd} \end{cases}$$

Then, R is a commutative multi-ring of order 3 with multi-unity, where all elements have different multiplicities.

Theorem 3.1. *For $n_1|y_1, n_2|y_2 \in A$ in a multi-ring A of order m:*

1. $n|x \otimes m|0 = m|0 = m|0 \otimes n|x$.
2. $n_1|y_1 \otimes (\ominus n_2|y_2) = (\ominus n_1|y_1) \otimes n_2|y_2 = \ominus(n_1|y_1 \otimes n_2|y_2)$
3. $(\ominus n_1|y_1) \otimes (\ominus n_2|y_2) = n_1|y_1 \otimes n_2|y_2$

Proof:
(1) For all $n|y \in A$,

$$n|y \otimes m|0 = n|y \otimes (m|0 \oplus m|0) = n|y \otimes m|0 \oplus n|y \otimes m|0$$

$$\implies n|y \otimes m|0 \oplus \{\ominus(n|y \otimes m|0)\} = n|y \otimes m|0 \oplus n|y \otimes m|0 \oplus \{\ominus(n|y \otimes m|0)\}$$

$$\implies m|0 = n|y \otimes m|0 \oplus [n|y \otimes m|0 \oplus \{\ominus(n|y \otimes m|0)\}] = n|y \otimes m|0 \oplus m|0$$

$$\implies n|y \otimes m|0 = m|0$$

Similarly, $m|0 \otimes n|y = m|0$.
(2) Using (1),

$$n_1|y_1 \otimes [n_2|y_2 \oplus (\ominus n_2|y_2)] = n_1|y_1 \otimes m|0 = m|0$$

$$\implies n_1|y_1 \otimes n_2|y_2 \oplus n_1|y_1 \otimes (\ominus n_2|y_2) = m|0$$

$$\implies n_1|y_1 \otimes (\ominus n_2|y_2) = \ominus(n_1|y_1 \otimes n_2|y_2)$$

Similarly, $(\ominus n_1|y_1) \otimes n_2|y_2 = \ominus(n_1|y_1 \otimes n_2|y_2)$.
(3)

$$(\ominus n_1|y_1) \otimes (\ominus n_2|y_2) = \ominus[n_1|y_1 \otimes (\ominus n_2|y_2)] = \ominus[\ominus(n_1|y_1 \otimes n_2|y_2)] = n_1|y_1 \otimes n_2|y_2$$

Definition 3.4. *Let A be a multi-ring of order m with multi-identity $m|1$ and $m|1 \neq m|0$. A member $n_1|u_1 \in A$ is called a multi-unit (or multi-invertible) if there exists $n_2|u_2 \in A$ such that $n_1|u_1 \otimes n_2|u_2 = m|1$. The element $n_2|u_2$ is called the inverse of $n_1|u_1$ and is denoted by $n_2|u_2^{-1}$.*

Definition 3.5. *Let A be a multi-ring of order m. An element $n_1|u_1 \in A$ is called a multi-zero divisor if $n_1|u_1 \neq m|0$ and there exists a nonzero element $n_2|u_2 \in A$ such that $n_1|u_1 \otimes n_2|u_2 = m|0$ or $n_2|u_2 \otimes n_1|u_1 = m|0$.*

Definition 3.6. *A multi-ring of order m is said to satisfy the multi-left (resp. multi-right) cancellation property if, for all $n_1|y_1, n_2|y_2, n_3|y_3 \in A$ with $n_1|y_1 \neq m|0$, the equality $n_1|y_1 \otimes n_2|y_2 = n_1|y_1 \otimes n_3|y_3$ (resp. $n_2|y_2 \otimes n_1|y_1 = n_3|y_3 \otimes n_1|y_1$) implies $n_2|y_2 = n_3|y_3$.*

Definition 3.7. *A commutative multi-ring of order m with multi-identity $m|1 \neq m|0$ is called a multi-integral domain if it has no multi-zero divisors.*

Example 3.5. The multi-ring

$$R = \{1|0, |n||n : n \in \mathbb{Z} \setminus \{0\}\},$$

defined in Example 3.3, is an example of a multi-integral domain with infinite order.

Definition 3.8. *A nonempty submultiset S of a multi-ring $(R, \oplus, \otimes)$ of order m is called a sub multi-ring of R if $(S, \oplus)$ is a multi-subgroup of the abelian multi-group $(R, \oplus)$ and S is closed under $\otimes$, i.e., for all $n_1|y_1, n_2|y_2 \in S$, we have $n_1|y_1 \otimes n_2|y_2 \in S$.*

The smallest sub multi-ring of R is $\{m|0\}$ and the greatest one is R itself.

Example 3.6. Consider the multi-ring

$$R = (Z_{10})^{10} = \{10|[0], 10|[1], 10|[2], \ldots, 10|[9]\},$$

under the operations defined in Example 3.1. Then the sub multiset

$$S = \{10|[0], 10|[2], 10|[4], 10|[6], 10|[8]\} \subset R$$

is a sub multi-ring of R.

Theorem 3.2. *Let $(R, \oplus, \otimes)$ be a multi-ring and S a nonempty sub multiset of R. Then S is a sub multi-ring of R if and only if*

$$n_1|y_1, n_2|y_2 \in S \implies n_1|y_1 \ominus n_2|y_2,\ n_1|y_1 \otimes n_2|y_2 \in S. \tag{3.1}$$

Proof: Let S be a sub multi-ring of R. Then $(S, \oplus)$ is a multi-subgroup and S is closed under $\otimes$, so (3.1) holds. Conversely, if (3.1) holds, then $(S, \oplus)$ is a multi-subgroup of $(R, \oplus)$ by [19, Theorem 3.3] and S is closed under $\otimes$. Hence, S is a sub multi-ring.

Theorem 3.3. *Let $\{S_\beta : \beta \in \mu\}$ be a collection of sub multi-rings of $(R, \oplus, \otimes)$. Then $S = \bigcap_{\beta\in\mu} S_\beta$ is a sub multi-ring of R.*

Proof: For each $\beta \in \mu$, $m|0 \in S_\beta$, so $m|0 \in S$ and S is nonempty. If $n_1|y_1, n_2|y_2 \in S$, then $n_1|y_1, n_2|y_2 \in S_\beta$ for all β, so $n_1|y_1 \ominus n_2|y_2, n_1|y_1 \otimes n_2|y_2 \in S_\beta$ for all β, hence $n_1|y_1 \ominus n_2|y_2, n_1|y_1 \otimes n_2|y_2 \in S$.

Remark 3.1. The union of two sub multi-rings may not be a sub multi-ring.
Consider the multi-ring

$$R = \{1|0, n|n : n \in \mathbb{Z} \setminus \{0\}\},$$

defined in Example 3.5. Let

$$S_1 = \{1|0, 2n|2n : n \in \mathbb{Z} \setminus \{0\}\}, \quad S_2 = \{1|0, 3n|3n : n \in \mathbb{Z} \setminus \{0\}\}.$$

Then S_1 and S_2 are sub multi-rings, but $S_1 \cup S_2$ is not, because, for example, $2|2 \in S_1$ and $3|3 \in S_2$, but $2|2 \oplus 3|3 = 5|5 \notin S_1 \cup S_2$.

4. Conclusions

In this paper, an endeavor has been made to investigate the significant algebraic structure of a ring in the multiset framework, and some basic properties of rings that hold in multi-rings were studied. The authors believe that there are still many theoretical cases, along with numerous applications, that remain to be explored with the aid of this concept.

5. References

1. Awolola, J. A. and Ibrahim, A. M. (2016). Some results on multigroups. *Quasigroups and Related Systems*, **24**, 169–177.
2. Cerf, V., Fernandez, E., Gostelow, K. and Volausky, S. (1971). Formal control and low properties of a model of computation. Report ENG 7178, Computer Science Department, University of California, Los Angeles, CA, December, p. 81.
3. Ejegwa, P. A. (2017). Upper and lower cuts of multigroups. *Prajna: International Journal of Mathematical Sciences and Applications*, **1**, 19–26.
4. Girish, K. P. and John, S. J. (2009). Relations and functions in multiset context. *Information Sciences*, **179**, 758–768.
5. Girish, K. P. and John, S. J. (2011). Rough multiset and its multiset topology. In: *Lecture Notes in Computer Science*, Springer, Berlin, Heidelberg, **6600**, 62–80.
6. Girish, K. P. and John, S. J. (2012). On multiset topologies. *Theory and Applications of Mathematics and Computer Science*, **2**, 37–52.
7. Girish, K. P. and John, S. J. (2012). Multiset topologies induced by multiset relations. *Information Sciences*, **188**, 298–313.
8. Ibrahim, A. M. and Singh, D. (2011). An outline of multiset space algebra. *International Journal of Algebra*, **5(31)**, 1515–1525.
9. Ibrahim, A. M. and Ejegwa, P. A. (2016). A survey on the concept of multigroups. *Journal of the Nigerian Association of Mathematical Physics*, **38**, 1–8.

10. Knuth, D. E. (1981). *The Art of Computer Programming, Vol. 2: Seminumerical Algorithms.* Addison-Wesley.

11. Nazmul, S. K., Ajumdar, P. M. and Samanta, S. K. (2013). On multisets and multigroups. *Annals of Fuzzy Mathematics and Informatics*, **3**, 643–656.

12. Suma, P. and John, S. J. (2019). Multiset approach to algebraic structures. In: *Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures*, IGI Global, Chapter 5, 78–90.

13. Suma, P. and John, S. J. (2021). Exploration of multiset ideals. *AIP Conference Proceedings*, **2336**, 05–12. https://doi.org/10.1063/5.0045816

14. Tella, Y. and Daniel, S. (2013). A study of group theory in the context of multiset theory. *International Journal of Science and Technology*, **2(8)**, 609–615.

15. Tripathy, B. C., Debnath, S. and Rakshit, D. (2018). On multiset group. *Proyecciones Journal of Mathematics*, **33**, 479–489.

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