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Best Proximity Points for Generalized $(\mathcal{F}, \mathcal{R})$-proximal Contractions

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ABSTRACT: We present the notion of generalized $(\mathcal{F}, \mathcal{R})$-proximal non-self contractions and prove best proximity point theorems in complete metric spaces endowed with an arbitrary binary relation. An example is given to vindicate our claims. We also show that the edge preserving structure is a particular case of the binary relation $\mathcal{R}$. Moreover, an application to variational inequality problem is given in order to demonstrate the efficacy of our results.

Key Words: Best proximity point, generalized $(\mathcal{F}, \mathcal{R})$-proximal contractions, binary relation.

Contents

1. Introduction

Banach contraction principle was established in 1922 for solving existence problems in nonlinear analysis, which was a major problem back then. It states that, every contraction self-mapping $\mathcal{T}$ on a complete metric space $(\mathcal{X}, d)$ has a unique fixed point. So far there has been many generalizations, extensions and applications of this principle in various metric space settings ([12], [23], [24]). One such salient generalization is to prove fixed point results in metric spaces equipped with an arbitrary binary relation. A lot of research has been done to obtain fixed point results by utilising different binary relations on metric spaces (see, [3,4,5,13,16,17,27,28,32,33]). In 1986, Turinici initiated this investigation by proving analogous variations of the Banach contraction principle ([29]- [31]). Alam and Imdad [1] used the relation theoretic versions of some standard metric notions such as continuity, convergence and completeness to prove fixed point results in metric spaces equipped with an arbitrary binary relation.

The notion of best proximity point for a non-self mapping $\mathcal{T}$ was introduced by Basha [10] in 2010. If $\mathcal{C}$ and $\mathcal{D}$ are nonempty subsets of a metric space $(\mathcal{X}, d)$ and $\mathcal{T} : \mathcal{C} \to \mathcal{D}$ is a non-self mapping. Then $c \in \mathcal{C}$ is said to be a best proximity point if $d(c, \mathcal{T}c) = d(\mathcal{C}, \mathcal{D})$, where $d(\mathcal{C}, \mathcal{D}) = \inf\{d(\mathcal{C}, \mathcal{D}) : c \in \mathcal{C}, d \in \mathcal{D}\}$. Numerous best proximity point results in different metric space settings have been proved by eminent mathematicians in the last decade (see, for instance [6,7,8,11,14,15,22,25,26]). In 2014, Wardowski [34] introduced a contractive mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ on a metric space $(\mathcal{X}, d)$, known as $\mathcal{F}$-contraction. He proved that if $(\mathcal{X}, d)$ is a complete metric space, then every $\mathcal{F}$-contraction has a unique fixed point.

We first introduce the notion of generalized $(\mathcal{F}, \mathcal{R})$-proximal non-self contractions and then prove some best proximity point theorems in complete metric spaces endowed with an arbitrary binary relation. We also give the relation between an arbitrary binary relation and a graphical structure. Towards the end, an application to variational inequality problems is provided in order to support our results.

2. Preliminaries

In this paper, $\mathcal{R}$ stands for a non-empty binary relation, $\mathbb{N}$ stands for the set of natural numbers, $\mathbb{R}$ stands for the set of real numbers and $\mathbb{R}^+$ stands for the set of positive real numbers.

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Definition 2.1 *[1] Let $x, y \in \mathcal{X}$, where $\mathcal{X}$ is a non-empty set endowed with a binary relation $\mathcal{R}$. Then, x and y are said to be $\mathcal{R}$-comparitive if either $(x, y) \in \mathcal{R}$ or $(y, x) \in \mathcal{R}$, denoted as $[x, y] \in \mathcal{R}$.*

Definition 2.2 *[20] Let $\mathcal{R}$ be a binary relation on a non-empty set $\mathcal{X}$.*

1. *The inverse relation of $\mathcal{R}$, that is, $\mathcal{R}^{-1}$ is defined as $\mathcal{R}^{-1} = \{(x, y) \in \mathcal{X}^2 | (y, x) \in \mathcal{R}\}$.*
2. *The symmetric closure of $\mathcal{R}$, that is, $\mathcal{R}^S$ is defined as the set $\mathcal{R} \cup \mathcal{R}^{-1}$. Clearly, $\mathcal{R}^S$ is the smallest symmetric relation on $\mathcal{X}$ containing $\mathcal{R}$.*

Theorem 2.1 *[1] For a binary relation $\mathcal{R}$ defined on a non-empty set $\mathcal{X}$, $(x, y) \in \mathcal{R}^S$ if and only if $[x, y] \in \mathcal{R}$.*

Definition 2.3 *[1] Let $\mathcal{X}$ be a non-empty set and $\mathcal{R}$ a binary relation on $\mathcal{X}$. A sequence $\{x_n\} \in \mathcal{X}$ is called $\mathcal{R}$-preserving if $(x_n, x_{n+1}) \in \mathcal{R}$ for all $n \in \mathbb{N} \cup \{0\}$.*

Definition 2.4 *[1] Let $\mathcal{X}$ be a non-empty set and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a mapping. A binary relation $\mathcal{R}$ on $\mathcal{X}$ is called $\mathcal{T}$-closed if for any $x, y \in X$, $(x, y) \in \mathcal{R}$ implies $(\mathcal{T}x, \mathcal{T}y) \in \mathcal{R}$.*

Definition 2.5 *[2] Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{R}$ a binary relation on $\mathcal{X}$. We say that $(\mathcal{X}, d)$ is $\mathcal{R}$-complete if every $\mathcal{R}$-preserving Cauchy sequence in $\mathcal{X}$ converges.*

Definition 2.6 *[2] Let $\mathcal{R}$ be a binary relation on a metric space $(\mathcal{X}, d)$. A self-mapping $\mathcal{T}$ on $\mathcal{X}$ is called $\mathcal{R}$-continuous at x if $\{\mathcal{T}x_n\} \to \mathcal{T}x$ for all $\mathcal{R}$-preserving sequence $\{x_n\} \subseteq \mathcal{X}$ with $\{x_n\} \to x$. Also if $\mathcal{T}$ is $\mathcal{R}$-continuous at every point of $\mathcal{X}$ then it is $\mathcal{R}$-continuous.*

Definition 2.7 *[34] Let $\mathfrak{F}$ be the set of functions $\mathcal{F} : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following:*
$(\mathcal{F}_1)$ $\mathcal{F}$ is strictly increasing;
$(\mathcal{F}_2)$ for every sequence $\{\alpha_n\}$ of positive numbers,

$$\lim_{n\to+\infty} \alpha_n = 0 \textit{ iff } \lim_{n\to+\infty} (\mathcal{F}\alpha_n) = -\infty;$$

$(\mathcal{F}_3)$ there exists $k \in (0, 1)$ so that $\lim\limits_{\alpha\to 0^+} \alpha^k \mathcal{F}(\alpha) = 0$.

Some functions belonging to $\mathfrak{F}$ are:

1. $\mathcal{F}(x) = \ln x$ for $x > 0$.
2. $\mathcal{F}(x) = -\frac{1}{\sqrt{x}}$ for $x > 0$.

Definition 2.8 *[34] A self-mapping $\mathcal{T}$ on a metric space $\mathcal{X}$ is called an $\mathcal{F}$-contraction if there exists $\mathcal{F} \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that*

$$\tau + \mathcal{F}(d(\mathcal{T}u, \mathcal{T}v)) \leq \mathcal{F}(d(u, v)),$$

for all $u, v \in X$ with $d(\mathcal{T}u, \mathcal{T}v) > 0$.

Henceforth, we give some notations for subsequent use. If C and D are non-empty subsets of $\mathcal{X}$, then

$$\begin{aligned} d(x, \mathcal{D}) &= \inf\{d(x, y) : y \in \mathcal{D}\}, \text{ where } x \in \mathcal{C}, \\ \mathcal{C}_0 &= \{x \in \mathcal{C} : d(x, y) = d(\mathcal{C}, \mathcal{D}) \text{ for some } y \in \mathcal{D}\}, \\ \mathcal{D}_0 &= \{y \in \mathcal{D} : d(x, y) = d(\mathcal{C}, \mathcal{D}) \text{ for some } x \in \mathcal{C}\}. \end{aligned}$$

Definition 2.9 *[21] Let $(\mathcal{X}, d)$ be a metric space. A pair of nonempty subsets $(\mathcal{C}, \mathcal{D})$ of (X, d), with $\mathcal{C}_0 \neq \emptyset$, is said to have the p-property if for every $x_1, x_2 \in \mathcal{C}$ and every $y_1, y_2 \in \mathcal{D}$*

$$\left.\begin{aligned} d(x_1, y_1) &= d(\mathcal{C}, \mathcal{D}) \\ d(x_2, y_2) &= d(\mathcal{C}, \mathcal{D}) \end{aligned}\right\} \textit{ implies } d(x_1, x_2) = d(y_1, y_2),$$

Definition 2.10 *[19] Let $(\mathcal{X},d)$ be a metric space endowed with a binary relation $\mathcal{R}\subseteq\mathcal{X}$. $(\mathcal{X},d,\mathcal{R})$ is said to be regular if for all sequence $\{x_n\}$ in $\mathcal{X}$ such that $\{x_n\}\to x$ and $(x_n,x_{n+1})\in\mathcal{R}$, for all $n\in\mathbb{N}$, we have $(x_n,x)\in\mathcal{R}$, for all $k\in\mathbb{N}$.*

Definition 2.11 *[9] A set $\mathcal{D}$ is said to be approximately compact with respect to $\mathcal{C}$, if every sequence $\{y_n\}$ of $\mathcal{D}$ with $d(x,y_n)\to d(x,\mathcal{D})$ for some $x\in\mathcal{C}$ has a convergent subsequence.*

3. Main results

Now, we define proximal $\mathcal{R}$-preserving mapping and generalized $(\mathcal{F},\mathcal{R})$-proximal contraction as follows.

Definition 3.1 *A mapping $\mathcal{T}:\mathcal{C}\to\mathcal{D}$ is said to be proximal $\mathcal{R}$-preserving if*

$$\left.\begin{array}{l}(x_1,x_2)\in\mathcal{R}\\ d(y_1,\mathcal{T}x_1)=d(\mathcal{C},\mathcal{D})\\ d(y_2,\mathcal{T}x_2)=d(\mathcal{C},\mathcal{D})\end{array}\right\}\ \textit{implies}\ (y_1,y_2)\in\mathcal{R},$$

Definition 3.2 *A mapping $\mathcal{T}:\mathcal{C}\to\mathcal{D}$ is said to be a generalized $(\mathcal{F},\mathcal{R})$-proximal contraction if there exists $\mathcal{F}\in\mathfrak{F}$ and $\tau>0$ such that*

$$\left.\begin{array}{l}d(x_1,\mathcal{T}y_1)=d(\mathcal{C},\mathcal{D})\\ d(x_2,\mathcal{T}y_2)=d(\mathcal{C},\mathcal{D})\end{array}\right\}\ \textit{implies}$$

$$\tau+\mathcal{F}(d(x_1,x_2))\leq\mathcal{F}(\max\{d(y_1,y_2),d(x_1,y_1),d(x_2,y_2),\frac{1}{2}[d(y_1,x_2)+d(y_2,x_1)]\}),$$

for all $x_1,x_2,y_1,y_2\in C, x_1\neq x_2$ and $(x_1,x_2),(y_1,y_2)\in\mathcal{R}$.

Theorem 3.1 *Let $(\mathcal{X},d)$ be a metric space, $\mathcal{C},\mathcal{D}$ be two non-empty closed subsets of $\mathcal{X}$ with $\mathcal{C}_0\neq\phi$ and $\mathcal{R}$ a relation on $\mathcal{X}$. Let $\mathcal{T}:\mathcal{C}\to\mathcal{D}$ satisfy the following assertions:*

1. *$\mathcal{T}(\mathcal{C}_0)\subseteq\mathcal{D}_0$ and $(\mathcal{X},d)$ is $\mathcal{R}$-complete;*
2. *$\mathcal{T}$ is proximal $\mathcal{R}$-preserving;*
3. *there exists $x_0,x_1\in\mathcal{C}_0$ such that*

$$d(x_1,\mathcal{T}x_0)=d(\mathcal{C},\mathcal{D})\ \textit{and}\ (x_0,x_1)\in\mathcal{R};$$

4. *$\mathcal{T}$ is $\mathcal{R}$-continuous;*
5. *$\mathcal{T}$ is a generalized $(\mathcal{F},\mathcal{R})$-proximal contraction.*

Then, $\mathcal{T}$ has a best proximity point.

Proof: As $\mathcal{T}(\mathcal{C}_0)\subseteq\mathcal{D}_0$, we have $x_2\in\mathcal{C}_0$ such that

$$d(x_2,\mathcal{T}x_1)=d(\mathcal{C},\mathcal{D})$$

Since $\mathcal{T}$ satisfies condition 3 and $\mathcal{T}$ is proximal $\mathcal{R}$-preserving, we obtain $(x_1,x_2)\in\mathcal{R}$, i.e.

$$d(x_2,\mathcal{T}x_1)=d(\mathcal{C},\mathcal{D}),\ \text{where}\ (x_1,x_2)\in\mathcal{R} \tag{3.1}$$

Again, as $\mathcal{T}(\mathcal{C}_0)\subseteq\mathcal{D}_0$, there exists $x_3\in\mathcal{C}_0$ such that

$$d(x_3,\mathcal{T}x_2)=d(\mathcal{C},\mathcal{D}). \tag{3.2}$$

From (3.1), (3.2) and using $\mathcal{T}$ is proximal $\mathcal{R}$-preserving, we get $(x_2,x_3)\in\mathcal{R}$, i.e.

$$d(x_3,\mathcal{T}x_2)=d(\mathcal{C},\mathcal{D}),\ \text{where}\ (x_2,x_3)\in\mathcal{R}.$$

Continuing the above process, we get

$$d(x_{n+1}, \mathcal{T}x_n) = d(\mathcal{C}, \mathcal{D}), \text{ where } (x_n, x_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Therefore the sequence $\{x_n\}$ is $\mathcal{R}$-preserving.

If for some $n_0, d(x_{n_0}, x_{n_0+1}) = 0$, then $x_{n_0} = x_{n_0+1}$ and hence from (3.3), $d(\mathcal{C}, \mathcal{D}) = d(x_{n_0}, \mathcal{T}x_{n_0})$, which completes the proof. Thus, for some $n \geq 0, d(x_n, x_{n+1}) > 0$. Since, $\mathcal{T}$ is a generalized $(\mathcal{F}, \mathcal{R})$-proximal contraction and taking $u_1 = x_n, u_2 = x_{n+1}, v_1 = x_{n-1}, v_2 = x_n$, we have

$$\begin{aligned}\tau + \mathcal{F}(d(x_n, x_{n+1})) &\leq \mathcal{F}(\max\{d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \frac{1}{2}[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]\}) \\ &\leq \mathcal{F}(\max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}). \end{aligned} \tag{3.4}$$

If $d(x_n, x_{n+1}) \geq d(x_{n-1}, x_n)$ for some $n \in \mathbb{N}$, then from equation (3.4) we have $\tau + F(x_n, x_{n+1}) \leq F(x_n, x_{n+1})$, which is a contradiction since $\tau > 0$. Thus $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$ and so from (3.4) we get

$$\tau + \mathcal{F}(d(x_n, x_{n+1})) \leq \mathcal{F}(d(x_{n-1}, x_n))$$

which implies

$$\begin{aligned}\mathcal{F}(d(x_n, x_{n+1})) &\leq \mathcal{F}(x_n, x_{n-1}) - \tau \leq ... \\ &\leq \mathcal{F}(d(x_0, x_1)) - n\tau, \text{ for all } n \in \mathbb{N} \end{aligned} \tag{3.5}$$

Set $t_n = d(x_n, x_{n+1})$. So, from (3.5), $\mathcal{F}(t_n) = -\infty$. Using the property $(\mathcal{F}_2)$, we get

$$t_n \to 0 \text{ as } n \to +\infty.$$

Now, let $k \in (0, 1)$ such that $\lim_{n\to+\infty} t_n^k \mathcal{F}(t_n) = 0$. By (3.5), the following holds for all $n \in \mathbb{N}$:

$$t_n^k \mathcal{F}(t_n) - t_n^k \mathcal{F}(t_0) \leq -n t_n^k \tau \leq 0. \tag{3.6}$$

Letting $n \to \infty$ in (3.6), we have

$$\lim_{n\to+\infty} n t_n^k = 0$$

Therefore $\lim_{n\to+\infty} n^{\frac{1}{k}} t_n = 0$, which implies that $\sum_{n=1}^{\infty} t_n$ is convergent. This further implies that $\{x_n\}$ is a Cauchy sequence. Because the space is $\mathcal{R}$-complete, and $\mathcal{C}$ is closed so the sequence $\{x_n\}$ converges to some element x^* in $\mathcal{C}$. If $\mathcal{T}$ is $\mathcal{R}$-continuous, then $\mathcal{T}x_n \to \mathcal{T}x^*$ as $n \to +\infty$ and

$$d(\mathcal{C}, \mathcal{D}) = \lim_{n\to+\infty} d(x_{n+1}, \mathcal{T}x_n) = d(x^*, \mathcal{T}x^*),$$

which implies that x^* is a best proximity point.

□

Theorem 3.2 *Let $(\mathcal{X}, d)$ be a metric space, $\mathcal{C}, \mathcal{D}$ be two non-empty closed subsets of $\mathcal{X}$ with $\mathcal{C}_0 \neq \phi$ and $\mathcal{D}$ be approximately compact with respect to $\mathcal{C}$. Let $\mathcal{R}$ be a relation on $\mathcal{X}$ and $\mathcal{T} : \mathcal{C} \to \mathcal{D}$ satisfy the following assertions:*

1. *$\mathcal{T}(\mathcal{C}_0) \subseteq \mathcal{D}_0$ and $(\mathcal{X}, d)$ is $\mathcal{R}$-complete;*
2. *$\mathcal{T}$ is proximal $\mathcal{R}$-preserving;*
3. *there exists $x_0, x_1 \in \mathcal{C}_0$ such that*

$$d(x_1, \mathcal{T}x_0) = d(\mathcal{C}, \mathcal{D}) \text{ and } (x_0, x_1) \in \mathcal{R};$$

4. *$(\mathcal{X}, d, \mathcal{R})$ is regular;*

5. *$\mathcal{T}$ is a generalized $(\mathcal{F},\mathcal{R})$-proximal contraction.*

Then, $\mathcal{T}$ has a best proximity point.

Proof: Following the proof of Theorem 3.1, we get a Cauchy sequence $\{x_n\} \subset \mathcal{C}$ such that (3.3) holds and $\{x_n\} \to x^*$ as $n \to +\infty$. Also,

$$\begin{aligned} d(x^*, \mathcal{D}) \leq d(x^*, Tx_n) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x^*, x_{n+1}) + d(\mathcal{C}, \mathcal{D}) \\ &\leq d(x^*, x_{n+1}) + d(x^*, \mathcal{D}). \end{aligned}$$

As $n \to +\infty$, we get $d(x^*, Tx_n) \to d(x^*, \mathcal{D})$, and since $\mathcal{D}$ is approximately compact with respect to $\mathcal{C}$, $\{Tx_n\}$ has a subsequence $\{Tx_{n_k}\}$ converging to some $y^* \in \mathcal{D}$. Thus $d(x^*, y^*) = \lim_{n\to+\infty} d(x_{n_k+1}, Tx_{n_k}) = d(\mathcal{C}, \mathcal{D})$. This implies that $x^* \in \mathcal{C}_0$. Since, $T(\mathcal{C}_0) \subseteq \mathcal{D}_0$, there exists some $t \in \mathcal{C}$ such that

$$d(t, Tx^*) = d(\mathcal{C}, \mathcal{D}) \tag{3.7}$$

As $(\mathcal{X}, d, \mathcal{R})$ is regular, we have $(x_n, x^*) \in \mathcal{R}$. Therefore for a subsequence $\{x_{m_k}\}$ of $\{x_n\}$ also, we have

$$(x_{m_k}, x^*) \in \mathcal{R}. \tag{3.8}$$

Also,

$$d(x_{m_k+1}, Tx_{m_k}) = d(\mathcal{C}, \mathcal{D}). \tag{3.9}$$

Using the idea that T is proximal $\mathcal{R}-$preserving along with (3.7),(3.8) and (3.9), we get $(x_{m_k+1}, t) \in \mathcal{R}$.

We claim that $d(t, x^*) = 0$. On the contrary, if $d(t, x^*) \neq 0$ then, there exists an $m_0 \in \mathbb{N}$ such that

$$d(t, x_{m_k+1}) > 0 \text{ for all } m_k \geq m_0.$$

Now, T is a generalized $(\mathcal{F},\mathcal{R})-$ proximal contraction, taking $x_1 = t, x_2 = x_{m_k+1}, y_1 = x^*$ and $y_2 = x_{m_k}$, we get

$$\tau + \mathcal{F}(d(t, x_{m_k+1})) \leq \mathcal{F}(\max\{d(x_{m_k}, x^*), d(x_{m_k+1}, x_{m_k}), d(t, x^*), \frac{1}{2}[d(x_{m_k+1}, x^*) + d(x_{m_k}, t)]\})$$

Using the fact that $\mathcal{F}$ is strictly increasing, we obtain

$$d(t, x_{m_k+1}) < \max\{d(x_{m_k}, x^*), d(x_{m_k+1}, x_{m_k}), d(t, x^*), \frac{1}{2}[d(x_{m_k+1}, x^*) + d(x_{m_k}, t)]\}$$

As $k \to \infty$, $d(t, x^*) < d(t, x^*)$, which is a contradiction. Therefore, $d(t, x^*) = 0$, i.e., $t = x^*$. So from (3.7), we deduce that $d(x^*, Tx^*) = d(\mathcal{C}, \mathcal{D})$, i.e., x^* is a best proximity point.

□

Theorem 3.3 *Suppose that the following condition holds along with the hypotheses of Theorem 3.1 or Theorem 3.2: for any $x, y \in \mathcal{C}$ with $(x, y) \notin \mathcal{R}$, there exists $z_0 \in \mathcal{C}_0$ such that $(x, z_0) \in \mathcal{R}$ and $(y, z_0) \in \mathcal{R}$. Then $\mathcal{T}$ has a unique best proximity point.*

Proof: Let there exists another best proximity point y^* of the mapping $\mathcal{T}$ such that

$$d(y^*, \mathcal{T}y^*) = d(x^*, \mathcal{T}x^*) = d(\mathcal{C}, \mathcal{D}). \tag{3.10}$$

Case-I
Let x^* and y^* be two distinct best proximity points of $\mathcal{T}$ such that $(x^*, y^*) \in \mathcal{R}$. Since $\mathcal{T}$ is a generalized $(\mathcal{F}, \mathcal{R})$-proximal contraction and taking $u_1 = v_1 = x^*$ and $u_2 = v_2 = y^*$, we have

$$\begin{aligned}
&\tau + \mathcal{F}(d(x^*, y^*)) \leq \mathcal{F}(\max\{d(x^*, y^*), d(x^*, x^*), d(y^*, y^*), \frac{1}{2}[d(x^*, y^*) + d(x^*, y^*)]\}) \\
&\implies \mathcal{F}(d(x^*, y^*)) < \mathcal{F}(d(x^*, y^*)).
\end{aligned}$$

Since $\mathcal{F}$ strictly increasing

$$d(x^*, y^*)) < d(x^*, y^*),$$

which is a contradiction. Therefore x^* and y^* must be same. Hence, $\mathcal{T}$ has a unique best proximity point.

Case-II
Let $(x^*, y^*) \notin \mathcal{R}$. By the given condition there exists $z_0 \in \mathcal{C}_0$ such that $(x^*, z_0) \in \mathcal{R}$ and $(y^*, z_0) \in \mathcal{R}$. Since $\mathcal{T}(\mathcal{C}_0) \subseteq \mathcal{D}_0$, there exists $z_1 \in \mathcal{C}_0$ such that $d(z_1, \mathcal{T}z_0) = d(\mathcal{C}, \mathcal{D})$. Also $d(x^*, \mathcal{T}x^*) = d(\mathcal{C}, \mathcal{D})$ and $(x^*, z_0) \in \mathcal{R}$. So by proximal $\mathcal{R}$-preserving condition we get that $(x^*, z_1) \in \mathcal{R}$. Therefore we get

$$d(z_1, \mathcal{T}z_0) = d(\mathcal{C}, \mathcal{D}) \text{ and } (x^*, z_1) \in \mathcal{R} \tag{3.11}$$

Now there exists $z_2 \in C_0$ such that $d(z_2, \mathcal{T}z_1) = d(\mathcal{C}, \mathcal{D})$. Also $d(x^*, \mathcal{T}x^*) = d(\mathcal{C}, \mathcal{D})$ and $(x^*, z_1) \in \mathcal{R}$. So by proximal $\mathcal{R}$-preserving condition we get that $(x^*, z_2) \in \mathcal{R}$. Therefore we get

$$d(z_2, \mathcal{T}z_1) = d(\mathcal{C}, \mathcal{D}) \text{ and } (x^*, z_2) \in \mathcal{R} \tag{3.12}$$

Continuing the above process we get a sequence $\{z_n\}$ such that

$$d(z_{n+1}, \mathcal{T}z_n) = d(\mathcal{C}, \mathcal{D}) \text{ and } (x^*, z_{n+1}) \in \mathcal{R} \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.13}$$

Now using (3.10), (3.13) and p-property, we get

$$d(z_{n+1}, x^*) = d(\mathcal{T}z_n, \mathcal{T}x^*) \text{ for all } n \in \mathbb{N} \cup \{0\}. \tag{3.14}$$

Suppose that $z_0 = x^*$, then from (3.14), we get

$$d(z_1, x^*) = d(\mathcal{T}z_0, \mathcal{T}x^*) = d(\mathcal{T}x^*, \mathcal{T}x^*) = 0,$$

which gives $z_1 = x^*$. Continuing the above process, we get $z_n = x^*$ for all $n \in \mathbb{N} \cup \{0\}$. Now suppose that $d(z_0, x^*) > 0$, then we get $d(z_{n+1}, x^*) > 0$. Since $\mathcal{T}$ is a generalized $(\mathcal{F}, \mathcal{R})$-proximal contraction and taking $u_1 = z_{n+1}, v_1 = z_n$ and $u_2 = v_2 = x^*$, we have

$$\tau + \mathcal{F}(d(z_{n+1}, x^*)) \leq \mathcal{F}(\max\{d(z_n, x^*), d(z_{n+1}, z_n), d(x^*, x^*), \frac{1}{2}[d(z_{n+1}, x^*) + d(x^*, z_n)]\}). \tag{3.15}$$

If $\max\{d(z_n, x^*), d(z_{n+1}, z_n)\} = d(z_{n+1}, z_n)$, then as $\mathcal{F}$ is strictly increasing, (3.15) implies

$$d(z_{n+1}, x^*) < d(z_{n+1}, z_n),$$

which implies that $d(z_{n+1}, x^*) < 0$ as $n \to +\infty$, which is a contradiction. Therefore $\max\{d(z_n, x^*), d(z_{n+1}, z_n)\} = d(z_n, x^*)$.

$$\begin{aligned}
\mathcal{F}(d(z_{n+1}, x^*)) &\leq \mathcal{F}(d(z_n, x^*)) - \tau \\
&\;\;. \\
&\;\;. \\
&\leq \mathcal{F}(d(z_0, x^*)) - (n+1)\tau.
\end{aligned}$$

As $n \to +\infty$, $\mathcal{F}(d(z_{n+1}, x^*)) \to -\infty$. Therefore from $\mathcal{F}_2$, we obtain $d(z_{n+1}, x^*) \to 0$ as $n \to +\infty$, i.e., $\{z_n\} \to x^*$. Similarly, we can show that $z_n \to y^*$ as $n \to +\infty$. Thus we obtain $x^* = y^*$ by the uniqueness of limit. □

Example 3.1 *Let $\mathcal{X}=\mathbb{R}^n$ and $\mathcal{R}=\{(x,y)|x_2\leq y_2,$ where $x=(x_1,x_2,...,x_n)$ and $y=(y_1,y_2,...,y_n)\}$ be binary relation, where $x,y\in\mathbb{R}^n$. Define a metric d on $\mathcal{X}$ by*

$$d(x,y)=\sum_{i=1}^{n}|x_i-y_i|,\ \text{where } x=(x_1,x_2,...,x_n) \text{ and } y=(y_1,y_2,...,y_n).$$

Clearly, $(\mathcal{X},d)$ is a $\mathcal{R}$-complete metric space. Let $\mathcal{F}:\mathbb{R}^+\to\mathbb{R}$ be given by $\mathcal{F}(x)=\ln x$. Clearly $\mathcal{F}$ satisfies the conditions $\mathcal{F}_1-\mathcal{F}_3$. Let $\mathcal{C}=\{(0,e,0,...,0)|e\geq 0\}$ and $\mathcal{D}=\{(1,g,0,...,0)|g\geq 0\}$, so we have $\mathcal{C}=\mathcal{C}_0$ and $\mathcal{D}=\mathcal{D}_0$. Define $\mathcal{T}:\mathcal{C}\to\mathcal{D}$ by

$$\mathcal{T}(0,e,0,...,0)=(1,\mathcal{S}(e),0,...,0) \text{ for all } (0,e,0,...,0)\in\mathcal{C}.$$

where $\mathcal{S}(e)=\frac{1}{5}-\frac{1}{e+5}$.

Clearly, $|\mathcal{S}(a)-\mathcal{S}(b)|\leq|a-b|$ for all $a,b\geq 0$. Also, $\mathcal{T}(\mathcal{C}_0)\subseteq\mathcal{D}_0$, $\mathcal{T}$ is proximal $\mathcal{R}-$preserving and $\mathcal{T}$ is $\mathcal{R}-$continuous. Now to show that $\mathcal{T}$ is a generalized $(\mathcal{F},\mathcal{R})-$proximal contraction, let $p=(0,e_1,0,...,0),q=(0,e_2,0,...,0),r=(0,S(e_1),0,...,0)$ and $s=(0,\mathcal{S}(e_2),0,...,0)$ be elements in $\mathcal{E}$. It is clear that $d(r,\mathcal{T}p)=d(\mathcal{C},\mathcal{D})$ and $d(s,\mathcal{T}q)=d(\mathcal{C},\mathcal{D})$. We obtain

$$\begin{aligned}d(r,s)=&|\mathcal{S}(e_1)-\mathcal{S}(e_2)|\\ \leq&e^{-\tau}|e_1-e_2|\\ \leq&e^{-\tau}\max\{|e_1-e_2|,|e_1-\mathcal{S}(e_1)|,|e_2-\mathcal{S}(e_2)|,\frac{1}{2}[|e_1-\mathcal{S}(e_2)|+|e_2-\mathcal{S}(e_1)|]\},\end{aligned}$$

where $e^{-\tau}=\frac{2}{5}$. Therefore $\mathcal{T}$ is a generalized $(\mathcal{F},\mathcal{R})-$proximal contraction with $\tau=\ln\frac{5}{2}$. Thus all the conditions of Theorem 3.1 are satisfied and hence $\mathcal{T}$ has a best proximity point $(0,0,...,0)$.

Corollary 3.3A *Let $(\mathcal{X},d)$ be a metric space and $\mathcal{R}$ a relation on $\mathcal{X}$. Let $\mathcal{T}:\mathcal{X}\to\mathcal{X}$ satisfy the following assertions:*

1. *$(\mathcal{X},d)$ is $\mathcal{R}$-complete.*
2. *there exists $x_0\in\mathcal{X}$ such that $(x_0,\mathcal{T}x_0)\in\mathcal{R}$.*
3. *$\mathcal{R}$ is $\mathcal{T}$-closed.*
4. *$\mathcal{T}$ is $\mathcal{R}$-continuous or $\mathcal{X}$ is $\mathcal{R}$-regular;*
5. *$\mathcal{T}$ satisfies the following contraction*

$$\tau+\mathcal{F}(d(\mathcal{T}x,\mathcal{T}y))\leq\mathcal{F}(\max\{d(x,y),d(x,Tx),d(y,Ty),\frac{1}{2}[d(x,Ty)+d(y,Tx)]\}).$$

for all $x,y\in\mathcal{X},\mathcal{T}x\neq\mathcal{T}y$, $(x,y)\in\mathcal{R}$ and $\tau>0$ for some $\mathcal{F}\in\mathfrak{F}$.

Then, $\mathcal{T}$ has a fixed point.

In 2008, Jachymski [18] introduced the graphical structure in metric spaces and proved some fixed point results for graphical variant of the Banach contraction principle. Let $\mathcal{X}$ be a non-empty set and Δ denotes the diagonal points of $\mathcal{X}\times\mathcal{X}$. Then, $\mathcal{G}$ is a directed graph whose vertex set $\mathcal{V}(\mathcal{G})$ coincides with $\mathcal{X}$, and the edge set $\mathcal{E}(\mathcal{G})$ contains its edges with all the loops, that is, $\Delta\subseteq\mathcal{E}(\mathcal{G})$. Moreover, we suppose that $\mathcal{G}$ has no parallel edges and therefore we can write $\mathcal{G}$ as a pair $(\mathcal{V}(\mathcal{G}),\mathcal{E}(\mathcal{G}))$. Also, we suppose $\mathcal{G}$ as a weighted graph by assigning to each edge the distance between its vertices. A path in $\mathcal{G}$ from x to y, where x and y are any vertices of the graph $\mathcal{G}$, of length $k(k\in\mathbb{N})$ is a sequence $\{x_i\}_{i=0}^{k}$ of $k+1$ vertices such that $x_0=x,x_k=y$ and $(x_{n-1},x_n)\in\mathcal{E}(\mathcal{G})$ for $i=1,2,3,...,k$. If there is a path between any two vertices of a graph $\mathcal{G}$ then it is said to be a connected graph, and $\mathcal{G}$ is said to be weakly connected if $\tilde{G}$ is connected.
For a metric space $(\mathcal{X},d)$, the triple $(\mathcal{X},d,\mathcal{G})$ denotes a $\mathcal{G}$-metric space where $\mathcal{G}$ is a graph on the non-empty set $\mathcal{X}$.

Remark 3.1 *The above results can be proved in a similar way for metric spaces endowed with graphs by defining* $\mathcal{R} = \{(x, y)|(x, y) \in \mathcal{E}(\mathcal{G}), x, y \in \mathcal{X}\}$, *where* $\mathcal{E}(\mathcal{G})$ *is the edge set of the directed graph* $\mathcal{G}$. *Jachymski [18] stated that the set of all edges of the directed graph contains all loops, which means that the set of all edges forms a reflexive binary relation on the underlying metric space. This shows the improvement of the relational-theoretic approach over the graphical approach. For example, if we take strict `<´as a binary relation, then the relevant fixed point theorem is obtained from relation-theoretic contraction principle. As the relation `<´is irreflexive, therefore we can never prove such a result from the graphical fixed point theorem. Thus, the edge preserving structure is a particular case of the binary relation* $\mathcal{R}$*([5]).*

4. Application

Now, we present an application of our results to a variational inequality problem. Let $\mathcal{H}$ be a real Hilbert space with induced norm $(||.||)$ and inner product $(\langle ., .\rangle)$. $\mathcal{K}$ be a non-empty, closed and convex subset of $\mathcal{H}$. A self operator $\mathcal{T}$ on $\mathcal{H}$ is called monotone if $\langle \mathcal{T}x - \mathcal{T}y, y - x\rangle \geq 0$. We consider the following monotone variational inequality problem.
Problem 1. Find $x \in \mathcal{K}$ such that $\langle \mathcal{Q}x, y - x\rangle \geq 0$ for all $y \in \mathcal{K}$, where $\mathcal{Q} : \mathcal{H} \to \mathcal{H}$ is monotone operator. A powerful tool for solving a variational inequality problem is the metric projection, say $\mathcal{P}_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ such that for each $x \in \mathcal{K}$, there exists a unique nearest point $\mathcal{P}_{\mathcal{K}}x \in \mathcal{K}$ such that $||x - \mathcal{P}_{\mathcal{K}}x|| \leq ||x - y||$ for all $y \in \mathcal{K}$.
Now, we see some important lemmas.

Lemma 4.1 *Let* $z \in \mathcal{H}$. *Then* $x = \mathcal{P}_{\mathcal{K}}(z)$ *if and only if* $\langle x - z, y - x\rangle \leq 0$ *for all* $y \in \mathcal{K}$.

Lemma 4.2 *Let* $\mathcal{Q} : \mathcal{K} \to \mathcal{K}$ *be monotone. Then* $x \in \mathcal{K}$ *is a solution of* $\langle \mathcal{Q}x, y - x\rangle \geq 0$ *for all* $y \in \mathcal{K}$ *if and only if* $x = \mathcal{P}_{\mathcal{K}}(x - \lambda \mathcal{Q}x), \lambda > 0$.

Our following result gives the solution of Problem 1.

Theorem 4.1 *Let* $\mathcal{K}$ *be a non-empty, closed non-empty subset of a real Hilbert space* $\mathcal{H}$ *and* $\mathcal{R}$ *a relation on* $\mathcal{H}$. *Suppose* $\mathcal{Q} : \mathcal{K} \to \mathcal{K}$, ${\mathcal{P}_{\mathcal{K}}}^* = \mathcal{P}_{\mathcal{K}}(\mathcal{I}_{\mathcal{K}} - \lambda \mathcal{Q}) : \mathcal{K} \to \mathcal{K}$, *where* $\mathcal{I}_{\mathcal{K}}$ *is the identity operator on* $\mathcal{K}$, *satisfy the following assertions:*

1. $(\mathcal{K}, d)$ *is* $\mathcal{R}$*-complete.*
2. *there exists* $x_0 \in \mathcal{K}$ *such that* $(x_0, {\mathcal{P}_{\mathcal{K}}}^* x_0) \in \mathcal{R}$.
3. $\mathcal{R}$ *is* ${\mathcal{P}_{\mathcal{K}}}^*$*-closed.*
4. ${\mathcal{P}_{\mathcal{K}}}^*$ *satisfies the following contraction*

$$\tau + \mathcal{F}(d({\mathcal{P}_{\mathcal{K}}}^* x, {\mathcal{P}_{\mathcal{K}}}^* y)) \leq \mathcal{F}(\max\{d(x, y), d(x, {\mathcal{P}_{\mathcal{K}}}^* x), d(y, {\mathcal{P}_{\mathcal{K}}}^* y), \frac{1}{2}[d(x, {\mathcal{P}_{\mathcal{K}}}^* y) + d(y, {\mathcal{P}_{\mathcal{K}}}^* x)]\}).$$

for all $x, y \in \mathcal{K}, {\mathcal{P}_{\mathcal{K}}}^* x \neq {\mathcal{P}_{\mathcal{K}}}^* y$, $(x, y) \in \mathcal{R}$ *and* $\tau > 0$ *for some* $\mathcal{F} \in \mathfrak{F}$.

Then there exists a unique element $x \in \mathcal{K}$ *such that* $\langle \mathcal{Q}x, y - x\rangle \geq 0$ *for all* $y \in \mathcal{K}$.

Proof: Define a binary relation $\mathcal{R} = \{(\mathcal{P}_{\mathcal{K}}(x), \mathcal{P}_{\mathcal{K}}(x)) : \mathcal{P}_{\mathcal{K}}(x) \leq \mathcal{P}_{\mathcal{K}}(x)$ with $\mathcal{P}_{\mathcal{K}}(x)\mathcal{P}_{\mathcal{K}}(x) \geq 0$ for all $x \in \mathcal{H}\}$ and $\mathcal{T} : \mathcal{K} \to \mathcal{K}$ by $\mathcal{T}x = \mathcal{P}_{\mathcal{K}}(x - \lambda \mathcal{Q}x)$ for all $x \in \mathcal{K}$, then $\mathcal{T}$ satisfies all the hypothesis of Corollary 3.3A and therefore $\mathcal{T}$ has a fixed point x. Hence by Lemma 4.2, $x \in \mathcal{K}$ is a solution of $\langle \mathcal{Q}x, y - x\rangle \geq 0$ for all $y \in \mathcal{K}$ if and only if x is a fixed point of $\mathcal{T}$. □

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References

1. Alam, A., Imdad, M., *Relation-theoretic contraction principle*, J. Fixed Point Theory Appl. 17(4), 693-702, (2015).
2. Alam, A., Imdad, M. *Relation-theoretic metrical coincidence theorems*, Filomat 31, 4421–4439, (2017).
3. Alam, A., Imdad, M., *Monotone generalized contractions in ordered metric spaces*, Bull. Korean Math. Soc. 53(1), 61-81, (2016).
4. Alam, A., Khan, A.R., Imdad, M., *Some coincidence theorems for generalized nonlinear contractions in ordered metric spaces with applications*, Fixed Point Theory Appl. 216, (2014).
5. Alam, A., George, R., Imdad, M., *Refinements to Relation-Theoretic Contraction Principle*, Axioms 11, 316, (2022).
6. Altun, I., Aslantas, M., Sahin, H., *KW-type nonlinear contractions and their best proximity points*, Numer. Funct. Anal. Optim. 24, 1-20, (2021).
7. Altun, I., Aslantas, M., Sahin, H., *Best proximity point results for p-proximal contractions*, Acta. Math. Hung. 162, 393-402, (2020).
8. Aslantas, M., Sahin, H., Altun, I., *Best proximity point theorems for cyclic p-contractions with some consequences and applications*, Nonlinear Anal. Model Control 26(1), 113-129, (2021).
9. Basha, S.S., Veeramani, P., *Best proximity pair theorems for multifunctions with open fibres*, J. Approx. Theory 103, 119–129, (2000).
10. Basha, S.S., *Extensions of Banach's contraction principle*, Numer. Funct. Anal. Optim. 31, 569-576, (2010).
11. Basha, S.S., *Best proximity points: optimal solutions*, J. Optim. Theory Appl. 151, 210-216, (2011).
12. Beg, I., Bartwal, A., Rawat, S., Dimri, R.C., *Best proximity points in noncommutative Banach spaces*, Comp. Appl. Math. 41, 41 (2022).
13. Debnath, P., Konwar, N., Radenovic, S., *Metric fixed point theory: Applications in science, engineering and behavioural sciences*, 314-316, (2021).
14. Dhivya, P., Radenović, S., Marudai, M., Bin-Mohsin, B., *Coupled fixed point and best proximity point results involving simulation functions*, Bol. Soc. Paran. Mat. 22, 1–15, (2004).
15. Eldered, A.A., Veeramani, P., *Existence and convergence for best proximity points*, J. Math. Anal. Appl. 33, 1001-1006, (2004).
16. Golubović, Z., Kadelburg, Z., Radenović, S., *Coupled coincidence points of mappings in ordered partial metric spaces*, Abstr. Appl. Anal. Article ID 192581, (2012).
17. Harjani, J., Sadarangani, K., *Fixed point theorems for weakly contractive mappings in partially ordered sets*, Nonlinear Anal. 71, 3403-3410, (2009).
18. Jachymski, J., *The contraction principle for mappings on a metric space with a graph*, Proc. Am. Math. Soc. 136(4), 1359-1373, (2008).
19. Karapınar, E., Roldán, A., Shahzad, N., Sintunavarat, W., *Discussion of coupled and tripled coincidence point theorems for ϕ-contractive mappings without the mixed g-monotone property*, Fixed Point Theory Appl. 2014(1), 1-16, (2014).
20. Lipschitz, S., *Schaum's Outlines of Theory and Problems of Set Theory and Related Topics*, McGraw-Hill, New York, (1964).
21. Raj, V.S. *A best proximity point theorem for weakly contractive non-self mappings*, Nonlinear Anal. 74(11), 4804–4808, (2011).
22. Raj, V.S., *Best proximity point theorems for nonself mappings*, Fixed Point Theory 14(2), 447-454, (2013).
23. Rawat, S., Dimri, R.C., Bartwal, A., *F-Bipolar metric spaces and fixed point theorems with applications*, J. Math. Comput. SCI-JM. 26(2), 184-195, (2022).
24. Rawat, S., Kukreti, S., Dimri, R.C., *Fixed point results for enriched ordered contractions in noncommutative Banach spaces*, J. Anal. 30, 1555-1566, (2022).
25. Sahin, H., Aslantas, M., Altun, I., *Feng-Liu type approach to best proximity point results for multivalued mappings*, J. Fixed Point Theory Appl. 22(1), 1-13, (2020).
26. Sultana, A., Vetrivel, V., *On the existence of best proximity points for generalized contractions*, Appl. Gen. Topol. 15(1), 55-63, (2014).
27. Samet, B., Turinici, M., *Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications*, Commun. Math. Anal. 13(2), 82-97, (2012).
28. Senapati, T., Dey, L.K., *Relation-theoretic metrical fixed point results via w-distance with applications*, J. Fixed Point Theory Appl. 19, 2945-2961, (2017).
29. Turinici, M., *Abstract comparison principles and multi-variable Gronwall-Bellman inequalities*, J. Math. Anal. Appl. 117(1), 100-127, (1986).

30. Turinici, M., *Contractive operators in relational metric spaces, Handbook of Functional Equations, Optimization and Its Appl.*, Springer, 95, 419-458, (2014).

31. Turinici, M., *Fixed points for monotone iteratively local contractions*, Dem. Math. 19(1), 171-180, (1986).

32. Turinici, M., *Nieto-López theorems in ordered metric spaces*, Math. Student 81(4), 219-229, (2012).

33. Turinici, M., *Ran-Reuring's theorems in ordered metric spaces*, J. Indian Math. Soc. 78, 207-214, (2011).

34. Wardowski, D., *Fixed points of a new type of contractive mappings in complete metric spaces*, Fixed Point Theory Appl. 2012, 94, (2012).

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