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Hyers - Ulam stability and continuous dependence of the solution of a nonlocal stochastic-integral fractional orders stochastic differential equation

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ABSTRACT: Stochastic problems have become an indispensable tool in modeling complex systems across various disciplines, including biology, chemistry, physics, economics, finance, mechanics and several areas. In this paper, we are concerning with the nonlocal problem of the integro-fractional orders stochastic differential equation

$$\frac{dX(t)}{dt} = f(t, D^{\alpha}X(t)) + g(t, B(t)), \quad t \in (0, T],$$

with the nonlocal stochastic-integral conditiom

$$X(\tau) \ = \ X_0 \ + \ \int_0^{T-\tau} \ h(s, D^{\beta}X(s))dW(s), \quad \tau \in [0, T]$$

where W is a standard Brownian motion, B is any Brownian motion and X_0 is a second order random variable. The existence of solution and its continuous dependencies on X_0, the functions $f(t, x)$, $g(t, x)$ and on the Brownian motion B will be discussed. Finally the Hyers - Ulam stability of the problem will be studied.

Key Words: Stochastic processes, Stochastic differential equations, existence of solutions, continuous dependence, Brownian motion, Brownian bridge process, Brownian Motion with Drift, nonlinear equations.

Contents

1. Introduction

Fractional differential equations and their applications have gotten extensive sighificant attention, they are widely utilized across various fields, interested researchers can explore the comprehensive studies in [6], [8], [13], [16], [26] and [30]. Many authors have devoted numerous efforts to investigating fractional stochastic differential equations as discussed in [1], [3], [5], [7], [12], [15], [18]. In particular, the existence and uniqueness of solutions to stochastic differential equations have been studied in detail by many authors

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as seen in [14], [17] and [22].
Let (Ω, G, μ) be a probability space, where Ω is a sample space, G is a σ-algebra of subsets of Ω, and μ is the probability measure (refer [4], [27]- [29]).
Let $I = [0, T]$ and $X(t; w) = \{ X(t),\ t \in I,\ w \in \Omega \}$ denote a second order stochastic process, such that

$$E(X^2(t)) < \infty,\ t \in I.$$

Let $C = C(I, L_2(\Omega))$ represent the class of all mean square second order continuous stochastic processes on I with the norm

$$\| X \|_C = \sup_{t\in I} \| X(t) \|_2, \quad \| X(t) \|_2 = \sqrt{E(X^2(t))}.$$

Let $W(t)$ is a standard brownian motion, $B(t),\ t \in [0, T]$ be any Brownian motion and $\alpha, \beta \in (0, 1],\ \beta \le \alpha$.
Here, we are concerned with the nonlocal problem of the integro-fractional orders stochastic differential equation

$$\frac{dX(t)}{dt} = f(t, D^\alpha X(t)) + g(t, B(t)), \quad t \in (0, T] \tag{1.1}$$

with the nonlocal stochastic-integral condition

$$X(\tau) = X_0 + \int_0^{T-\tau} h(s, D^\beta X(s)) dW(s), \quad \tau \in [0, T], \tag{1.2}$$

where X_0 is a second order random variable.
The existence of at least one solutions $X \in C$ will be demostrated. Conditions ensuring the uniqueness of the solution will be provided. The continuous dependencies of the unique solution on X_0, and on the functions $f(t, x),\ g(t, x)$ and on the Brownian motion B will be proved. Additionally, the Hyers - Ulam stability of the problem (1.1)-(1.2) will also be established.
Remark As a consequence of our results we can prove that
(i) When $\tau = 0$ in (1.2), our problem will be the nonlocal stochastic-integral problem of the differential equation (1.1) with the nonlocal stochastic-integral condition

$$X(0) = X_0 + \int_0^T h(s, D^\beta X(s)) dW(s).$$

(ii) When $\tau = T$ in (1.2), our problem will be the backward problem of the differential equation (1.1) with the backward condition

$$X(T) = X_0.$$

One of our motivation here is to generalize our work in [14]. In [14], the nonlocal problem of the stochastic differential equation

$$\frac{dX}{dt} = f(t, X(t)) + B(t), \quad t \in (0, T]$$

with the nonlocal random initial condition

$$X(0) + \sum_{k=1}^{n} a_k X(\tau_k) = X_0, \quad \tau_k \in (0, T)$$

and consequently the with the nonlocal mean square Riemann-Stieltjes random integral condition

$$X(0) + \int_0^T X(s)\ dv(s) = X_0.$$

was studied, where X_0 is a second order random variable, $B(t)$ is the standard Brownian motion and a_k are positive real numbers.

1.1. Examples of Brownian motion

Definition 1.1 *(Brownian motion with Drift) [22], [25]*
A Brownian Motion B is called a Brownian motion with Drift μ and volatility σ if it can be written as

$$B(t) = \mu t + \sigma W(t), \qquad t \in R_+$$

where $W(t)$ is a standard Brownian motion.

Definition 1.2 *(Brownian motion started at A) [23]*
A process $B(t)$ is called a Brownian motion started at A, $A \in L_2(\Omega)$ if it can be written as

$$B(t) \ = \ A \ + \ W(t)$$

where $W(t)$ is a standard Brownian motion.

Definition 1.3 *(Brownian bridge) [24]*
A Brownian motion B is called a Brownian bridge if it can be written as

$$B(t) = a(1-t) + bt + (1-t)\int_0^t \frac{dW(s)}{1-s}, \qquad t \in [0,1), \quad a, b \in R$$

where $W(t)$ is a standard Brownian motion.

2. Solution of the problem

Consider the problem (1.1) - (1.2) with the following assumptions.

1 The functions $f, g, h : I = [0,T] \times L_2(\Omega) \to L_2(\Omega)$ are measurable in $t \in I \ \ \forall x \in L_2(\Omega)$ and continuous in $x \in L_2(\Omega) \ \forall t \in I$, and there exists a constant $b > 0$, and a second order process $a(t) \in L_2(\Omega)$, $a = \sup_{t \in I} ||a(t)||_2$ such that

$$max\{||f(t, x(t))||_2, ||g(t, x(t)||_2, ||h(t, x(t)||_2 \leq a + b \ \parallel x(t) \parallel_2 .$$

2 $bT^{1-\alpha} < \Gamma(2-\alpha)$.

Now, we have the following equivalent lemma.

Lemma 2.1 *The nonlocal problem (1.1)-(1.2) is equivalent to the stochastic integral equation*

$$X(t) \ = \ X_0 \ + \ \int_0^{T-\tau} \ h(s, I^{\alpha-\beta} Y(s)) dW(s) \ - \ I^{\alpha} Y(\tau) \ + \ I^{\alpha} Y(t), \qquad t \in [0,T] \tag{2.1}$$

where $Y(t)$ is the solution of the fractional-order random integral equation

$$Y(t) \ = \ I^{1-\alpha}[f(s, Y(s)) + g(s, B(s))]. \tag{2.2}$$

Proof: Let X be a solution of (1.1). Operating by $I^{1-\alpha}$ on equations (1.1), we obtain

$$D^{\alpha} X(t) = I^{1-\alpha} \frac{dX(t)}{dt} = I^{1-\alpha}(f(t, D^{\alpha} X(t)) + g(t, B(t))).$$

Let

$$D^{\alpha} X(t) \ = \ Y(t) \in C([0,T], L_2(\Omega)),$$

then

$$Y(t) \ = \ I^{1-\alpha}(f(t, D^{\alpha} X(t)) + g(t, B(t)))$$
$$X(t) \ = \ X(0) \ + \ I^{\alpha} Y(t), \quad \text{and} \quad X(\tau) \ = \ X(0) \ + \ I^{\alpha} Y(\tau).$$

From which we can deduce that

$$X(0) \;= X(\tau) \;-\; I^{\alpha}Y(\tau),$$

then

$$X(t) \;=\; X(\tau) \;-\; I^{\alpha}Y(\tau) \;+\; I^{\alpha}Y(t).$$

But

$$D^{\beta}X(t) \;=\; I^{1-\beta}\frac{d}{dt}X(t) \;=\; I^{\alpha-\beta}I^{1-\alpha}\frac{d}{dt}X(t) \;=\; I^{\alpha-\beta}Y(t).$$

Then we obtain (2.1)

$$X(t) \;=\; X_0 \;+\; \int_0^{T-\tau} \; h(s, I^{\alpha-\beta}Y(s))dW(s) \;-\; I^{\alpha}Y(\tau) \;+\; I^{\alpha}Y(t), \qquad t \in [0,T]$$

where $Y(t)$ is the solution of the fractional-order random integral equation (2.2)

$$Y(t) \;=\; \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s, Y(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s, B(s))ds. \tag{2.3}$$

Conversely, let $Y(t)$ be a solution of (2.3). Then from (2.1) and (2.2) we obtain

$$\begin{aligned} X(t) \;&=\; X_0 \;+\; \int_0^{T-\tau} \; h(s, I^{\alpha-\beta}Y(s))dW(s) \;-\; I^{\alpha}Y(\tau) + I^{\alpha}\; I^{1-\alpha}\; [f(t, Y(t)) + g(t, B(t))] \\ &=\; X(\tau) \;-\; I^{\alpha}Y(\tau) \;+\; \int_0^t \; [f(s, D^{\alpha}X(s)) + g(s, B(s))]\; ds \end{aligned}$$

Thus, we get

$$\frac{d}{dt}X(t) \;=\; f(t, D^{\alpha}X(t)) + g(t, B(t))$$

and

$$X(\tau) \;=\; X_0 \;+\; \int_0^{T-\tau} \; h(s, D^{\beta}\; X(s))\; dW(s).$$

Then we have proved the equivalence between the problem (1.1)-(1.2) and the equations (2.1) and (2.2). □

3. Existence of solution

Theorem 3.1 *Let the assumptions (i)-(iii) be satisfied, then the fractional-order integral equation (2.2) has at least one solution $Y(t) \in C$.*

Proof: Consider the set Q such that

$$Q \;=\; \{Y \in C : ||Y||_C \leq r\} \subset C.$$

Define the mapping $FY(t)$ where

$$FY(t) = I^{1-\alpha}\,[f(t, Y(t) + g(t, B(t))]$$

Let $Y \in Q$, then

$$\begin{aligned}
||FY||_2 &\leq \int\limits_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel f(s,Y(s) \parallel_2 ds + \int\limits_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel g(s,B(s)) \parallel_2 ds \\
&\leq \int\limits_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} [||a(t)||_2 + b \parallel Y(t) \parallel_2] \, ds + \int\limits_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} [||a(t)||_2 + b \parallel B(t) \parallel_2] \, ds \\
&\leq [2a + b \parallel Y \parallel_C] \int\limits_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds + b \int\limits_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel B \parallel_C ds \\
&\leq [2a + b \parallel Y \parallel_C + b \parallel B \parallel_C] \, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} = r
\end{aligned}$$

where

$$r \leq [2a + br + b \parallel B \parallel_C] \, \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Thus

$$r \leq \frac{[2a + b \parallel B \parallel_C] \, T^{1-\alpha}}{\Gamma(2-\alpha) - [bT^{1-\alpha}]}.$$

That proves $F : Q \to Q$ and the class $\{FQ\}$ is uniformly bounded on Q.
Now, considering $t_1, t_2 \in [0,T]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
||FY(t_2) - FY(t_1)||_2 &\leq ||\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y(s))ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y(s))ds||_2 \\
&+ ||\int_0^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds - \int_0^{t_1} \frac{(t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds||_2 \\
&\leq ||\int_0^{t_1} \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y(s))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y(s))ds||_2 \\
&+ ||\int_0^{t_1} \frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds||_2.
\end{aligned}$$

Then

$$\begin{aligned}
||FY(t_2) - FY(t_1)||_2 &\leq [2a + br] \, [\int_0^{t_1} |\frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}|ds + \int_{t_1}^{t_2} |\frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}|ds] \\
&+ b||B||_C \, [\int_0^{t_1} |\frac{(t_2-s)^{-\alpha} - (t_1-s)^{-\alpha}}{\Gamma(1-\alpha)}|ds + ||B||_C \int_{t_1}^{t_2} |\frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)}|ds] \\
&= [2a + b_1 r] \, [\int_0^{t_1} \frac{(t_2-s)^{\alpha} - (t_1-s)^{\alpha}}{(t_2-s)^{\alpha}(t_1-s)^{\alpha}\Gamma(1-\alpha)} ds + \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} ds] \\
&+ b||B||_C [\int_0^{t_1} \frac{(t_2-s)^{\alpha} - (t_1-s)^{\alpha}}{(t_2-s)^{\alpha}(t_1-s)^{\alpha}\Gamma(1-\alpha)} ds + ||B||_C \int_{t_1}^{t_2} \frac{(t_2-s)^{-\alpha}}{\Gamma(1-\alpha)} ds].
\end{aligned}$$

This proves the equi-continuity of the class $\{FQ\}$ on Q.
Now, let $Y_n \in Q$, $Y_n \to Y$ $w.p.1$ (see [4])

$$\begin{aligned}
\underset{n\to\infty}{l.i.m}\ FY_n &= \underset{n\to\infty}{l.i.m}\left[\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y_n(s))ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds\right] \\
&= \underset{n\to\infty}{l.i.m}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y_n(s)ds + \underset{n\to\infty}{l.i.m}\int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds \\
&= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,\underset{n\to\infty}{l.i.m}\ Y_n(s)ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds \\
&= \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} f(s,Y(s)ds + \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} g(s,B(s))ds = FY.
\end{aligned}$$

This proves that $\{FY\}$ is continuous. Consequently, the closure of $\{FY\}$ is compact (see [4]). Thus, equation (2.3) has a solution $Y \in C$. □

Now for the problem (1.1)-(1.2), we have the following theorem.

Theorem 3.2 *Let the assumptions (i)-(iii) be satisfied, then the problem (1.1)-(1.2) has at least one solution $X \in C$ given by (2.1).*

Proof: From Lemma 1, the solution of the problem (1.1)-(1.2) is given by (2.1)

$$X(t) \;=\; X_0 \;+\; \int_0^{T-\tau} h(s, I^{\alpha-\beta}Y(s))dW(s) \;-\; I^\alpha Y(\tau) \;+\; I^\alpha Y(t), \qquad t\in[0,T]$$

where Y is given by (2.2).
Now, let Y be a solution of (2.2), then we have

$$\begin{aligned}
\|X(t)\|_2 &\le \|X_0\|_2 + \sqrt{\int_0^{T-\tau} \|\, h(s, I^{\alpha-\beta}\, Y(s))\,)\|_2^2\, ds} + I^\alpha\|Y(\tau)\|_2 + I^\alpha\|Y(t)\|_2 \\
&\le \|X_0\|_2 + \sqrt{\int_0^{T-\tau} (a+b\|I^{\alpha-\beta}Y(s))\,\|_2)^2\, ds} + I^\alpha\|Y(\tau)\|_2 + I^\alpha\|Y(t)\|_2 \\
&\le \|X_0\|_2 + \sqrt{\int_0^{T-\tau} (a+b\|Y\|_C I^{\alpha-\beta}(1))^2\, ds} + \|Y\|_C I^\alpha(1) + \|Y\|_C I^\alpha(1) \\
&\le \|X_0\|_2 + \sqrt{\int_0^{T-\tau} \left(a+b\|Y\|_C \frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)^2 ds} + 2\|Y\|_c \frac{t^\alpha}{\Gamma(\alpha+1)} \\
&\le \|X_0\|_2 + \sqrt{\int_0^{T-\tau} \left(a+b\|Y\|_C \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)^2 ds} + 2\|Y\|_c \frac{T^\alpha}{\Gamma(\alpha+1)} \\
&\le \|X_0\|_2 + \left(a+b\|Y\|_C \frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\sqrt{T-\tau} + 2\|Y\|_c \frac{T^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Then

$$\|X\|_C \;\le\; \|X_0\|_2 + \left(a+br\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right)\sqrt{T-\tau} + 2r\frac{T^\alpha}{\Gamma(\alpha+1)}$$

So, the solution X of the problem (1.1)-(1.2) exists and $X \in C([0,T], L_2(\Omega))$. □

4. Uniqueness Theorem

For discussing the uniqueness of the solution $Y \in C([0,T], L_2(\Omega))$ of fractional order integral equation (2.2), consider the following assumption

iv- The functions $f, g, h : I \times L_2(\Omega) \to L_2(\Omega)$, are measurable in $t \in I$ $\forall x \in L_2(\Omega)$ and satisfy the Lipschitz condition such that

$$\max\{\| f(t,x(t)) - f(t,y(t) \|_2, \| g(t,x(t)) - g(t,y(t) \|_2, \| h(t,x(t)) - h(t,y(t) \|_2\} \leq b \| x(t) - y(t) \|_2,$$

and

$$a(t) \ = \ \max\{f(t,0), g(t,0), h(t,0)\}$$

Theorem 4.1 *Let the assumptions (ii)- (iv) be satisfied, then the solution of the integral equation (2.3) and consequently, the problem (1.1)-(1.2) is unique.*

Proof: From assumption (iv) we can deduce that

$$||f(t,X)||_2 - ||f(t,0)||_2 \leq ||f(t,X) - f(t,0)||_2 \leq b\ ||x(t)||_2$$

and

$$||f(t,X)||_2 \leq a + b\ ||X(t)||_2.$$

Then the assumptions of Theorem (3.1) are satisfied and (2.3) has at least one solution.
Let Y_1 and Y_2 be two solutions of (2.3), then

$$\begin{aligned} ||Y_1(t) - Y_2(t)||_2 \ &\leq \ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||f(s, Y_1(s) - f(s, Y_2(s)||_2 ds \\ &\leq \ b||Y_1 - Y_2||_C \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds \\ &\leq \ b||Y_1 - Y_2||_C \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}. \end{aligned}$$

Then

$$(1 - b\frac{T^{1-\alpha}}{\Gamma(2-\alpha)})||Y_1 - Y_2||_C \leq 0 \qquad \Rightarrow \qquad ||Y_1 - Y_2||_C \leq 0$$

and this implies that

$$||Y_1 - Y_2||_C \ = \ 0 \quad \Rightarrow \quad Y_1(t) \ = \ Y_2(t).$$

Then the solution of fractional order integral equation (2.3) is unique.
Let X_1, X_2 be two solution of (2.1), then

$$X_1(t) - X_2(t) \ = \ \int_0^{T-\tau} [h(s, I^{\alpha-\beta}\, Y_1(s)) \ - \ h(s, I^{\alpha-\beta}\, Y_2(s))]dW(s) \ - \ I^{\alpha}(Y_1(\tau) - Y_2(\tau)) \ + \ I^{\alpha}(Y_1(t) - Y_2(t)),$$

then

$$\begin{aligned} ||\ X_1(t) - X_2(t)||_2 \ &\leq \ \ ||\int_0^{T-\tau} [h(s, I^{\alpha-\beta}\, Y_1(s)) \ - \ h(s, I^{\alpha-\beta}\, Y_2(s))]dW(s)||_2 \\ &+ \ I^{\alpha}||Y_1(\tau) - Y_2(\tau)||_2 \ + \ I^{\alpha}||Y_1(t) - Y_2(t)||_2 \\ &\leq \ \sqrt{\int_0^{T-\tau} ||h(s, I^{\alpha-\beta}\, Y_1(s)) \ - \ h(s, I^{\alpha-\beta}\, Y_2(s))||_2^2 ds} \\ &+ \ I^{\alpha}||Y_1(\tau) - Y_2(\tau)||_2 \ + \ I^{\alpha}||Y_1(t) - Y_2(t)||_2 \\ &\leq \ b\sqrt{\int_0^{T-\tau} (I^{\alpha-\beta}||Y_1(s) \ - \ Y_2(s))||_2)^2 ds} \\ &+ \ I^{\alpha}||Y_1(\tau) - Y_2(\tau)||_2 \ + \ I^{\alpha}||Y_1(t) - Y_2(t)||_2. \end{aligned}$$

So

$$|| X_1 - X_2||_C \leq \frac{bT^{\alpha-\beta}\sqrt{T-\tau}}{\Gamma(1+\alpha-\beta)}||Y_1 - Y_2||_C + \frac{2T^{\alpha}}{\Gamma(1+\alpha)}||Y_1 - Y_2||_C$$

Hence from the uniqueness of Y, we obtain

$$|| X_1 - X_2||_C = 0.$$

Consequently, the solution (2.1) of the initial value problem (1.1)-(1.2)

$$X(t) = X_0 + \int_0^{T-\tau} h(s, I^{\alpha-\beta}\, Y(s))\, dW(s) - I^{\alpha}Y(t) + I^{\alpha}Y(t) \in C(I, L_2(\Omega))$$

is unique one. □

5. Continuous Dependence

5.1. Continuous Dependence on the initial data X_0

Definition 5.1 *The solution $X \in C$ of the problem (1.1)-(1.2) depends continuously on the initial data X_0 if, $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that*

$$||X_0 - X_0^*||_2 \leq \delta \qquad \Rightarrow \qquad ||X - X^*||_C \leq \epsilon$$

where X^* is the solution of

$$X^*(t) = X_0^* + \int_0^{T-\tau} h(s, I^{\alpha-\beta}\, Y^*(s))\, dW(s) - I^{\alpha}Y^*(\tau) + I^{\alpha}Y^*(t),$$

and

$$Y^*(t) = I^{1-\alpha}\, [f(t, Y^*(t)) + g(t, B(t))].$$

Consider now the following theorem.

Theorem 5.1 *The unique solution of the problem (1.1)-(1.2) depends continuously on the initial data X_0.*

Proof: First of all, we know that $Y(t)$ doesn't depend on X_0, thus, we get

$$||Y(t) - Y^*(t)||_2 = 0$$

Now

$$\begin{aligned} || X(t) - X^*(t)||_2 &\leq ||X_0 - X_0^*||_2 + ||\int_0^{T-\tau} [h(s, I^{\alpha-\beta}\, Y(s)) - h(s, I^{\alpha-\beta}\, Y(s)^*)]dW(s)||_2 \\ &+ I^{\alpha}||Y(\tau) - Y^*(\tau)||_2 + I^{\alpha}||Y(t) - Y^*(t)||_2 \\ &\leq \delta = \epsilon \end{aligned}$$

and the results follows. □

5.2. Continuous Dependence on the stochastic function $f(t,x)$

Definition 5.2 *The solution $X \in C$ of the problem (1.1)-(1.2) depends continuously on the stochastic function $f(t,x)$ if, $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that*

$$||f(s,x(s) - f^*(s,x^*(s))||_2 \ \leq \delta \qquad \Rightarrow \qquad ||X - X^*||_C \leq \epsilon$$

where X^* is the solution of

$$X^*(t) \ = \ X_0 \ + \int_0^{T-\tau} \ h(s, I^{\alpha-\beta}\ Y^*(s))\ dW(s) \ - \ I^{\alpha}Y^*(\tau) \ + \ I^{\alpha}Y^*(t),$$

and

$$Y^*(t) \ = \ I^{1-\alpha}\ [f^*(t, Y^*(t)) + g(t, B(t))].$$

Consider now the following theorem.

Theorem 5.2 *The unique solution of the problem (1.1)-(1.2) depends continuously on $f(t,x)$.*

Proof: First of all we have

$$\begin{aligned} ||Y(t) - Y^*(t)||_2 \ &\leq \ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||f^*(s,Y(s)) - f(s,Y^*(s))||_2 ds \\ &\leq \ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}\ ||f^*(s,Y(s)) - f^*(s,Y^*(s)) + f^*(s,Y^*(s)) \\ &- \ \ f(s,Y(s)) + f(s,Y(s)) - f(s,Y^*(s))||_2\, ds \\ &\leq \ 2b_1||Y - Y^*||_C \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds + \delta \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds \end{aligned}$$

Thus, we get

$$||Y(t) - Y^*(t)||_2 \leq 2b||Y - Y^*||_C \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + \delta\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} = 2bT^*\ ||Y - Y^*||_C + \delta T^*$$

where, $T^* = \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}$, then

$$(1 - 2bT^*)||Y - Y^*||_C \leq T^*\delta$$

and

$$||Y - Y^*||_C \leq \frac{T^*\delta}{(1-2bT^*)} \ = \ \epsilon_1.$$

Now

$$\begin{aligned} ||\ X(t) - X^*(t)||_2 \ &\leq \ \ || \int_0^{T-\tau} \ [h(s, I^{\alpha-\beta}\ Y(s)) \ - \ h(s, I^{\alpha-\beta}\ Y^*(s))]dW(s)||_2 \\ &+ \ \ I^{\alpha}||Y(\tau) - Y^*(\tau)||_2 \ + \ I^{\alpha}||Y(t) - Y^*(t)||_2 \\ &\leq \ \ \sqrt{\int_0^{T-\tau} \ ||h(s, I^{\alpha-\beta}\ Y(s)^*) \ - \ h(s, I^{\alpha-\beta}\ Y^*(s))||_2^2 ds} \\ &+ \ \ I^{\alpha}||Y(\tau) - Y^*(\tau)||_2 \ + \ I^{\alpha}||Y(t) - Y^*(t)||_2 \\ &\leq \ \ b\sqrt{\int_0^{T-\tau} \ (I^{\alpha-\beta}||Y(s) \ - \ \ Y^*(s)||_2)^2 ds} \\ &+ \ \ I^{\alpha}||Y(\tau) - Y^*(\tau)||_2 \ + \ I^{\alpha}||Y(t) - Y^*(t)||_2. \end{aligned}$$

Then

$$\begin{aligned}||X - X^*||_C &\leq \frac{bT^{\alpha-\beta}\sqrt{T-\tau}}{\Gamma(1-\alpha+\beta)}||\ Y -\ Y^*||_C \ + \ \frac{2T^{\alpha}}{\Gamma(1+\alpha)}||Y - Y^*||_C \\ &\leq \epsilon_1(b\sqrt{T-\tau}\frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)} \ + \ \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)}) \\ &\leq \epsilon\end{aligned}$$

and the results follows. □

5.3. Continuous Dependence on the stochastic function $g(t, x)$

Definition 5.3 *The solution* $X \in C$ *of the problem (1.1)-(1.2) depends continuously on the stochastic function* $g(t, x)$ *if,* $\forall \ \epsilon > 0, \exists \ \delta > 0$ *such that*

$$||g(t, x(t)) - g^*(t, x(t))||_2 \ \leq \delta \qquad \Rightarrow \qquad ||X - X^*||_C \leq \epsilon$$

where X^* *is the solution of*

$$X^*(t) \ = \ X_0 \ + \ \int_0^{T-\tau} \ h(s, I^{\alpha-\beta} \ Y^*(s)) \ dW(s) \ - \ I^{\alpha}Y^*(\tau) \ + \ I^{\alpha}Y^*(t),$$

and

$$Y^*(t) \ = \ I^{1-\alpha} \ [f(t, Y^*(t)) + g^*(t, B(t))].$$

Consider now the following theorem.

Theorem 5.3 *The unique solution of the problem (1.1)-(1.2) depends continuously on* $g(t, x)$.

Proof: First of all we have

$$\begin{aligned}||Y(t) - Y^*(t)||_2 \ &\leq \ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}||f(s, Y(s)) - f(s, Y^*(s))||_2 ds \\ &+ \ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \ || \ g(t, B(t)) - g^*(t, B(t))||_2 ds \\ &\leq \ b_1||Y - Y^*||_C \ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}ds + b_2\delta \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}ds\end{aligned}$$

Thus, we get

$$||Y(t) - Y^*(t)||_2 \leq b||Y - Y^*||_C \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + b\delta \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)}ds = bT^* \ ||Y - Y^*||_C \ + \ bT^*\delta,$$

then

$$(1 - bT^*)||Y - Y^*||_C \leq bT^*\delta$$

and

$$||Y - Y^*||_C \leq \frac{bT^*\delta}{(1 - bT^*)} \ = \ \epsilon_2.$$

Then Thus as proved in section (5.2), we get

$$\begin{aligned}||X - X^*||_C &\leq \frac{bT^{\alpha-\beta}\sqrt{T-\tau}}{\Gamma(1-\alpha+\beta)}||\ Y -\ Y^*||_C \ + \ \frac{2T^{\alpha}}{\Gamma(1+\alpha)}||Y - Y^*||_C \\ &\leq \epsilon_2(b\sqrt{T-\tau}\frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)} \ + \ \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)}) \\ &\leq \epsilon\end{aligned}$$

and the results follows. □

5.4. Continuous Dependence on the Brownian motions

Definition 5.4 *The solution $X \in C$ of the problem (1.1)-(1.2) depends continuously on the Brownian Motion B if, $\forall\ \epsilon > 0, \exists\ \delta > 0$ such that*

$$||B(t) - B^*(t)||_2 \le \delta \qquad \Rightarrow \qquad ||X - X^*||_C \le \epsilon$$

where X^* is the solution of

$$X^*(t) = X_0 + \int_0^{T-\tau} h(s, I^{\alpha-\beta}\, Y^*(s))\, dW(s) - I^\alpha Y^*(\tau) + I^\alpha Y^*(t),$$

and

$$Y^*(t) = I^{1-\alpha}\, [f(t, Y^*(t)) + g(t, B^*(t))].$$

Consider now the following theorem.

Theorem 5.4 *The unique solution of the problem (1.1)-(1.2) depends continuously on $B(t)$.*

Proof: First of all we have

$$\begin{aligned} ||Y(t) - Y^*(t)||_2 &\le \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||f(s, Y(s)) - f(s, Y^*(s))||_2 ds \\ &+ \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \parallel g(t, B(t)) - g(t, B^*(t))||_2 ds \\ &\le b_1 ||Y - Y^*||_C \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ds + b_2 \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||B(t) - B^*(t)||_2 ds \end{aligned}$$

Thus, we get

$$\begin{aligned} ||Y(t) - Y^*(t)||_2 &\le b||Y - Y^*||_C \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} + b \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} ||B(t) - B^*(t)||_2 ds \\ &\le bT^*\ ||Y - Y^*||_C + b \int_0^t \frac{(t-s)^{-\alpha}}{\Gamma(1-\alpha)} \delta ds \\ &= bT^*\ ||Y - Y^*||_C + bT^* \delta, \end{aligned}$$

then

$$(1 - bT^*)||Y - Y^*||_C \le bT^* \delta$$

and

$$||Y - Y^*||_C \le \frac{bT^* \delta}{(1 - bT^*)} = \epsilon_3.$$

Now Thus as proved in sections (5.2) and (5.3), we get

$$\begin{aligned} ||X - X^*||_C &\le \frac{bT^{\alpha-\beta}\sqrt{T-\tau}}{\Gamma(1-\alpha+\beta)} ||\ Y - \ Y^*||_C + \frac{2T^\alpha}{\Gamma(1+\alpha)} ||Y - Y^*||_C \\ &\le \epsilon_3 (b\sqrt{T-\tau} \frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)} + \frac{2T^{\alpha+1}}{\Gamma(\alpha+1)}) \\ &\le \epsilon \end{aligned}$$

and the results follows. $\square$

5.5. Examples

(I) Let $B(t)=\mu t+\sigma W(t)$ be the Brownian motion with drift, $B^*(t)=\mu^* t+\sigma^* W(t)$ and W is a standard Brownian motion, then $\forall\ \epsilon>0, \exists\ \delta>0$ such that

$$\max\{|\mu-\mu^*|, |\sigma-\sigma^*|\}\leq\delta,$$

then

$$||B(t)-B^*(t)||_2=t\ |\mu-\mu^*|+||W(t)||_2|\sigma-\sigma^*|\leq\delta(T+\sqrt{T})=\delta.$$

Therefore, our results in Theorems (3.1), (3.2), (4.1) and(5.4) can be applied for the Brownian motion with drift.

(II) Let W be a standard Brownian motion and

$$B(t)=a(1-t)+bt+(1-t)\int_0^t\frac{dW(s)}{1-s},\qquad t\in[0,1)$$

and

$$B^*(t)=a^*(1-t)+b^*t+(1-t)\int_0^t\frac{dW(s)}{1-s},\qquad t\in[0,T)$$

where

$$Max\ \{a-a^*,\quad b-b^*\}\ \leq\delta.$$

So, we can get

$$||B-B^*||_2\quad=\quad|(a-a^*)(1-t)+(b-b^*)t|\leq\ \delta|(1-t)+t|\ =\ \delta.$$

Then our results in Theorems (3.1), (3.2), (4.1) and(5.4) can be applied for the Brownian bridge.
(III) At last, let W be a standard Brownian motion, A be a second order random variable $A\in L_2(\Omega)$ and

$$B(t)\ =\ A\ +W(t)$$

be the Brownian motion started at $A\in L_2(\Omega)$.
Let

$$B^*(t)\ =\ A^*\ +\ W(t),\qquad ||A-A^*||_2\leq\delta,$$

then we can get

$$||B-B^*||_2\ =\ ||A-A^*||_2\leq\delta.$$

Accordingly, our results in Theorems (3.1), (3.2), (4.1) and(5.4) are applicable for the Brownian motion started at $A\in L_2(\Omega)$.

6. Hyers-Ulam stability

In this section, we have the following definition.

Definition 6.1 *([2], [9]-[11], [19]-[21]) Let the solution $(Y\in C(I,L_2(\Omega)))$ of the integral equation (2.2) and consequently, the solution $(X\in C(I,L_2(\Omega)))$ of the problem (1.1)-(1.2), be exists uniquely. The problem (1.1)-(1.2) is said to be Hyers-Ulam stable, if $\forall\epsilon>0\ \exists\delta>0$, such that for any approximate $\delta-$approximate solution $X_s\in C([0,T],L_2(\Omega))$ of (1.1)-(1.2) such that*

$$\left|\left|\frac{d}{dt}X_s(t)-[f(t,D^\alpha X_s(t))+g(t,B(t))]\right|\right|_2\leq\delta,\tag{6.1}$$

then we have

$$\parallel X-X_s\parallel_C<\ \epsilon.$$

Now, we have the following theorem.

Theorem 6.1 *Let the assumptions outlined in Theorem (3.1) be hold. Then the problem (1.1)-(1.2) is Hyers-Ulam stable.*

Proof: Firstly, from Lemma (2.1), we have $Y(t) = D^{\alpha}X(t)$, which given by (2.2). Let $Y_s(t)$ be the ($\delta-$approximate)solution to (2.2), now

$$||Y_s(t) - I^{1-\alpha}(f(t, Y_s(t)) + g(t, B(t))||_2 = ||I^{1-\alpha}\ \frac{d}{dt}X_s(t) - I^{1-\alpha}\ [f(t, D^{\alpha}X_s(t)) + g(t, B(t))]||_2$$

$$\leq\ I^{1-\alpha}\ ||\frac{d}{dt}X_s(t) - [f(t, D^{\alpha}X_s(t)) + g(t, B(t))]||_2 \leq I^{1-\alpha}\ \delta\ \leq \frac{\delta\ t^{1-\alpha}}{\Gamma(2-\alpha)}.$$

Now

$$\begin{aligned}
\parallel Y(t) - Y_s(t) \parallel_2 &= \big|\big|I^{1-\alpha}[f(t, Y(t)) + g(t, B(t))] - I^{1-\alpha}[g(t, Y_s(t)) + g(t, B(t))]\big|\big|_2 \\
&= \big|\big|I^{1-\alpha}[f(t, Y(t)) + g(t, B(t))] - I^{1-\alpha}[g(t, Y_s(t)) + g(t, B(t))] \\
&+\ \ I^{1-\alpha}[g(t, Y_s(t)) + g(t, B(t))] - Y_s(t)\big|\big|_2 \\
&\leq\ I^{1-\alpha}\,||f(t, Y(t)) - f(t, Y_s(t))||_2 + \big|\big|I^{1-\alpha}[f(t, Y_s(t)) + g(t, B_s(t))] - Y_s(t)\big|\big|_2 \\
&\leq\ I^{1-\alpha} \parallel f(t, Y(t)) - f(t, Y_s(t)) \parallel_2 + \delta\ \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\leq\ b_1 \parallel Y - Y_s \parallel_C \frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{\delta\ t^{1-\alpha}}{\Gamma(2-\alpha)} \\
&\leq\ b_1 T^*||Y - Y_s||_C + \delta\ T^*, \qquad T^* = \frac{T^{1-\alpha}}{\Gamma(2-\alpha)}.
\end{aligned}$$

Thus

$$\parallel Y - Y_s \parallel_C \leq \frac{\delta T^*}{(1 - b_1 T^*)} = \epsilon_1.$$

and

$$\begin{aligned}
||X(t) - X_s(t)||_2 &\leq\ b\sqrt{T}(\frac{T^{\alpha-\beta}}{\Gamma(1-\alpha+\beta)}||\ Y -\ Y_s||_C)\ +\ \frac{T^{\alpha}}{\Gamma(1+\alpha)}||Y - Y_s||_C \\
&\leq\ ||\ Y -\ Y_s||_C(b\frac{T^{\alpha-\beta}}{\Gamma(1-\alpha+\beta)}\ +\ \frac{T^{\alpha}}{\Gamma(1+\alpha)}||Y - Y_s||_C) \\
&\leq\ \epsilon_1(b\sqrt{T}\frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)}\ +\ \frac{T^{\alpha+1}}{\Gamma(\alpha+1)}) = \epsilon
\end{aligned}$$

Then we obtain our result

$$||X - X_s||_C \leq \epsilon.$$

□

7. Conclusions

In this paper, in Theorem 3.1, we proved the existence of solutions $X \in C([0, T], L_2(\Omega))$ of the nonlocal stochastic-integral fractional orders stochastic differential equation

$$\frac{dX(t)}{dt} = f(t, D^{\alpha}X(t)) + g(t, B(t)), \quad t \in (0, T],$$

with the nonlocal stochastic-integral condition

$$X(\tau) = X_0\ +\ \int_0^{T-\tau}\ h(s, D^{\beta}X(s))dW(s)$$

where B represents any Brownian motion, W be the standard Brownian motion and X_0 denote a second order random variable.Theorem (4.1) outlines the sufficient condition required to guarantee the uniqueness of the solution. The stability of the problem in the sense of Hyers - Ulam, is established in Theorem (6.1). Moreover, the continuous dependence of the solution on X_0, $f(t,x)$, $g(t,x)$ and on the Brownian motion B are proved in Theorems (5.1), (5.2), (5.3) and(5.4). The three spatial cases Brownian bridge process, the Brownian motion with Drift and the Brownian motion started at A are examined.

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References

1. Abouagwa M., Bantan R.A.R., Almutiry W., Khalaf A.D. and Elgarhy M., *Mixed Caputo fractional neutral stochastic differential equations with impulses and variable delay*, Fractal Fract., 5, 239, 1-19, 2021.

2. Agarwal R.P., Bing Xu. b. and Weinian Zhang., *Stability of functional equations in single variable*, J. Math. Anal. Appl., 288, 852-869, 2003.

3. Ahmad M., Zada A., Ghaderi M., George R. and Rezapour S., *On the existence and stability of a neutral stochastic fractional differential system*, Fractal Fract., 6, 203, 1-16, 2022.

4. Curtain R.F. and Pritchard A.J., *Functional Analysis in Modern Applied Mathematics*, cambride press: Cambride, MA, USA, 1977.

5. Elborai M.M., *On some stochastic fractional integro-differential equations*, Adv. Dyn Sys. Appl, 1, 49-57; ISSN 0973-5321, 2006

6. Elborai M.M., Abdou A.A., El-Sayed W. and Awed Sh.I., *Numerical methods for solving integro partial differential equation with fractional order*, Journal of Positive School Psychology, 6, 4, 2124-2134, 2022.

7. El-Sayed A.M.A., *On stochastic fractional calculus operators*, J. Frac. Calculu. Appl., 6, 1, 101-109, 2015.

8. El-Sayed A.M.A., Arafa A. and Haggag A. *Mathematical Models for the novel coronavirus (2019-nCOV) with clinical data using fractional operator*, Numer Methods Partial Differential Eq., 1-22, 2022.

9. El-Sayed A.M.A., Al-Barg E.M. and Ebead H. R. *A fixed point of a constrained problem of state-dependent functional equation*, Adv. Fixed Point Theory, 14, ISSN:1927-6303, 1-12, 2024.

10. El-Sayed A.M.A., Alhamali A.A.A., Hamdallah E.M.A. and Edead H. R. *Qualitative aspecta of a factional-order integro-differential equation with a quadratic functional integro-differential constraint*, fractal and fractional, 7,835, 1-16, 2023.

11. El-Sayed A.M.A., Ba-Ali M.M.S. and Hamdallah E.M.A. *An investigation of a nonlinear Delay functional equation with a quadratic functional integral constraint*, Mathematics, 1, 14475, 1-24, 2023

12. El-Sayed A.M.A. and El-Gendy M.E.I., *Continuous dependence of the solution of random fractional order differential equation with nonlocal condition*, Fractional Differential Calculus, 7, 135-149, 2017.

13. El-Sayed A.M.A., Abd-Elsalam Sh. A. and El-Gendy M.E.I. *Distributed state-dependent with conjugate feedback control*, Int. J. Anal. Appl., 22, ISSN:2292-8639, 1-15, 2024

14. El-Sayed A.M.A. and El-Gendy M.E.I. *Solvability of a stochastic differential equation with nonlocal and integral condition*, Differential Equations And Control Processes, 3, 47-59, 2017.

15. El-Sayed A.M.A. and Fouad H.A., *On a coupled system of stochastic Ito differential and the arbitrary (fractional)order differential equations with nonlocal random and stochastic integral conditions. Mathematics, MDPI, 9, 20:2571, 1-14, 2021.*

16. *El-Sayed A.M.A and Gaafar F.M.,* Fractional calculus and some intermediate physical processes*, Applied Mathematics and Computation, 2003, 144, 1, 117- 126.*

17. *El-Sayed A.M.A., Gaafar F. and El-Gendy M.,* Continuous dependence of the solution of Ito stochastic differential equation with nonlocal conditions*, Applied Mathematical Sciences, 10, 40, 1971-1982, 2016.*

18. *El-Sayed A.M.A. and Hashim. H.G.,* Stochastic Ito differential and integral of fractional-orders*, Journal of Fractional Calculus and Application*, 13, 2, 251-258, 2022.

19. El-Sayed A.M.A., Hashim. H.G. and Al-Issa Sh. M. *A comprehensive view of the solvability of non-local fractional orders pantograph equation with a fractal-fractional feedback control*, Mathematics, 9,7, 19276-19298, 2024.

20. Hyers D.H., *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. USA, 1941, 27, 222 - 224, 1941.

21. Hyers D.H., *The stability of homomorphisms and related topics. In global analysis-analysis on manifolds*, Teubner-Texte Math., 57, 140-153, 1983.

22. Kafash B., Lalehzari R., Delavarkhalafi A. and Mahmoudi E., *Application of stochastic differential system in chemical reactions via simulation*, MATCH Commum. Math. Comput. Chem., 71, 265-277, 2014.

23. Knill O., *Probability Theory and Stochastic Process with Applications*, Narinder Kumar Lijhara for Overseas Press India Private Limited, 2009.

24. Oksendal B., *Stochastic Differential Equations, An Introduction and Applications*, Springer-Verlag Heidelberg New York, 2000.

25. Posch O., *Advanced Macroeconomics*, University of Hamburg, 2010.

26. Rida S.Z., El-Sayed A.M.A. and Arafa A.A.M., *Effect of bacterial memory dependent growth by using fractional derivatives reaction-diffusion chemotactic model*, J. Stat. Phys., 140, 797-811, 2010.

27. Soong T.T., *Random Differential Equations in Science and Engineering*, Academic Press: New York, NY, USA, 1973.

28. Wong E., *Stochastic Processes*, Informations and Dynamical Systems; McGraw-Hill: New York, NY, USA, 1971.

29. Wong E., *Introduction to Random Processes*, Springer: Berlin, Germany, 1980.

30. Wang B. H., Wang Y., Dai C. Q. and Chen Y.X., *Dynamical characteristic of analytical fractional solutions for the space-time fractional Fokas-Lenells equation*, Alex Eng. J., 59, 4699-4707, 2020.

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