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Boros Integral Involving the Product of Special Functions and the Incomplete I-Function

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ABSTRACT: In this present research, we developed a three parameter Boros integral formula for the incomplete I-function along with the generalized multi-index Mittag-Leffler function (MLF) and Srivastava polynomial. The derived outcomes are of a generic nature and may yield Boros integrals engaging the incomplete $\overline{H}$-function and incomplete H-function as a specific instance.

Key Words: Incomplete I-function, Mellin-Barnes integrals contour, Boros integral, Generalized multi-index Mittag-Leffler function..

Contents

1. Introduction and Preliminaries

Recently, a new category of incomplete I-functions ${}^{\gamma}I_{p,\,q}^{m,\,n}(y)$ and ${}^{\Gamma}I_{p,\,q}^{m,\,n}(y)$ has been introduced by Jangid et al. [7], which is the generalization of Rathie's I-function [13] and it is described as:

$$\begin{aligned}{}^{\gamma}I_{p,\,q}^{m,\,n}(y) &= {}^{\gamma}I_{p,\,q}^{m,\,n}\left[y \,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_2,\varsigma_2;\mathcal{F}_2),\cdots,(f_p,\varsigma_p;\mathcal{F}_p)\\(g_1,\varrho_1;\mathcal{G}_1),\cdots,(g_q,\varrho_q;\mathcal{G}_q)\end{matrix}\right]\\ &= {}^{\gamma}I_{p,\,q}^{m,\,n}\left[y \,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\pounds}\phi(r,t)\,y^r\,dr, \end{aligned} \tag{1.1}$$

and

$$\begin{aligned}{}^{\Gamma}I_{p,\,q}^{m,\,n}(y) &= {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[y \,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_2,\varsigma_2;\mathcal{F}_2),\cdots,(f_p,\varsigma_p;\mathcal{F}_p)\\(g_1,\varrho_1;\mathcal{G}_1),\cdots,(g_q,\varrho_q;\mathcal{G}_q)\end{matrix}\right]\\ &= {}^{\Gamma}I_{p,\,q}^{m,\,n}\left[y \,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right] = \frac{1}{2\pi i}\int_{\pounds}\Phi(r,t)\,y^r\,dr, \end{aligned} \tag{1.2}$$

$\forall\ y \neq 0$, where

$$\phi(r,t) = \frac{\{\gamma(1-f_1+\varsigma_1 r,\,t)\}^{\mathcal{F}_1}\prod\limits_{j'=1}^{m}\{\Gamma(g_{j'}-\varrho_{j'}r)\}^{\mathcal{G}_{j'}}\prod\limits_{j'=2}^{n}\{\Gamma(1-f_{j'}+\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}}{\prod\limits_{j'=n+1}^{p}\{\Gamma(f_{j'}-\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}\prod\limits_{j'=m+1}^{q}\{\Gamma(1-g_{j'}+\varrho_{j'}r)\}^{\mathcal{G}_{j'}}}, \tag{1.3}$$

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and

$$\Phi(r,t)=\frac{\{\Gamma(1-f_1+\varsigma_1 r,\,t)\}^{\mathcal{F}_1}\prod_{j'=1}^{m}\{\Gamma(g_{j'}-\varrho_{j'}r)\}^{\mathcal{G}_{j'}}\prod_{j'=2}^{n}\{\Gamma(1-f_{j'}+\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}}{\prod_{j'=n+1}^{p}\{\Gamma(f_{j'}-\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}\prod_{j'=m+1}^{q}\{\Gamma(1-g_{j'}+\varrho_{j'}r)\}^{\mathcal{G}_{j'}}}, \tag{1.4}$$

where, $\Gamma(.,t)$ and $\gamma(.,t)$ are the upper and lower incomplete gamma function described in (1.7) and (1.6) respectively.
The incomplete I-functions ${}^{\gamma}I^{m,\,n}_{p,\,q}(y)$ and ${}^{\Gamma}I^{m,\,n}_{p,\,q}(y)$ exist $\forall\, y\geq 0$ in accordance with Rathie's parameters and contour mentioned in [13] with,

$$|arg(y)|<\Delta\frac{\pi}{2},$$

where,

$$\Delta>0,$$

and

$$\Delta=\sum_{l'=1}^{m}\mathcal{G}_{l'}\,\varrho_{l'}-\sum_{l'=m+1}^{q}\mathcal{G}_{l'}\,\varrho_{l'}+\sum_{l'=1}^{n}\mathcal{F}_{l'}\,\varsigma_{l'}-\sum_{l'=n+1}^{p}\mathcal{F}_{l'}\,\varsigma_{l'}\,.$$

For $\mathcal{F}_1=1$, the following relation is satisfied by the incomplete I-functions:

$${}^{\gamma}I^{m,\,n}_{p,\,q}(y)+\,{}^{\Gamma}I^{m,\,n}_{p,\,q}(y)=I^{m,\,n}_{p,\,q}(y), \tag{1.5}$$

for the well known Rathie's I-function [13]. Some additional properties regarding the incomplete I-function can be found in [1].

The $\gamma(s,u)$ and $\Gamma(s,u)$ are explained in the following manner:

$$\gamma(s,u)=\int_0^u v^{s-1}\,e^{-v}\,dv,\qquad \big(u\geq 0\,\text{and}\,\Re(s)>0\big), \tag{1.6}$$

and

$$\Gamma(s,u)=\int_u^\infty v^{s-1}\,e^{-v}\,dv,\qquad \big(u\geq 0\text{ and }\Re(s)>0\text{ when }u=0\big), \tag{1.7}$$

recognized as the lower and upper incomplete gamma functions respectively.
The following relation is satisfied by the incomplete gamma functions.

$$\gamma(s,u)+\Gamma(s,u)=\Gamma(s),\qquad \big(\Re(s)>0\big). \tag{1.8}$$

A general class of polynomials was studied by the Srivastava [12], described in the following manner:

$$S^U_V[t]=\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\,t^R\,, \tag{1.9}$$

where $U\in\mathbb{Z}^+$ and $A_{V,\,R}$ are real or complex numbers arbitrary constant.
The notations $[k]$ indicates the floor function and $(\kappa)_\mu$ signify the Pochhammer symbol described in the manner mentioned below:

$$(\kappa)_\mu=\frac{\Gamma(\kappa+\mu)}{\Gamma(\kappa)}\quad\forall\,\mu\in\mathbb{C}-\{0\}\text{ and }(\kappa)_0=1,$$

in the form of the Gamma function.

Saxena and Nishimoto [15,16] invented the multi-index MLF (see [6]), described as:

$$E_{\eta,k}\left[\left(\zeta_{j'},\delta_{j'}\right)_{1,m'};z\right]=E_{\eta,k}\left[(\zeta_1,\delta_1),\cdots,(\zeta_{m'},\delta_{m'});z\right]=\sum_{w=0}^{\infty}\frac{(\eta)_{kw}\,z^w}{\prod_{j'=1}^{m'}\Gamma\left(\zeta'_j+\delta'_j w\right)\,w!}, \tag{1.10}$$

where $\max\{0, \Re(k)-1\} < \Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right)$, z, η, k, $\zeta_{j'}$, $\delta_{j'} \in \mathbb{C}$ *and* $\Re(\zeta_{j'}) > 0\ \forall\, 1 \leq j' \leq m'$.

For $\eta = k = 1$, Kiryakova [9] provided a description of the multi-index MLF in the form given below:

$$E_{1,1}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}, z\right] = \sum_{w=0}^{\infty} \frac{z^w}{\prod_{j'=1}^{m'} \Gamma\left(\zeta_{j'} + \delta_{j'} w\right)}, \tag{1.11}$$

where $\Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > 0,\ z,\ \zeta_{j'},\ \delta_{j'} \in \mathbb{C},\ \ and\ \ \Re(\zeta_{j'}) > 0\ \forall\, 1 \leq j' \leq m'$.

For $m' = 1$, (1.10) simplifies to Srivastava and Tomovski's [17] generalized MLF given below:

$$E_{\eta,k}\left[(\zeta,\delta), z\right] = E_{\zeta,\delta}^{\eta,k}(z) = \sum_{w=0}^{\infty} \frac{(\eta)_{kw}\, z^w}{\Gamma(\zeta + \delta w)\ w!}, \tag{1.12}$$

where $z, \eta, k, \zeta, \delta \in \mathbb{C}$, and $\Re(\delta) > \max\{\Re(k)-1, 0\}$.

For $k = 1$, (1.12) simplifies to Prabhakar's [11] generalized MLF given below:

$$E_{\eta,1}\left[(\zeta,\delta), z\right] = E_{\zeta,\delta}^{\eta,1}(z) = \sum_{w=0}^{\infty} \frac{(\eta)_{w}\, z^w}{\Gamma(\zeta + \delta w)\ w!}, \tag{1.13}$$

where $\min\{\Re(\zeta), \Re(\delta)\} > 0$ and $z, \eta, \zeta, \delta \in \mathbb{C}$.
For $\eta = 1$, (1.13) reduces to the MLF defined by Wiman [18] in the manner given below:

$$E_{1,1}\left[(\zeta,\delta), z\right] = E_{\zeta,\delta}^{1,1}(z) = \sum_{w=0}^{\infty} \frac{z^w}{\Gamma(\zeta + \delta w)}, \tag{1.14}$$

where $\min\{\Re(\zeta), \Re(\delta)\} > 0$ and $z, \zeta, \delta \in \mathbb{C}$.
Finally, for $\zeta = 1$, the MLF [5] is attained, defined as:

$$E_{\delta}(z) = E_{1,\delta}^{1,1}(z) = \sum_{w=0}^{\infty} \frac{z^w}{\Gamma(1 + \delta w)}, \tag{1.15}$$

where $\Re(\delta) > 0$ and $z, \delta \in \mathbb{C}$.

Lemma 1.1 *Let $e > 0$, $f \geq 0$, $d > -\sqrt{ef}$, and $P' > \frac{1}{2}$, we have the integral depending upon the three parameters, see [10].*

$$\int_0^{\infty} \left[\frac{h^2}{eh^4 + 2dh^2 + f}\right]^{P'} dh = \frac{B\left(P' - \frac{1}{2}, \frac{1}{2}\right)}{2^{P'+1/2}\sqrt{e}\left[d + \sqrt{ef}\right]^{P'-1/2}}, \tag{1.16}$$

where $B(m, n)$ denotes the Beta function. Equation (1.16) can also be expressed in the following way, by using the relation $B(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$.

$$\int_0^{\infty} \left[\frac{h^2}{eh^4 + 2dh^2 + f}\right]^{P'} dh = \frac{\sqrt{\pi}\,\Gamma\left(P' - \frac{1}{2}\right)}{\Gamma(P')\sqrt{e}\,2^{P'+\frac{1}{2}}(d + \sqrt{ef})^{P'-\frac{1}{2}}}. \tag{1.17}$$

Concerning the proof, see Boros and Moll [4] and Quershi et al. [12].

2. Main Results

For $X = \frac{h^2}{eh^4+2dh^2+f}$, the following is the outcomes:

Theorem 2.1 *For* $f \geq 0, e > 0\,, d > -\sqrt{ef},\, P^{'} > \frac{1}{2},\, z, \eta, k, \zeta_{j'}, \delta_{j'} \in \mathbb{C}, \Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max$ $\{0, \Re(k) - 1\}$, *and* $\Re(\zeta_{j'}) > 0\, \forall j^{'} = 1, \cdots, m^{'}$, *then the following result holds:*

$$\int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d + \sqrt{ef}\right)\right]^{P^{'}-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\zeta_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d + \sqrt{ef})}\right]^w$$
$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d + \sqrt{ef})]^{-g} \,\middle|\, \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P^{'} - w,\, g;\, 1\right), (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,p} \\ (1 - P^{'} - w,\, g;\, 1), (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{1,\,q} \end{matrix}\right]. \tag{2.1}$$

Proof: The LHS of equation (2.1) is:

$$H^{'} = \int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] {}^{\Gamma}I_{p,\,q}^{m,\,n}\left(X^g\,y\right) dh. \tag{2.2}$$

Replace Incomplete I-function and multi-index MLF by (1.2) and (1.10) respectively, we get:

$$H^{'} = \int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}\,(t\,X)^w}{\prod_{j'=1}^{m'} \Gamma\left(\zeta_{j'} + \delta_{j'} w\right)\, w!} \frac{1}{2\pi i} \int_{\pounds} \Phi(r,t)\, \left(X^g\,y\right)^r\, dr, \tag{2.3}$$

where $\Phi(r,t)$ is given by equation (1.4).
Change the integration order in the above equation yields:

$$H^{'} = \sum_{w=0}^{\infty} \frac{(\eta)_{kw}\,(t)^w}{\prod_{j'=1}^{m'} \Gamma\left(\zeta_{j'} + \delta_{j'} w\right)\, w!}$$
$$\times \frac{1}{2\pi i} \int_{\pounds} \frac{\{\Gamma(1 - f_1 + \varsigma_1 r,\, t)\}^{\mathcal{F}_1} \prod_{j'=1}^{m} \{\Gamma(g_{j'} - \varrho_{j'} r)\}^{\mathcal{G}'_j} \prod_{j'=2}^{n} \{\Gamma(1 - f_{j'} + \varsigma_{j'} r)\}^{\mathcal{F}_{j'}}}{\prod_{j'=n+1}^{p} \{\Gamma(f_{j'} - \varsigma_{j'} r)\}^{\mathcal{F}_{j'}} \prod_{j'=m+1}^{q} \{\Gamma(1 - g_{j'} + \varrho_{j'} r)\}^{\mathcal{G}_{j'}}}$$
$$\times\, y^r \int_0^\infty X^{P^{'}+w+gr} dh\, dr, \tag{2.4}$$

Now with the help of Lemma 1, evaluate the integral:

$$\int_0^\infty X^{P^{'}+w+gr} dh = \frac{\sqrt{\pi}\,\Gamma(P^{'} + w + gr - \frac{1}{2})}{\Gamma(P^{'} + w + gr)\sqrt{e}\, 2^{P^{'}+w+gr+\frac{1}{2}}(d + \sqrt{ef})^{P^{'}+w+gr-\frac{1}{2}}}. \tag{2.5}$$

Put (2.5) in (2.4), we get:

$$H^{'} = \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d + \sqrt{ef}\right)\right]^{P^{'}-\frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\zeta_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d + \sqrt{ef})}\right]^w$$
$$\times \frac{1}{2\pi i} \int_{\pounds} \frac{\{\Gamma(1 - f_1 + \varsigma_1 r, t)\}^{\mathcal{F}_1} \{\Gamma\left(P^{'} + w + gr - \frac{1}{2}\right)\} \prod_{j'=1}^{m} \{\Gamma(g_{j'} - \varrho_{j'} r)\}^{\mathcal{G}_{j'}}}{\prod_{j'=n+1}^{p} \{\Gamma(f_{j'} - \varsigma_{j'} r)\}^{\mathcal{F}_{j'}} \prod_{j'=m+1}^{q} \{\Gamma(1 - g_{j'} + \varrho_{j'} r)\}^{\mathcal{G}_{j'}}}$$
$$\times \frac{\prod_{j'=2}^{n} \{\Gamma(1 - f_{j'} + \varsigma_{j'} r)\}^{\mathcal{F}_{j'}}}{\{\Gamma\left(P^{'} + w + gr\right)\}} \times y^r [2(d + \sqrt{ef})]^{-gr}\, dr. \tag{2.6}$$

Now convert equation (2.6) in incomplete I-function to obtain the required result. $\square$

Theorem 2.2 *For $f \geq 0, e > 0, d > -\sqrt{ef}$, $P^{'} > \frac{1}{2}$, $z, \eta, k, \zeta_{j'}, \delta_{j'} \in \mathbb{C}, \Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max \{0, \Re(k) - 1\}$, and $\Re(\zeta_{j'}) > 0 \, \forall j^{'} = 1, \cdots, m^{'}$, then the following result holds:*

$$\int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] {}^{\gamma}I_{p,\,q}^{m,\,n}(X^g\, y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d + \sqrt{ef}\right)\right]^{P^{'}-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\zeta_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d + \sqrt{ef})}\right]^w$$
$$\times\, {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d + \sqrt{ef})]^{-g} \,\middle|\, \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P^{'} - w,\, g;\, 1\right), (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,p} \\ (1 - P^{'} - w,\, g;\, 1), (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{1,\,q} \end{matrix}\right]. \tag{2.7}$$

Theorem 2.2 is demonstrated under the same circumstances and in the same manner as Theorem 2.1.

Theorem 2.3 *For $f \geq 0, e > 0, d > -\sqrt{ef}$, $P^{'} > \frac{1}{2}$, $z, \eta, k, \zeta_{j'}, \delta_{j'} \in \mathbb{C}, \Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max \{0, \Re(k) - 1\}$, and $\Re(\zeta_{j'}) > 0 \, \forall j^{'} = 1, \cdots, m^{'}$, the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in \mathbb{Z}^+$, then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] S_V^U[lX]\, {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\, y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d + \sqrt{ef}\right)\right]^{P^{'}-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\zeta_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d + \sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\, A_{V,R}}{R!} \left[\frac{l}{2(d + \sqrt{ef})}\right]^R$$
$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d + \sqrt{ef})]^{-g} \,\middle|\, \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P^{'} - w - R,\, g;\, 1\right), (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,p} \\ (1 - P^{'} - w - R,\, g;\, 1), (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{1,\,q} \end{matrix}\right]. \tag{2.8}$$

Proof: The LHS of equation (2.8) is:

$$H = \int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] S_V^U[lX]\, {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\, y)dh. \tag{2.9}$$

Replace Incomplete I-function, multi-index MLF and Srivastava polynomial by (1.2), (1.10) and (1.9) respectively, we obtain:

$$H = \int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P^{'}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}\, (t\, X)^w}{\prod_{j'=1}^{m'} \Gamma\left(\zeta_{j'} + \delta_{j'} w\right)\, w!}$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}}{R!} A_{V,R}\, (l\, X)^R \frac{1}{2\pi i} \int_{\pounds} \Phi(r,t)\, (X^g\, y)^r\, dr, \tag{2.10}$$

where $\Phi(r,t)$ is given by equation (1.4).
Change the integration order in the above equation yields:

$$H=\sum_{w=0}^{\infty}\frac{(\eta)_{kw}\,(t)^{w}}{\prod_{j'=1}^{m'}\Gamma\left(\zeta_{j'}+\delta_{j'}w\right)\,w!}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}}{R!}A_{V,R}\,l^{R}$$
$$\times\frac{1}{2\pi i}\int_{\pounds}\frac{\{\Gamma(1-f_1+\varsigma_1 r,\,t)\}^{\mathcal{F}_1}\prod_{j'=1}^{m}\{\Gamma(g_{j'}-\varrho_{j'}r)\}^{\mathcal{G}_{j'}}\prod_{j'=2}^{n}\{\Gamma(1-f_{j'}+\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}}{\prod_{j'=n+1}^{p}\{\Gamma(f_{j'}-\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}\prod_{j'=m+1}^{q}\{\Gamma(1-g_{j'}+\varrho_{j'}r)\}^{\mathcal{G}_{j'}}}y^{r}$$
$$\times\int_{0}^{\infty}X^{P'+w+R+gr}dh\,dr. \tag{2.11}$$

Now with the help of Lemma 1, evaluate the integral:

$$\int_{0}^{\infty}X^{P'+w+R+gr}dh=\frac{\sqrt{\pi}\,\Gamma(P'+w+R+gr-\frac{1}{2})}{\Gamma(P'+w+R+gr)\sqrt{e}\,2^{P'+w+R+gr+\frac{1}{2}}(d+\sqrt{ef})^{P'+w+R+gr-\frac{1}{2}}}. \tag{2.12}$$

Put (2.12) in equation (2.11), we get:

$$H=\frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}}\sum_{w=0}^{\infty}\frac{(\eta)_{kw}}{\prod_{j'=1}^{m'}\Gamma(\zeta_{j'}+\delta_{j'}w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^{w}\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}$$
$$\times\left[\frac{l}{2(d+\sqrt{ef})}\right]^{R}\frac{1}{2\pi i}\int_{\pounds}\frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\{\Gamma\left(P'+w+R+gr-\frac{1}{2}\right)\}}{\prod_{j'=n+1}^{p}\{\Gamma(f_{j'}-\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}\prod_{j'=m+1}^{q}\{\Gamma(1-g_{j'}+\varrho_{j'}r)\}^{\mathcal{G}_{j'}}}$$
$$\times\frac{\prod_{j'=1}^{m}\{\Gamma(g_{j'}-\varrho_{j'}r)\}^{\mathcal{G}_{j'}}\prod_{j'=2}^{n}\{\Gamma(1-f_{j'}+\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}}{\{\Gamma\left(P'+w+R+gr\right)\}}y^{r}[2(d+\sqrt{ef})]^{-gr}\,dr. \tag{2.13}$$

Now convert equation (2.13) in incomplete I-function to obtain the desired result. □

Theorem 2.4 *For $f\geq 0, e>0, d>-\sqrt{ef}$, $P'>\frac{1}{2}$, $z,\eta,k,\zeta_{j'},\delta_{j'}\in\mathbb{C}, \Re\left(\sum_{j'=1}^{m'}\delta_{j'}\right)>\max\{0,\Re(k)-1\}$, and $\Re(\zeta_{j'})>0\,\forall j'=1,\cdots,m'$, the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in\mathbb{Z}^{+}$, then we have the following result:*

$$\int_{0}^{\infty}\left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'}E_{\eta,k}\left[\left(\zeta_{j'},\delta_{j'}\right)_{1,m'};tX\right]\,S_V^U[lX]\,{}^{\gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$=\frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}}\times\sum_{w=0}^{\infty}\frac{(\eta)_{kw}}{\prod_{j'=1}^{m'}\Gamma(\zeta_{j'}+\delta_{j'}w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^{w}$$
$$\times\sum_{R=0}^{[V/U]}\frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^{R}$$
$$\times{}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\,\middle|\,\begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\(1-P'-w-R,\,g;\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right]. \tag{2.14}$$

Theorem 2.4 is demonstrated under the same circumstances and in the same manner as Theorem 2.3.

Remark 2.1 If we set $U=1, A_{V,0}=1$ and $A_{V,R}=0\;\forall\;R\neq 0$ in Theorem 2.3 and 2.4, then the result is same as that of Theorem 2.1 and 2.2.

3. Special Case

In this section, As a particular instance of Theorem 2.3 and Theorem 2.4, we establish the Boros integral for the multiplication of Srivastava polynomial, Generalized Multi-Index MLF with the incomplete $\overline{I}$-function and the incomplete $\overline{H}$-function. Further, some special value will be given to Srivastava polynomial in order to get the outcomes in the form of Hermite and Laguerre polynomials. If we provide the parameter of particular features, we get the following special cases to delineate the use of fundamental outcomes.

(i) Incomplete $\bar{I}$- function: If we put $\mathcal{G}_{j'} = 1$ for $1 \le j' \le m$ in equation (1.2) and making use of the connection, that is (see [8,3]):

$$
\begin{aligned}
{}^{\Gamma}\overline{I}^{m,\,n}_{p,\,q}(y) &= {}^{\Gamma}\overline{I}^{m,\,n}_{p,\,q}\left[y \,\middle|\, \begin{array}{c} (f_1, \varsigma_1; \mathcal{F}_1 : t), (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,p} \\ (g_{j'}, \varrho_{j'}; 1)_{1,\,m}, (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{m+1,\,q} \end{array} \right] \\
&= \frac{1}{2\pi i} \int_{\pounds} \overline{\phi}(r,t) y^{r}\, dr,
\end{aligned}
\tag{3.1}
$$

where,

$$
\overline{\phi}(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r, t)\}^{\mathcal{F}_1} \prod\limits_{j'=1}^{m} \Gamma(g_{j'} - \varrho_{j'} r) \prod\limits_{j'=2}^{n} \{\Gamma(1-f_{j'}+\varsigma_{j'} r)\}^{\mathcal{F}_{j'}}}{\prod\limits_{j'=n+1}^{p} \{\Gamma(f_{j'} - \varsigma_{j'} r)\}^{\mathcal{F}_{j'}} \prod\limits_{j'=m+1}^{q} \{\Gamma(1-g_{j'}+\varrho_{j'} r)\}^{\mathcal{G}_{j'}}},
\tag{3.2}
$$

in (2.8) and (2.14), then we obtain the corollaries as follows:

Corollary 3.1 *For $e > 0,\ f \ge 0,\ d > -\sqrt{ef},\ P' > \frac{1}{2}$,$\Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max\{0, \Re(k) - 1\}$, $\Re(\zeta_{j'}) > 0\, \forall j' = 1, \cdots, m'$, the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in \mathbb{Z}^{+}$, then we have the following result:*

$$
\begin{aligned}
&\int_0^{\infty} \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P'} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'} ; tX\right] S_V^U[lX]\ {}^{\Gamma}\overline{I}^{m,\,n}_{p,\,q}(X^g\, y) dh \\
&= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\zeta_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d+\sqrt{ef})}\right]^{w} \\
&\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\, A_{V,R}}{R!} \left[\frac{l}{2(d+\sqrt{ef})}\right]^{R} \\
&\times {}^{\Gamma}\overline{I}^{m,\,n+1}_{p+1,\,q+1}\left[y[2(d+\sqrt{ef})]^{-g} \,\middle|\, \begin{array}{c} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P' - w - R,\, g;\, 1\right), (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,p} \\ (1 - P' - w - R,\, g;\, 1), (g_{j'}, \varrho_{j};\, 1)_{1,\,m},\, (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{m+1,\,q} \end{array} \right].
\end{aligned}
\tag{3.3}
$$

Corollary 3.2 *For $e > 0,\ f \ge 0,\ d > -\sqrt{ef},\ P' > \frac{1}{2}$,$\Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max\{0, \Re(k) - 1\}$, $\Re(\zeta_{j'}) > 0\, \forall j' = 1, \cdots, m'$, the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U \in \mathbb{Z}^{+}$, then* *we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{\eta,k}\left[\left(\zeta_{j'},\delta_{j'}\right)_{1,m'};tX\right] S_V^U[lX]\ {}^{\gamma}\bar{I}_{p,\,q}^{m,\,n}(X^g\, y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'}\Gamma(\zeta_{j'}+\delta_{j'}w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\ {}^{\gamma}\bar{I}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\ (1-P'-w-R,\,g;\,1),(g_{j'},\varrho_j;\,1)_{1,\,m},\,(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{m+1,\,q}\end{matrix}\right]. \tag{3.4}$$

(ii) Incomplete $\bar{H}$-function: If we put $\mathcal{F}_{j'}=1\ \ \forall\ j'=n+1,n+2,...,p$ and $\mathcal{G}_{j'}=1\ \ \forall\ j'=1,2,...,m$ in (1.2) and making use of the connection, that is (see [8])

$$\overline{\Gamma}_{p,\,q}^{m,\,n}(y) = \overline{\Gamma}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,n},(f_{j'},\varsigma_{j'};1)_{n+1,\,p}\\ (g_{j'},\varrho_{j'};1)_{1,\,m},(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{m+1,\,q}\end{matrix}\right]$$
$$= \overline{\Gamma}_{p,\,q}^{m,\,n}\left[y\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,n},(f_{j'},\varsigma_{j'})_{n+1,\,p}\\ (g_{j'},\varrho_{j'})_{1,\,m},(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{m+1,\,q}\end{matrix}\right]$$
$$= \frac{1}{2\pi i}\int_{\pounds}\overline{\psi}(r,t)y^r\ dr, \tag{3.5}$$

where,

$$\overline{\psi}(r,t) = \frac{\{\Gamma(1-f_1+\varsigma_1 r,t)\}^{\mathcal{F}_1}\prod\limits_{j'=1}^{m}\Gamma(g_{j'}-\varrho_{j'}r)\prod\limits_{j'=2}^{n}\{\Gamma(1-f_{j'}+\varsigma_{j'}r)\}^{\mathcal{F}_{j'}}}{\prod\limits_{j'=n+1}^{p}\Gamma(f_{j'}-\varsigma_{j'}r)\prod\limits_{j'=m+1}^{q}\{\Gamma(1-g_{j'}+\varrho_{j'}r)\}^{\mathcal{G}_{j'}}}, \tag{3.6}$$

in (2.8) and (2.14), then we obtain the corollaries as follows.

Corollary 3.3 *For $e>0,\ f\geq 0,\ d>-\sqrt{ef},\ P'>\frac{1}{2}$,$\Re\left(\sum_{j'=1}^{m'}\delta_{j'}\right)>\max\{0,\Re(k)-1\}$, $\Re(\zeta_{j'})>0\,\forall j'=1,\cdots,m'$, the coefficient $A_{V,R}$ are real or complex arbitrary constants and $U\in\mathbb{Z}^+$, then we have the following result:*

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{\eta,k}\left[\left(\zeta_{j'},\delta_{j'}\right)_{1,m'};tX\right] S_V^U[lX]\ \bar{\Gamma}_{p,\,q}^{m,\,n}(X^g\, y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'}\Gamma(\alpha_{j'}+\delta_{j'}w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\ \bar{\Gamma}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g,\,1\right),\\ (1-P'-w-R,\,g,\,1),(g_{j'},\varrho_{j'};1)_{1,\,m},\end{matrix}\right.$$
$$\left.\begin{matrix}(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,n},\,(f_{j'},\varsigma_{j'};1)_{n+1,\,p}\\ (g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{m+1,\,q}\end{matrix}\right]. \tag{3.7}$$

Corollary 3.4 *For* $e > 0,\ f \geq 0,\ d > -\sqrt{ef},\ P' > \frac{1}{2}$ $,\Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max\{0, \Re(k) - 1\}$, $\Re(\zeta_{j'}) > 0\,\forall j' = 1, \cdots, m'$, *the coefficient* $A_{V,R}$ *are real or complex arbitrary constants and* $U \in \mathbb{Z}^+$, *then we have the following result:*

$$
\begin{aligned}
&\int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P'} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] S_V^U[lX]\ \bar{\gamma}_{p,\,q}^{m,\,n}(X^g\, y)dh \\
&= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d + \sqrt{ef}\right)\right]^{P' - \frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\alpha_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d + \sqrt{ef})}\right]^w \\
&\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\, A_{V,R}}{R!} \left[\frac{l}{2(d + \sqrt{ef})}\right]^R \\
&\times \bar{\gamma}_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d + \sqrt{ef})]^{-g} \,\middle|\, \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - P' - w - R,\, g,\, 1\right), \\ (1 - P' - w - R,\, g,\, 1), (g_{j'}, \varrho_{j'}; 1)_{1,\,m}, \end{matrix}\right. \\
&\qquad\qquad \left. \begin{matrix} (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,n},\ (f_{j'}, \varsigma_{j'}; 1)_{n+1,\,p} \\ (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{m+1,\,q} \end{matrix}\right]. \qquad (3.8)
\end{aligned}
$$

(iii) Hermite Polynomial: If we set $A_{V,\,R} = (-1)^R$ and $U = 2$ in (1.9) then $S_V^2[t] \to t^{V/2} H_V\left(\frac{1}{2\sqrt{t}}\right)$ and making use of the connection, that is (see [14]):

$$
H_V(t) = \sum_{R=0}^{[V/2]} (-1)^R \frac{V!}{R!(V-2R)!}(2t)^{V-2R}, \qquad (3.9)
$$

in (2.8) and (2.14), then we obtain the corollaries as follows.

Corollary 3.5 *For* $e > 0,\ f \geq 0,\ d > -\sqrt{ef},\ P' > \frac{1}{2}$ $,\Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max\{0, \Re(k) - 1\}$, $\Re(\zeta_{j'}) > 0\,\forall j' = 1, \cdots, m'$, *then we have the following result:*

$$
\begin{aligned}
&\int_0^\infty \left(\frac{h^2}{eh^4 + 2dh^2 + f}\right)^{P'} E_{\eta,k}\left[\left(\zeta_{j'}, \delta_{j'}\right)_{1,m'}; tX\right] (Xl)^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{Xl}}\right) {}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\, y)dh \\
&= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d + \sqrt{ef}\right)\right]^{P' - \frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'} \Gamma(\alpha_{j'} + \delta_{j'} w) w!} \left[\frac{t}{2(d + \sqrt{ef})}\right]^w \\
&\times \sum_{R=0}^{[V/2]} \frac{(-1)^R\, V!}{R!\,(V-2R)!} \left[\frac{l}{2(d + \sqrt{ef})}\right]^R \\
&\times {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d + \sqrt{ef})]^{-g} \,\middle|\, \begin{matrix} (f_1, \varsigma_1; \mathcal{F}_1 : t), \left(\frac{3}{2} - w - P' - R,\, g,\, 1\right), (f_{j'}, \varsigma_{j'}; \mathcal{F}_{j'})_{2,\,p} \\ (1 - P' - w - R,\, g,\, 1), (g_{j'}, \varrho_{j'}; \mathcal{G}_{j'})_{1,\,q} \end{matrix}\right]. \qquad (3.10)
\end{aligned}
$$

Corollary 3.6 *For* $e > 0,\ f \geq 0,\ d > -\sqrt{ef},\ P' > \frac{1}{2}$ $,\Re\left(\sum_{j'=1}^{m'} \delta_{j'}\right) > \max\{0, \Re(k) - 1\}$,

$\Re(\zeta_{j'}) > 0 \, \forall j' = 1, \cdots, m'$, then we have the following result:

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{\eta,k}\left[\left(\zeta_{j'},\delta_{j'}\right)_{1,m'};tX\right] (Xl)^{\frac{V}{2}} H_V\left(\frac{1}{2\sqrt{Xl}}\right) {}^{\gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\prod_{j'=1}^{m'}\Gamma(\alpha_{j'}+\delta_{j'}w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/2]} \frac{(-1)^R\,V!}{R!\,(V-2R)!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\, {}^{\gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-w-P'-R,\,g,\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\ (1-P'-w-R,\,g,\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right]. \quad (3.11)$$

For $\eta = k = 1$, we get special case of the multi-index MLF defined by Kiryakova [9] as in (1.11), we have the following corollary.

Corollary 3.7

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{1,1}\left[\left(\zeta_{j'},\delta_{j'}\right)_{1,m'};tX\right] S_V^U[lX]\,{}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{1}{\prod_{j'=1}^{m'}\Gamma(\alpha_{j'}+\delta_{j'}w)}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\ (1-P'-w-R,\,g;\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right], \quad (3.12)$$

under the identical circumstances as that of Theorem 2.3 for $\eta = k = 1$.

For $m = 1$, we get special case of the multi-index MLF defined by Srivastava and Tomovski [17] as in (1.12), we have the following corollary.

Corollary 3.8

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{\eta,k}^{\zeta,\delta}(tX)\; S_V^U[lX]\,{}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{(\eta)_{kw}}{\Gamma(\zeta+\delta w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\, {}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\,\middle|\, \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\ (1-P'-w-R,\,g;\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right], \quad (3.13)$$

under the identical circumstances as that of Theorem 2.3 for $m = 1$.

For $k = 1$, we obtain a specific instance of Prabhakar's multi-index MLF [11] as a special case of Corollary 3.8 as in (1.13), we have the following corollary.

Corollary 3.9

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{\eta,1}^{\zeta,\delta}(tX)\ S_V^U[lX]\,{}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{(\eta)_w}{\Gamma(\zeta+\delta w)w!}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\,{}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\ (1-P'-w-R,\,g;\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right], \quad (3.14)$$

under the identical circumstances as that of Theorem 2.3.

For $\eta=1$, we obtain a specific instance of Wilman's multi-index MLF [18] as a special case of Corollary 3.9 as in (1.14), we have the following corollary.

Corollary 3.10

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{1,1}^{\zeta,\delta}(tX)\ S_V^U[lX]\,{}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{1}{\Gamma(\zeta+\delta w)}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\,{}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p},\\ (1-P'-w-R,\,g;\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right], \quad (3.15)$$

under the identical circumstances as that of Theorem 2.3.

For $\zeta=1$, we obtain specific instance of multi-index MLF described in (1.15) as a special case of corollary 3.10, we have the following Corollary.

Corollary 3.11

$$\int_0^\infty \left(\frac{h^2}{eh^4+2dh^2+f}\right)^{P'} E_{\delta}(tX)\ S_V^U[lX]\,{}^{\Gamma}I_{p,\,q}^{m,\,n}(X^g\,y)dh$$
$$= \frac{\sqrt{\pi}}{2\sqrt{e}\left[2\left(d+\sqrt{ef}\right)\right]^{P'-\frac{1}{2}}} \times \sum_{w=0}^{\infty} \frac{1}{\Gamma(1+\delta w)}\left[\frac{t}{2(d+\sqrt{ef})}\right]^w$$
$$\times \sum_{R=0}^{[V/U]} \frac{(-V)_{UR}\,A_{V,R}}{R!}\left[\frac{l}{2(d+\sqrt{ef})}\right]^R$$
$$\times\,{}^{\Gamma}I_{p+1,\,q+1}^{m,\,n+1}\left[y[2(d+\sqrt{ef})]^{-g}\ \middle|\ \begin{matrix}(f_1,\varsigma_1;\mathcal{F}_1:t),\left(\frac{3}{2}-P'-w-R,\,g;\,1\right),(f_{j'},\varsigma_{j'};\mathcal{F}_{j'})_{2,\,p}\\ (1-P'-w-R,\,g;\,1),(g_{j'},\varrho_{j'};\mathcal{G}_{j'})_{1,\,q}\end{matrix}\right], \quad (3.16)$$

under the identical circumstances as that of Theorem 2.3.

Remark 3.1

- If we set $U = 1, A_{V,0} = 1$ and $A_{V,R} = 0 \ \forall \ R \neq 0$ for the the first four corollaries (Corollary 3.1-Corollary 3.4) then it becomes the special case for Theorem 2.1 and 2.2.
- If we put $m^{'} = k = \eta = \zeta = 1, \delta = 0,$ and $t = 0$ in Theorem 2.1 and 2.3, then we get Theorem 1 and 3 respectively, mentioned in Bhatter et al. [2].
- The incomplete I-function ${}^{\gamma}I_{p,\,q}^{m,\,n}(z)$ has the identical relations to the previous four corollaries (Corollary 3.8-Corollary 3.11). Identical outcomes can be obtained for the incomplete $\overline{I}$-function, incomplete $\overline{H}$-function, incomplete H-function, and incomplete hypergeometric function.

4. Concluding Remarks

In this article, we obtain the Boros integral with three parameter for the incomplete I-function which is the extension of the I-function investigated by Jangid et al. [7] for the product of incomplete I-function, Srivastava Polynomial and multi-index Mittag-Leffler function. As the incomplete I-function generalize variety of incomplete functions like: I-function, Meijer G-function, hypergeometric function, H-function, $\overline{I}$-function and many other functions. Also, Srivastava polynomial generalize various other polynomial like: Hermite polynomial, Jacobi polynomial, Laguerre polynomial, Gegenbauer polynomial, Legendre polynomial, Tchebycheff polynomial, Gould-Hopper polynomial and several other polynomials. Simillarly, multi- index Mittag-Leffler function generalizes many other Mittag-Leffler type function and we can get a variety of integrals involving a few special function by specialising the Boros integral's parameters. Hence our main findings are useful and can be used to establish the number of different Boros integral forms linked with numerous kinds of special functions and polynomials.

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