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Infinitely many solutions for a class of fractional equations via variant fountain theorems

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ABSTRACT: This article is concerned with a class of fractional type equation involving an anisotropic operator and potential of the form

$$(-\Delta_x)^s u - \Delta_y u + \Phi(x,y)u = g(x,y,u),\ (x,y) \in \mathbb{R}^n \times \mathbb{R}^m.$$

By means of the variational method and the variant fountain theorems, we investigate the existence of infinitely many high or small energy solutions without the usual assumption of coerciveness on the potential Φ in the different cases when the nonlinear term is either asymptotically linear or superquadratic growth.

Key Words: Potential BO-ZK equation, infinitely many solutions, variational method.

Contents

1. Introduction and main results

In recent years, a special attention has been focused on the study of the fractional partial differential equations involving fractional and nonlocal operators (isotropic & anisotropic) due to its significant applications in the modeling of many phenomena in various fields of science and engineering, such as the thin obstacle problem, optimization, phase transitions, finance, stratified materials, crystal dislocation, anomalous diffusion, soft thin films, semipermeable membranes, flame propagation and ultra-relativistic limits of quantum mechanics, we refer to [1,8,9,17]. Recently, from the nonlinear analysis point of view special attention has been paid to the study of equations involing Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) fractional anisotropic operator $(-\Delta_x)^s - \Delta_y$, we refer to [12,13,18,19]. For example, [18] M. Massar, by applying a version of the symmetric mountain pass theorem and using the variational method he proved the existence of infinitely many radial solutions for a class fractional Kirchhoff-type equation. In [19] based on critical point and by exploiting some original arguments developed in [3,2,4], the existence of infinitely many solutions for the following fractional Kirchhoff-type equation

$$(1 + b[u]_s^2)((-\Delta_x)^s u - \Delta_y u) + V(x,y)u = f(x,y,u),\ (x,y) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m, \tag{1.1}$$

where $b > 0$, $[u]_s^2 = \int_{\mathbb{R}^N} |(-\Delta_x)^{\frac{s}{2}} u|^2 + |\nabla_y u|^2 dxdy$ and $V : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a potential function, was discussed.

In this article, we aim to discuss the existence of infinitely many solutions for the following fractional equation involving (BO-ZK) operator

$$(-\Delta_x)^s u - \Delta_y u + \Phi(x,y)u = g(x,y,u),\ (x,y) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^m, \tag{1.2}$$

where $n, m \geq 1$, $s \in (0,1)$, $-\Delta_y$ denotes the usual Laplacian in the $y-$ variable, $(-\Delta_x)^s$ denotes the fractional Laplacian in the $x-$ variable which is defined via the Fourier transform by $\mathcal{F}\left[(-\Delta_x)^s u\right](\xi,\eta) =$

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$|\xi|^{2s}\mathcal{F}[u](\xi,\eta)$, where $\mathcal{F}[u]$ is the Fourier transform of u; if u is smooth enough and rapidly decreasing, it can be expressed by

$$(-\Delta_x)^s u(x,y) = C(s,n,m) \lim_{\epsilon\to 0^+} \int_{\mathbb{R}^n\setminus B_\epsilon[x]} \frac{u(x,y)-u(z,y)}{|x-z|^{n+2s}}dz, \text{ for all } (x,y)\in\mathbb{R}^n\times\mathbb{R}^m,$$

with $B_\epsilon[x]$ is the $\epsilon-$ ball in $\mathbb{R}^n$ centered at x and $C(s,n,m)$ is a normalization positive constant, $\Phi:\mathbb{R}^N\to(0,+\infty)$ is a continuous potential function and $f\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$ is a nonlinear function.

When $\Phi\equiv 1$ and $g(x,y,u)=u^p$, Eq. (1.2) goes back to the study of solitary waves of the generalized Benjamin-Ono-Zakharov-Kuznetsov (BO-ZK) equation

$$u_t + \partial_{x_1}\left((-\Delta_x)^s u - \Delta_y u + u^p\right) = 0, \quad (x,y)\in\mathbb{R}^n\times\mathbb{R}^m, \tag{1.3}$$

see [20] for some local and global well-posedness results on this equation. In the cases $n=m=1$ and $s=\frac{1}{2}$, Eq.(1.3) reduces to the equation

$$u_t - \mathcal{H}_x u_{xx} + u_{yyx} + (u^p)_x = 0, \quad (x,y)\in\mathbb{R}^2, \tag{1.4}$$

where $\mathcal{H}_x$ denotes the Hilbert transform in x which is defined by

$$\mathcal{H}_x u(x,y,t) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{u(z,y,t)}{x-z}dz, \quad \text{for all } (x,y,t)\in\mathbb{R}^2\times\mathbb{R}^+,$$

here $P.V.$ is a commonly used abbreviation in the Cauchy principal value. Eq.(1.4) was introduced by Jorge et al. [15] and Latorre et al. [16] appears in the study of electromigration in thin nanoconductors on a dielectric substrate. The (BO-ZK) operator also has been used by some researchers in [6,5,10] for studying of toy models parabolic equations for which local diffusions occur only in certain directions and nonlocal diffusions. We refer the reader to [14,7] for more information and properties (regularity & rigidity) of this operator.

The main mathematical difficulty to study our nonlocal problem arises from the lack of compactness in critical Sobolev embedding. In order to overcome this difficulty of the noncompact embedding, Esfahani [13] has established a new compact embedding theorem for the subspace of the fractional Sobolev-Liouville space $\mathcal{H}^s(\mathbb{R}^N)$. Furthermore, when $g(x,y,u)=f(u)$, the author has obtained the existence of positive and sign-changing solutions of equation (1.2) by using the variational method combined with a variant of deformation lemma and some original arguments in [2,3,4] provided coerciveness on the potential V and $\inf V(x,y)>0$ and suitable assumptions on the nonlinearity f.

We will prove in the very article the existence of infinitely many high or small energy of equation (1.2) by using the variational method and two variant fountain theorems established in [22], without the following usual assumption of coerciveness on the potential Φ :
There exists $r>0$, such that

$$\lim_{|z|\to+\infty} meas\left(\{(x,y)\in B_r(z) : \Phi(x,y)\leq M\}\right) = 0 \text{ for all } M>0,$$

where $meas$ denotes the Lebesgue measure on $\mathbb{R}^N$ and $B_r(z)$ is the r-ball in $\mathbb{R}^N$ of centre z and radius r, (see [13]).

In order to state our results, we need to make some hypotheses on Φ and g. For the potential Φ, we assume that

(Φ_1) $\Phi\in C(\mathbb{R}^N,(0,+\infty))$ and $\Phi_0 := \inf_{(x,y)\in\mathbb{R}^N}\Phi(x,y) > 0.$

For the nonlinearity g, we will consider two cases.
Asymptotically linear case:

(g_0) $g\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, $g(x,y,t)t\geq 0$ for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}$;

(g_1) $\lim_{t\to 0}\dfrac{g(x,y,t)}{t}=0$ uniformly in $(x,y)\in\mathbb{R}^N$;

(g_2) There exists a positive function $\alpha\in L^{\frac{2}{2-q}}(\mathbb{R}^N)$ with $q\in(1,2)$ such that

$$|g(x,y,t)|\le\alpha(x,y)(1+|t|^{q-1})\text{ for all }(x,y,t)\in\mathbb{R}^N\times\mathbb{R};$$

(g_3) $\lim_{t\to+\infty}\dfrac{G(x,y,t)}{|t|^\theta}\ge\delta>0$ uniformly in $(x,y)\in\mathbb{R}^N$, for some $\theta\in[1,q)$, where $G(x,y,t)=\int_0^t g(x,y,\tau)d\tau$.

Superquadratic case:

(g_0) $g\in C(\mathbb{R}^N\times\mathbb{R},\mathbb{R})$, $g(x,y,t)t\ge 0$ for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}$;

(g_1) $\lim_{t\to 0}\dfrac{g(x,y,t)}{t}=0$ uniformly in $(x,y)\in\mathbb{R}^N$;

(G_2) There exist $p\in(2,p_s^*)$ and a positive function $\beta\in L^\nu(\mathbb{R}^N)$ with $p_s^*:=\frac{2(n+ms)}{n+(m-2)s}$ and $\nu>\frac{2(n+ms)}{2(n+ms)-p(n+(m-2)s)}$ such that

$$|g(x,y,t)|\le\beta(x,y)(1+|t|^{p-1})\text{ for all }(x,y,t)\in\mathbb{R}^N\times\mathbb{R};$$

(G_3) $\lim_{t\to+\infty}\dfrac{G(x,y,t)}{|t|^2}=+\infty$ uniformly in $(x,y)\in\mathbb{R}^N$;

(G_4) There exists $\mu\ge 1$ such that $\mathcal{G}(x,y,rt)\le\mu\mathcal{G}(x,y,t)$ for all $(x,y,t)\in\mathbb{R}^N\times\mathbb{R}$ and $r\in[0,1]$, where $\mathcal{G}(x,y,t)=tg(x,y,t)-2G(x,y,t)$.

Due to the dependence of potential, let us recall our suitable working Sobolev space. Let $\mathcal{H}^s(\mathbb{R}^N)$ be the fractional Sobolev-Liouville space

$$\mathcal{H}^s(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N):\iint_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{s}{2}}u|^2+|\nabla_y u|^2+u^2\right)dxdy<\infty\right\},$$

with the norm

$$\|u\|_{\mathcal{H}^s(\mathbb{R}^N)}=\left(\iint_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{s}{2}}u|^2+|\nabla_y u|^2+u^2\right)dxdy\right)^{\frac{1}{2}}.$$

The homogeneous fractional Sobolev-Liouville space $\mathcal{D}^s(\mathbb{R}^N)$ is defined by

$$\mathcal{D}^s(\mathbb{R}^N)=\left\{u\in L^2(\mathbb{R}^N):[u]_s<\infty\right\},$$

where

$$[u]_s=\left(\iint_{\mathbb{R}^N}\left(|(-\Delta_x)^{\frac{s}{2}}u|^2+|\nabla_y u|^2\right)dxdy\right)^{\frac{1}{2}}.$$

We search the solutions in the following subspace

$$X_s=\left\{u\in\mathcal{D}^s(\mathbb{R}^N):\iint_{\mathbb{R}^N}\Phi(x,y)u^2dxdy<\infty\right\},$$

equipped with the inner product

$$(u,v)_{X_s}=\iint_{\mathbb{R}^N}\left((-\Delta_x)^{\frac{s}{2}}u(-\Delta_x)^{\frac{s}{2}}v+\nabla_y u\nabla_y v+\Phi(x,y)uv\right)dxdy,$$

which induces the norm

$$\|u\|_{X_s}=(u,u)_{X_s}^{\frac{1}{2}}.$$

Now, we give the sense in which we will take a solution to equation (1.2).

Definition 1.1 *A function $u \in X_s$ is called weak solution of (1.2) if*

$$\iint_{\mathbb{R}^N} \left((-\Delta_x)^{\frac{s}{2}} u(-\Delta_x)^{\frac{s}{2}}\psi + \nabla_y u \nabla_y \psi + \Phi(x,y)u\psi\right) dxdy = \iint_{\mathbb{R}^N} g(x,y,u)\psi dxdy, \text{ for all } \psi \in X_s.$$

The main results can be described by the following two theorems.

Theorem 1.1 (Asymptotically linear) *Assume that (Φ_1) and $(g_0)-(g_3)$ hold, and g is odd in t. Then equation (1.2) admits infinitely many solutions $\{u_k\}_1^\infty$ verifying*

$$\frac{1}{2}\iint_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}} u_k|^2 + |\nabla_y u_k|^2 + \Phi(x,y)u_k^2\right) dxdy - \iint_{\mathbb{R}^N} G(x,y,u_k)dxdy \to 0^- \text{ as } k \to +\infty.$$

Theorem 1.2 (Superquadratic) *Assume that (Φ_1), (g_0), (g_1) and $(G_2)-(G_4)$ hold, and g is odd in t. Then equation (1.2) admits infinitely many solutions $\{u_k\}_1^\infty$ verifying*

$$\frac{1}{2}\iint_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}} u_k|^2 + |\nabla_y u_k|^2 + \Phi(x,y)u_k^2\right) dxdy - \iint_{\mathbb{R}^N} G(x,y,u_k)dxdy \to +\infty \text{ as } k \to +\infty.$$

2. Auxiliary Results

In this section, we give some preliminary results for the proof of Theorems 1.1 and 1.2.

Lemma 2.1 ([11]) *The fractional Sobolev-Liouville space $\mathcal{H}^s(\mathbb{R}^N)$ is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, p_s^*]$; and is compactly embedded into $L^q(\mathbb{R}^N)$ for $q \in (2, p_s^*)$.*

Remark 2.1 *In view of (Φ_1) and the previous lemma, the space X_s is continuously embedded into $L^q(\mathbb{R}^N)$ for $q \in [2, p_s^*]$; and is compactly embedded into $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, p_s^*)$.*

To obtain infinitely many solutions, we will employ two variant Fountain theorems [22]. So, we first recall some basic notations.

Let $(E, \|\cdot\|)$ be a Banach space and $\{E_j\}$ be a sequence of the finite dimensional subspaces of E such that $E = \overline{\oplus_{j=0}^\infty E_j}$. Set

$$Y_k = \oplus_{j=0}^k E_j \text{ and } Z_k = \overline{\oplus_{j=k}^\infty E_j}.$$

Let $\{\varphi_\lambda : E \to \mathbb{R},\ \lambda \in [1,2]\}$ be a family of C^1 functionals such that $\varphi_\lambda(u) = A(u) - \lambda B(u)$ and $\varphi_\lambda(-u) = \varphi_\lambda(u)$, where $A, B : E \to \mathbb{R}$ are two functionals.

Theorem 2.1 *Assume that φ_λ satisfies the following assertions:*
(i) φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$;
(ii) $B(u) \geq 0$ and $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E;
(iii) There exist $\rho_k > r_k > 0$ such that

$$\alpha_k(\lambda) := \inf_{u \in Z_k, \|u\| = \rho_k} \varphi_\lambda(u) \geq 0 > \beta_k(\lambda) := \max_{u \in Y_k, \|u\| = r_k} \varphi_\lambda(u) \text{ for all } \lambda \in [1,2],$$

$$\eta_k(\lambda) := \inf_{u \in Z_k, \|u\| \leq \rho_k} \varphi_\lambda(u) \to 0 \text{ as } k \to +\infty \text{ uniformly for } \lambda \in [1,2].$$

Then, there exists $\lambda_n \to 1$ and $u_{\lambda_n} \in Y_n$ such that

$$\varphi'_{\lambda_n}|_{Y_n}(u_{\lambda_n}) = 0,\ \varphi_{\lambda_n}(u_{\lambda_n}) \to \sigma_k \in [\eta_k(2), \beta_k(1)] \text{ as } n \to \infty.$$

In particular, if $\{u_{\lambda_n}\}$ has a convergent subsequence for any k, then φ_1 has infinitely many nontrivial critical points $\{u_k\}_{k=1}^\infty \in E \setminus \{0\}$ satisfying $\varphi_1(u_k) \to 0^-$ as $k \to \infty$.

Theorem 2.2 *Assume that φ_λ satisfies the following assertions:*
(i) φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$;
(ii) $B(u) \geq 0$ and either $B(u) \to \infty$ or $A(u) \to \infty$ as $\|u\| \to \infty$ (or $B(u) \leq 0$ and $B(u) \to -\infty$ as

$\|u\| \to \infty$);
(iii) There exist $\rho_k > r_k > 0$ such that

$$\beta_k(\lambda) := \inf_{u\in Z_k, \|u\|=r_k} \varphi_\lambda(u) > \alpha_k(\lambda) := \max_{u\in Y_k, \|u\|=\rho_k} \varphi_\lambda(u) \text{ for all } \lambda \in [1,2].$$

Then

$$\beta_k(\lambda) \le \sigma_k(\lambda) := \inf_{\gamma\in\Gamma_k} \max_{u\in B_k} \varphi_\lambda(\gamma(u)) \text{ for all } \lambda \in [1,2],$$

where $\Gamma_k = \{\gamma \in C(B_k, X) : \gamma \text{ is odd }, \gamma|_{\partial B_k} = id\} (k \ge 2)$ and $B_k = \{u \in Y_k : \|u\| \le \rho_k\}$. Moreover, for a.e. $\lambda \in [1,2]$, there exist a sequence $\{u_n^k(\lambda)\}_{n=1}^\infty$ such that

$$\sup_n \|u_n^k(\lambda)\| < \infty, \ \ \varphi'_\lambda(u_n^k(\lambda)) \to 0 \text{ and } \varphi_\lambda(u_n^k(\lambda)) \to \sigma_k(\lambda) \text{ as } n \to \infty.$$

Obviously, the energy functional associated to (1.2) defined on X_s by

$$I(u) = \frac{1}{2}\iint_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}}u|^2 + |\nabla_y u|^2 + \Phi(x,y)u^2\right) dxdy - \iint_{\mathbb{R}^N} G(x,y,u)dxdy,$$

is of class C^1 and its critical points are solutions of (1.2). To apply Theorems 2.1 and 2.2, we will consider the perturbed functional

$$I_\lambda(u) = A(u) - \lambda B(u),$$

where

$$A(u) = \frac{1}{2}\iint_{\mathbb{R}^N} \left(|(-\Delta_x)^{\frac{s}{2}}u|^2 + |\nabla_y u|^2 + \Phi(x,y)u^2\right) dxdy$$

and

$$B(u) = \iint_{\mathbb{R}^N} G(x,y,u)dxdy.$$

Since X_s is reflexive and separable, there exist $\{e_j\}_{j=1}^\infty \subset X_s$ and $\{e_j^*\}_{j=1}^\infty \subset X_s^*$ such that

$$X_s = \overline{span\{e_j : j = 1, 2 \dots\}}, \quad X_s^* = \overline{span\{e_j^* : j = 1, 2 \dots\}}$$

and

$$\langle e_i^*, e_j\rangle = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \ne j. \end{cases}$$

For convenience, we set $X_{s_j} = span\{e_j\}$, $Y_k = \oplus_{j=1}^k X_{s_j}$ and $Z_k = \overline{\oplus_{j=k}^\infty X_{s_j}}$.

3. Proof of main theorems

Proof of Theorem 1.1. Clearly, for any $\lambda \in [1,2]$, I_λ is an even functional on X_s. By $(g_0) - (g_3)$, we see that I_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1,2]$. Evidently, $I_\lambda(-u) = I_\lambda(u)$ for all $(\lambda, u) \in [1,2] \times X_s$. Similar to [21], one can show that B is nonnegative and $B(u) \to \infty$ as $\|u\|_{X_s} \to \infty$ on any finite dimensional subspace of X_s, and Moreover, there exist real sequences $0 < r_k < \rho_k$ such that

$$\alpha_k(\lambda) := \inf_{u\in Z_k, \|u\|_{X_s}=\rho_k} I_\lambda(u) \ge 0, \quad \beta_k(\lambda) =: \max_{u\in Y_k, \|u\|_{X_s}=r_k} I_\lambda(u) < 0$$

and

$$\eta_k(\lambda) := \inf_{u\in Z_k, \|u\|_{X_s}\le\rho_k} I_\lambda(u) \to 0 \text{ as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

From Theorem 2.1, there exists $\lambda_n \to 1$ and $u_n = u_{\lambda_n} \in Y_n$ such that

$$I'_{\lambda_n}|_{Y_n}(u_n) = 0, \ I_n(u_n) \to \sigma_k \in [\eta_k(2), \beta_k(1)] \text{ as } n \to \infty.$$

We split the rest of proof in two step.
Step 1: $\{u_n\}$ is bounded. By $(g_0) - (g_2)$, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|g(x,y,t)| \le \varepsilon|t| + C_\varepsilon \alpha(x,y)|t|^{q-1} \text{ for all } (x,y,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{3.1}$$

Therefore, by (3.1), (Φ_1), the Sobolev embedding and the Hölder inequality, we have

$$\begin{aligned}\|u_n\|^2_{X_s} &= 2I_{\lambda_n}(u_n) + 2\lambda_n \iint_{\mathbb{R}^N} G(x,y,u_n)dxdy\\ &\leq C_1 + 4\varepsilon\|u_n\|^2_{L^2(\mathbb{R}^N)} + 4C_\varepsilon \iint_{\mathbb{R}^N} \alpha(x,y)|u_n|^q dxdy\\ &\leq C_1 + 4\varepsilon C_2^2\|u_n\|^2_{X_s} + 4C_\varepsilon \Phi_0^{-\frac{q}{2}}\|\alpha\|_{L^{\frac{2}{2-q}}(\mathbb{R}^N)}\left(\iint_{\mathbb{R}^N}\Phi(x,y)|u_n|^2dxdy\right)^{\frac{q}{2}}\\ &\leq C_1 + 4\varepsilon C_2^2\|u_n\|^2_{X_s} + 4C_\varepsilon \Phi_0^{-\frac{q}{2}}\|\alpha\|_{L^{\frac{2}{2-q}}(\mathbb{R}^N)}\|u_n\|^q_{X_s},\end{aligned}$$

which shows that $\{u_n\}$ is bounded in X_s, since $1 < q < 2$.

Step 2: $u_n \to u$ in X_s. Since $\{u_n\}$ is bounded X_s, up to a subsequence, $u_n \rightharpoonup u$ for some $u \in X_s$. Let us consider the orthogonal projection operator $P_n : X_s \to Y_n$. Then

$$I'_{\lambda_n}(u_n) = I'_{\lambda_n}|_{Y_n}(u_n) = 0.$$

Thus

$$\begin{aligned}\|u_n - P_n u\|^2_{X_s} &= \langle I'_{\lambda_n}(u_n), u_n - P_n u\rangle - \langle I'(P_n u), u_n - P_n u\rangle\\ &\quad + \lambda_n \iint_{\mathbb{R}^N} g(x,y,u_n)(u_n - P_n u)dxdy - \iint_{\mathbb{R}^N} g(x,y,P_n u)(u_n - u)dxdy\\ &= -\langle I'(P_n u), u_n - P_n u\rangle + \lambda_n \iint_{\mathbb{R}^N} g(x,y,u_n)(u_n - P_n u)dxdy\\ &\quad - \iint_{\mathbb{R}^N} g(x,y,P_n u)(u_n - u)dxdy. \end{aligned} \tag{3.2}$$

Using the fact that $\alpha \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$, we can find $R_\varepsilon > 0$ such that

$$\|\alpha\|_{L^{\frac{2}{2-q}}(B^c_{R_\varepsilon})} < \frac{\varepsilon}{C_\varepsilon}, \tag{3.3}$$

where $B^c_{R_\varepsilon} = \mathbb{R}^N \backslash B_{R_\varepsilon}$ and $B_{R_\varepsilon} = \{(x,y) \in \mathbb{R}^N : |(x,y)| \leq R_\varepsilon\}$.
According to Remark 2.1, we have $u_n \to u$ in $L^2(B_{R_\varepsilon})$, hence for n sufficiently large,

$$\|u_n - u\|_{L^2(B_{R_\varepsilon})} < \frac{\varepsilon}{C_\varepsilon\|\alpha\|_{L^{\frac{2}{2-q}}(\mathbb{R}^N)}}. \tag{3.4}$$

From (3.1), (3.3)-(3.4), the Hölder inequality and the Sobolev embedding, we entail

$$\begin{aligned}\iint_{\mathbb{R}^N}|g(x,y,P_n u)(u_n - u)|dxdy &\leq \varepsilon \iint_{\mathbb{R}^N}|P_n u|\ |u_n - u|dxdy + C_\varepsilon \iint_{B_{R_\varepsilon}} \alpha(x,y)|P_n u|^{q-1}|u_n - u|dxdy\\ &\quad + C_\varepsilon \iint_{B^c_{R_\varepsilon}} \alpha(x,y)|P_n u|^{q-1}|u_n - u|dxdy\\ &\leq \varepsilon\|P_n u\|_{L^2(\mathbb{R}^N)}\|u_n - u\|_{L^2(\mathbb{R}^N)}\\ &\quad + C_\varepsilon\|\alpha\|_{L^{\frac{2}{2-q}}(\mathbb{R}^N)}\|P_n u\|^{q-1}_{L^2(B_{R_\varepsilon})}\|u_n - u\|_{L^2(B_{R_\varepsilon})}\\ &\quad + C_\varepsilon\|\alpha\|_{L^{\frac{2}{2-q}}(B^c_{R_\varepsilon})}\|P_n u\|^{q-1}_{L^2(B^c_{R_\varepsilon})}\|u_n - u\|_{L^2(B^c_{R_\varepsilon})}\\ &\leq \varepsilon C_2\|P_n u\|_{X_s}\|u_n - u\|_{X_s} + \varepsilon C_3\|P_n u\|^{q-1}_{X_s} + \varepsilon C_4\|P_n u\|^{q-1}_{X_s}\|u_n - u\|_{X_s}\\ &\leq \varepsilon\left(C_2\|u\|_{X_s}\|u_n - u\|_{X_s} + C_3\|u\|^{q-1}_{X_s} + C_4\|u\|^{q-1}_{X_s}\|u_n - u\|_{X_s}\right). \end{aligned} \tag{3.5}$$

Since $\{u_n\}$ is bounded in X_s, from (3.5) we deduce that

$$\lim_{n\to+\infty}\iint_{\mathbb{R}^N} g(x,y,P_n u)(u_n - u)dxdy = 0. \tag{3.6}$$

Similarly, we can show that

$$\lim_{n\to+\infty}\lambda_n\iint_{\mathbb{R}^N} g(x,y,u_n)(u_n-P_nu)dxdy=0. \tag{3.7}$$

On the other hand, we have $P_nu\to u$ and $(u_n-P_nu)\rightharpoonup 0$ as $n\to+\infty$, thus

$$\lim_{n\to+\infty}\langle I'(P_nu),u_n-P_nu\rangle=0. \tag{3.8}$$

It holds from (3.2) and (3.6)-(3.8) that $(u_n-P_nu)\to 0$ in X_s. Hence $u_n\to u$ in X_s.

Applying Theorem 2.1, we conclude that $I=I_1$ has infinitely many nontrivial critical points, and consequently (1.2) admits infinitely many nontrivial solutions $\{u_k\}_{k=1}^{\infty}$ verifying $I(u_k)\to 0^-$ as $k\to\infty$. The proof of Theorem 1.1 is finished.

Proof of Theorem 1.2. As in above, under conditions of Theorem 1.2, all assertions given in Theorem 2.2 are satisfied. Then, for all most every $\lambda\in[1,2]$, there exists $\{u_n^k(\lambda)\}_{n=1}^{\infty}$ such that

$$\sup_n\|u_n^k(\lambda)\|_{X_s}<\infty,\ I'_\lambda(u_n^k(\lambda))\to 0,\ I_\lambda(u_n^k(\lambda))\to\sigma_k(\lambda)\text{ as }n\to\infty. \tag{3.9}$$

Let

$$b_k=\sup_{u\in Z_k,\|u\|_{X_s}=1}\iint_{\mathbb{R}^N}\beta(x,y)|u|^pdxdy.$$

Then the sequence $\{b_k\}_{k=1}^{\infty}$ is nonnegative and nonincreasing, thus $b_k\to b\geq 0$ as $k\to\infty$. By definition of b_k, for any $k=1,2,\ldots$, there is $u_k\in Z_k$ with $\|u_k\|_{X_s}=1$ such that

$$b_k-\frac{1}{k}\leq\iint_{\mathbb{R}^N}\beta(x,y)|u_k|^pdxdy. \tag{3.10}$$

For some $u\in X_s$, we have $u_k\rightharpoonup u$. Therefore

$$\langle e_j^*,u\rangle=\lim_{k\to+\infty}\langle e_j^*,u_k\rangle=0\text{ for }j=1,2,\ldots,$$

and hence $u=0$. By Remark 2.1, there exists $C_5>0$ such that

$$\|u\|_{L^{\frac{\nu p}{\nu-1}}(\mathbb{R}^N)}\leq C_5\|u\|_{X_s}.$$

Let $B_k=\{(x,y)\in\mathbb{R}^N:|(x,y)|<k\}$. Since $\beta\in L^\nu(\mathbb{R}^N)$, for $\varepsilon>0$, we can find $k_\varepsilon>0$ such that

$$\|\beta\|_{L^\nu(B^c_{k_\varepsilon})}\leq\frac{\varepsilon}{2C_5^p}.$$

Furthermore, we have $u_k\to 0$ in $L^{\frac{\nu p}{\nu-1}}(B_{k_\varepsilon})$, thus for some integer $\overline{k}_\varepsilon$ such that

$$\|u_k\|^p_{L^{\frac{\nu p}{\nu-1}}(B_{k_\varepsilon})}\leq\frac{\varepsilon}{2\|\beta\|_{L^\nu(\mathbb{R}^N)}}\text{ for all }k\geq\overline{k}_\varepsilon.$$

From the last two inequality, we obtain

$$\begin{aligned}\iint_{\mathbb{R}^N}\beta(x,y)|u_k|^pdxdy &= \iint_{B_{k_\varepsilon}}\beta(x,y)|u_k|^pdxdy+\iint_{B^c_{k_\varepsilon}}\beta(x,y)|u_k|^pdxdy\\ &\leq \|\beta\|_{L^\nu(\mathbb{R}^N)}\|u_k\|^p_{L^{\frac{\nu p}{\nu-1}}(B_{k_\varepsilon})}+\|\beta\|_{L^\nu(B^c_{k_\varepsilon})}\|u_k\|^p_{L^{\frac{\nu p}{\nu-1}}(\mathbb{R}^N)}\\ &\leq \varepsilon,\end{aligned}$$

which means that $\iint_{\mathbb{R}^N}\beta(x,y)|u_k|^pdxdy\to 0$ as $k\to\infty$. It follows then from (3.10) that $b_k\to 0$ as $k\to\infty$. By (g_0), (g_1) and (G_2), for any $\varepsilon>0$ there exists $C_\varepsilon>0$ such

$$|G(x,y,t)|\leq\varepsilon|t|^2+C_\varepsilon\beta(x,y)|t|^p\text{ for all }(x,y,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{3.11}$$

Then

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{2}\|u\|_{X_s}^2 - \lambda\varepsilon\|u\|_{L^2(\mathbb{R}^2)}^2 - \lambda C_\varepsilon \iint_{\mathbb{R}^N} \beta(x,y)|u|^p dxdy \\ &\geq \frac{1}{2}\|u\|_{X_s}^2 - \lambda\varepsilon C_6^2\|u\|_{X_s}^2 - \lambda C_\varepsilon \|u\|^p \\ &\geq \frac{1}{2}\|u\|_{X_s}^2 - 2\varepsilon C_6^2\|u\|_{X_s}^2 - 2C_\varepsilon \|u\|^p \\ &\geq \frac{1}{2}\|u\|_{X_s}^2 - 2\varepsilon C_6^2\|u\|_{X_s}^2 - 2C_\varepsilon b_k \|u\|_{X_s}^p, \end{aligned}$$

thus for $0 < \varepsilon \leq \frac{1}{8C_6^2}$ and $u \in Z_k$,

$$I_\lambda(u) \geq \frac{1}{4}\|u\|_{X_s}^2 - 2C_\varepsilon b_k \|u\|_{X_s}^p.$$

So, taking $r_k = (16C_\varepsilon b_k)^{\frac{1}{2-p}}$, we have

$$I_\lambda(u) \geq \frac{1}{8}(16C_\varepsilon b_k)^{\frac{2}{2-p}} > 0 \text{ for all } u \in Z_k \text{ with } \|u\|_{X_s} = r_k,$$

thus

$$\beta_k(\lambda) = \inf_{u\in Z_k, \|u\|=r_k} I_\lambda(u) \geq \frac{1}{8}(16C_\varepsilon b_k)^{\frac{2}{2-p}} \text{ for all } \lambda \in [1,2].$$

According to Theorem 2.2, we deduce that

$$\sigma_k(\lambda) = \inf_{\gamma\in\Gamma}\max_{u\in B_k} I_\lambda(\gamma(u)) \geq \beta_k(\lambda) \geq \overline{\beta}_k := \frac{1}{8}(16C_\varepsilon b_k)^{\frac{2}{2-p}} \to \infty \text{ as } k \to \infty.$$

Observe that

$$\sigma_k(\lambda) \leq \overline{\sigma}_k := \max_{u\in B_k} I_1(u),$$

therefore for any $k \geq k_0 > 0$,

$$\overline{\beta}_k \leq \sigma_k(\lambda) \leq \overline{\sigma}_k. \tag{3.12}$$

In view of Theorem 2.2, for a sequence $\lambda_m \to 1$, $\{u_n(\lambda_m)\}_{n=1}^\infty$ is bounded. Then following arguments in proof of Theorem 1.1, we see that up to a subsequence, for any $m \geq 1$ and $k \geq k_0$, $u_n(\lambda_m) \to u^k(\lambda_m)$ as $n \to \infty$. Hence, from (3.9) and (3.12), we get

$$I'_{\lambda_m}(u^k(\lambda_m)) = 0 \text{ and } \overline{\beta}_k \leq I_{\lambda_m}(u^k(\lambda_m)) \leq \overline{\sigma}_k \text{ for all } k \geq k_0. \tag{3.13}$$

Now, we claim that $\{u^k(\lambda_m)\}_{m=1}^\infty$ is bounded in X_s. In otherwise, we let $v_m = \frac{u^k(\lambda_m)}{\|u^k(\lambda_m)\|_{X_s}}$. Then, going to a subsequence, $v_m \rightharpoonup v$ in X_s, $v_m \to v$ in $L^r_{loc}(\mathbb{R}^N)$ for any $r \in [2, p^*)$ and $v_m(x) \to v(x)$ for a.e. $x \in \mathbb{R}^N$. If $v \equiv 0$, by (3.11) we obtain

$$\lim_{m\to+\infty} \iint_{\mathbb{R}^N} G(x,y,v_m)dxdy = 0. \tag{3.14}$$

Let $\{t_m\} \subset [0,1]$ such that

$$I_{\lambda_m}(t_m u^k(\lambda_m)) = \max_{t\in[0,1]} I_{\lambda_m}(tu^k(\lambda_m)).$$

For an arbitrary $R > 0$ and m sufficiently large, we have $\frac{R}{\|u^k(\lambda_m)\|_{X_s}} \in (0,1)$. Then

$$\begin{aligned} I_{\lambda_m}(t_m u^k(\lambda_m)) &\geq I_{\lambda_m}(Rv_m) \\ &= \frac{R^2}{2} - \lambda_m \iint_{\mathbb{R}^N} G(x,y,Rv_m)dxdy. \end{aligned}$$

Letting $m \to \infty$ and using (3.14), we obtain

$$\lim_{m\to\infty} I_{\lambda_m}(t_m u^k(\lambda_m)) = +\infty. \tag{3.15}$$

Note that

$$\langle I'_{\lambda_m}(t_m u^k(\lambda_m)), t_m u^k(\lambda_m)\rangle = 0.$$

Therefore, by (3.13) and (G_4)

$$\begin{aligned}
\frac{1}{\mu} I_{\lambda_m}(t_m u^k(\lambda_m)) &= \frac{1}{\mu}\left(I_{\lambda_m}(t_m u^k(\lambda_m)) - \frac{1}{2}\langle I'_{\lambda_m}(t_m u^k(\lambda_m)), t_m u^k(\lambda_m)\rangle\right)\\
&= \frac{\lambda_m}{2\mu}\iint_{\mathbb{R}^N}\left[t_m u^k(\lambda_m) g(x,y,t_m u^k(\lambda_m)) - 2G(x,y,t_m u^k(\lambda_m))\right] dxdy\\
&\le \frac{\lambda_m}{2}\iint_{\mathbb{R}^N}\left[u^k(\lambda_m) g(x,y,u^k(\lambda_m)) - 2G(x,y,u^k(\lambda_m))\right] dxdy\\
&= I_{\lambda_m}(u^k(\lambda_m)) - \frac{1}{2}\langle I'_{\lambda_m}(u^k(\lambda_m)), u^k(\lambda_m)\rangle\\
&= I_{\lambda_m}(u^k(\lambda_m))\\
&\le \overline{\sigma}_k \text{ for all } k \ge k_0.
\end{aligned}$$

This contradicts (3.15), and hence $v \not\equiv 0$. Let $\Omega_0 = \{(x,y) \in \mathbb{R}^N : v(x,y) \neq 0\}$. Dividing $I_{\lambda_m}(u^k(\lambda_m))$ by $\|u^k(\lambda_m)\|^2_{X_s}$, using (G_3) and Fatou's Lemma, we get

$$\begin{aligned}
\frac{1}{2} &= \frac{I_{\lambda_m}(u^k(\lambda_m))}{\|u^k(\lambda_m)\|^2_{X_s}} + \lambda_m \iint_{\mathbb{R}^N} \frac{G\left(x,y,u^k(\lambda_m)\right)}{\|u^k(\lambda_m)\|^2_{X_s}} dxdy\\
&= \frac{I_{\lambda_m}(u^k(\lambda_m))}{\|u^k(\lambda_m)\|^2_{X_s}} + \lambda_m \iint_{\Omega_0} \frac{G\left(x,y,u^k(\lambda_m)\right)}{|u^k(\lambda_m)|^2}|v_m(x,y)|^2 dxdy\\
&\ge \liminf_{m\to\infty} \iint_{\Omega_0} \frac{G\left(x,y,u^k(\lambda_m)\right)}{|u^k(\lambda_m)|^2}|v_m(x,y)|^2 dxdy\\
&\ge \iint_{\Omega_0} \liminf_{m\to\infty} \frac{G\left(x,y,u^k(\lambda_m)\right)}{|u^k(\lambda_m)|^2}|v_m(x,y)|^2 dxdy = +\infty.
\end{aligned}$$

This is impossible. Consequently $\{u^k(\lambda_m)\}$ is bounded. Then, as in the proof of Theorem 1.1, $\{u^k(\lambda_m)\}_{m=1}^\infty$ has a subsequence $u^k(\lambda_m) \to u^k$ in X_s for all $k \ge k_0$, and therefore u^k is a critical point of $I_1 = I$ with $\overline{\beta}_k \le I(u^k) \le \overline{\sigma}_k$. Thus problem (1.2) admits infinitely many solutions with high energy, since $\beta_k \to \infty$ as $k \to \infty$. The proof of Theorem 1.2 is finished.

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