Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–5.
ISSN-0037-8712
doi:10.5269/bspm.66433

A Remark on Autocommuting Probability of Finite Groups

Parama Dutta* and Rajat Kanti Nath

ABSTRACT: Let G be a finite group and let $\mathrm{Aut}(G)$ be its automorphism group. The autocommuting probability of G, denoted by $\Pr(G, \mathrm{Aut}(G))$, is the probability that a randomly chosen automorphism of G fixes a randomly chosen element of$G$. In this paper, we characterize all finite groups G such that $\Pr(G, \mathrm{Aut}(G)) = \frac{p+q-1}{pq}$, where p, q are the smallest primes dividing $|\mathrm{Aut}(G)|$, $|G|$ respectively. We shall also show that there is no finite group G such that $\Pr(G, \mathrm{Aut}(G)) = \frac{q^2+p-1}{pq^2}$, where p, q are primes as mentioned above.

Key Words: Automorphism group, autocommuting probability.

Contents

1. Introduction

Let G be a finite group and let $\mathrm{Aut}(G)$ be its automorphism group. Let $x \in G$ and $\alpha \in \mathrm{Aut}(G)$. The autocommutator of x and α, denoted by $[x, \alpha]$, is defined as $x^{-1}\alpha(x)$. We write $L(G) = \{x \in G : [x, \alpha] = 1 \text{ for all } \alpha \in \mathrm{Aut}(G)\}$ to denote the absolute center of G. The concepts of autocommutator and absolute center were introduced by Hegarty [8] in the year 1994. Let $\mathrm{orb}_G(x) = \{\alpha(x) : \alpha \in \mathrm{Aut}(G)\}$ and $\mathrm{orb}_G(G) = \{\mathrm{orb}_G(x) : x \in G\}$. Note that $\mathrm{orb}_G(x)$ is also known as fusion class of x in G. The autocommuting probability of G, denoted by $\Pr(G, \mathrm{Aut}(G))$, is the probability that the autocommutator of a randomly chosen pair of elements (x, α) is equal to the identity element of G. Thus

$$\Pr(G, \mathrm{Aut}(G)) = \frac{|\{(x, \alpha) \in G \times \mathrm{Aut}(G) : [x, \alpha] = 1\}|}{|G||\mathrm{Aut}(G)|}.$$

Autocommuting probability of G is also known as the probability that an automorphism of G fixes an element of G. Sherman [10] introduced this notion in the year 1975. Results on $\Pr(G, \mathrm{Aut}(G))$ and its generalizations can be found in [1,3,4,5,6,7]. In [3], it was shown that

$$\Pr(G, \mathrm{Aut}(G)) = \frac{|\mathrm{orb}_G(G)|}{|G|}. \tag{1.1}$$

Sherman [10] studied $\Pr(G, \mathrm{Aut}(G))$ for some classes of finite abelian groups. In [1], Arora and Karan have computed $\Pr(G, \mathrm{Aut}(G))$ for certain classes of finite groups and characterized finite abelian groups such that $\Pr(G, \mathrm{Aut}(G)) = \frac{2}{p}$ for any prime p (see [1, Theorem 5]). Arora and Karan [1], also mentioned that there is no finite group such that $\Pr(G, \mathrm{Aut}(G)) = \frac{1}{p}$, where p is the smallest prime dividing $|\mathrm{Aut}(G)|$. However, in 2020, Goyal, Kalra and Gumber [7] have computed $\Pr(G, \mathrm{Aut}(G))$ for certain classes of finite p-groups and characterized finite abelian groups such that $\Pr(G, \mathrm{Aut}(G)) = \frac{1}{p}$.

If p and q are the smallest primes dividing $|\mathrm{Aut}(G)|$ and $|G|$ respectively, then in [3, Corollary 3.5] it was shown that

$$\Pr(G, \mathrm{Aut}(G)) \leq \frac{p+q-1}{pq}. \tag{1.2}$$

* Corresponding author.
2010 *Mathematics Subject Classification:* 20D60, 20P05, 20F28.
Submitted December 24, 2022. Published October 30, 2025

Moreover, if G is non-abelian then by [3, Corollary 3.6], we have

$$\Pr(G, \mathrm{Aut}(G)) \leq \frac{q^2+p-1}{pq^2}. \tag{1.3}$$

In particular, if $p = q$ then $\Pr(G, \mathrm{Aut}(G)) \leq \frac{2p-1}{p^2} \leq \frac{3}{4}$ and $\Pr(G, \mathrm{Aut}(G)) \leq \frac{p^2+p-1}{p^3} \leq \frac{5}{8}$ (if G is non-abelian).

In [6], it was proved that if G is abelian then $\Pr(G, \mathrm{Aut}(G)) = \frac{3}{4}$ if and only if $G \cong \mathbb{Z}_4$, and $\frac{5}{8} < \Pr(G, \mathrm{Aut}(G)) < \frac{3}{4}$ if and only if $G \cong \mathbb{Z}_6$ or $\mathbb{Z}_3$. They also proved that $\frac{5}{8}$ can not be realize as autocommuting probability of finite groups. In view of these, following question arise naturally.

Question 1.1 *Can we characterize all finite groups such that equality holds in* (1.2) *and* (1.3)*?*

In this paper we answer the above question and show that $\Pr(G, \mathrm{Aut}(G)) = \frac{p+q-1}{pq}$ if and only if G is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_4$. We shall also show that there is no finite group G such that $\Pr(G, \mathrm{Aut}(G)) = \frac{q^2+p-1}{pq^2}$.

2. Main Results

In proving our main results, we shall need the following lemmas.

Lemma 2.1 [3, Theorem 4.1] *Let G be a finite group with* $\Pr(G, \mathrm{Aut}(G)) = \frac{p+q-1}{pq}$ *for some primes p and q. If p and q are the smallest primes dividing $|Aut(G)|$ and $|G|$ respectively, then $\frac{G}{L(G)} \cong \mathbb{Z}_q$.*

Lemma 2.2 [3, Theorem 4.2] *Let G be a finite non-abelian group such that* $\Pr(G, \mathrm{Aut}(G)) = \frac{q^2+p-1}{pq^2}$ *for some primes p and q. If p and q are the smallest primes dividing $|\mathrm{Aut}(G)|$ and $|G|$ respectively, then $\frac{G}{L(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$.*

Lemma 2.3 [2, Theorem 2.2] *A finite group G is cyclic if and only if $\frac{G}{L(G)}$ is cyclic. Moreover, if $\frac{G}{L(G)} \cong \mathbb{Z}_n$, then either $G \cong \mathbb{Z}_{2n}$, if n is even, or $G \cong \mathbb{Z}_n$ or $\mathbb{Z}_{2n}$ if n is odd.*

Lemma 2.4 [2, Theorem 3.1] *If G is a finite group such that $\frac{G}{L(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then G is isomorphic to one of the following groups*

1. $\mathbb{Z}_p \times \mathbb{Z}_p$
2. $\mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_2$ *(p is odd)*
3. $\mathbb{Z}_4 \times \mathbb{Z}_2$
4. D_8
5. Q_8
6. $\langle x, y : x^4 = y^4 = 1,\ y^{-1}xy = x^{-1}\rangle$.

Lemma 2.5 [1, Proposition 7] *Let G be a group of order p^2 (p-prime). Then* $\Pr(G, \mathrm{Aut}(G)) = \frac{k}{p^2}$*, where k is either* 2 *or* 3*.*

Lemma 2.6 [3, Proposition 2.7] *Let G and H be two finite groups such that $gcd(|G|, |H|) = 1$. Then*

$$\Pr(G \times H, \mathrm{Aut}(G \times H)) = \Pr(G, \mathrm{Aut}(G))\Pr(H, \mathrm{Aut}(H)).$$

Lemma 2.7 [9, Lemma 2.1] *Let G_1 and G_2 be two finite groups with relatively prime orders. Then*

$$\mathrm{Aut}(G_1 \times G_2) = \mathrm{Aut}(G_1) \times \mathrm{Aut}(G_2).$$

Now we prove the main results of this paper. We begin with the following characterization.

Theorem 2.1 *Let G be a finite group and let p, q be the smallest primes dividing $|\operatorname{Aut}(G)|$, $|G|$ respectively. Then $\Pr(G, \operatorname{Aut}(G)) = \frac{p+q-1}{pq}$ if and only if G is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_4$.*

Proof: If $\Pr(G, \operatorname{Aut}(G)) = \frac{p+q-1}{pq}$, where p and q are the smallest primes dividing $|\operatorname{Aut}(G)|$ and $|G|$ respectively, then by Lemma 2.1 we get $\frac{G}{L(G)} \cong \mathbb{Z}_q$. By Lemma 2.3, we have

$$G \cong \begin{cases} \mathbb{Z}_4, & \text{if } q = 2 \\ \mathbb{Z}_q \text{ or } \mathbb{Z}_{2q}, & \text{if } q \text{ is odd,} \end{cases}$$

noting that q is the smallest prime dividing $|G|$. If q is an odd prime and $G \cong \mathbb{Z}_q$ or $\mathbb{Z}_{2q}$ then, by Lemma 2.6, we have $\Pr(\mathbb{Z}_{2q}, \operatorname{Aut}(\mathbb{Z}_{2q})) = \frac{2}{q}$. Also $\Pr(\mathbb{Z}_q, \operatorname{Aut}(\mathbb{Z}_q)) = \frac{2}{q}$. Therefore

$$\frac{p+q-1}{pq} = \frac{2}{q},$$

which gives $p = 2$, $q = 3$ and so $G \cong \mathbb{Z}_3$. Thus G is isomorphic to $\mathbb{Z}_3$ or $\mathbb{Z}_4$.

Conversely, it is easy to see that if $G \cong \mathbb{Z}_3$ then $p = 2$, $q = 3$ and $\Pr(G, \operatorname{Aut}(G)) = \frac{2}{3} = \frac{p+q-1}{pq}$. Also, if $G \cong \mathbb{Z}_4$ then $p = 2 = q$ and $\Pr(G, \operatorname{Aut}(G)) = \frac{3}{4} = \frac{p+q-1}{pq}$. □

In [6, Theorem 1], Goyal, Kalra and Gumber proved that there is no finite group G such that $\Pr(G, \operatorname{Aut}(G)) = \frac{5}{8}$. In the following theorem we give more values that can not be realize as autocommuting probability of finite groups.

Theorem 2.2 *There is no finite group G such that*

$$\Pr(G, \operatorname{Aut}(G)) = \frac{q^2+p-1}{pq^2},$$

where p and q are the smallest primes dividing $|\operatorname{Aut}(G)|$ and $|G|$ respectively.

Proof: Let $\Pr(G, \operatorname{Aut}(G)) = \frac{q^2+p-1}{pq^2}$, where p and q are the smallest primes dividing $|\operatorname{Aut}(G)|$ and $|G|$ respectively. Then, by Lemma 2.2, we have $\frac{G}{L(G)} \cong \mathbb{Z}_q \times \mathbb{Z}_q$. Now, using Lemma 2.4, we get $G \cong \mathbb{Z}_q \times \mathbb{Z}_q$, $\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_2$ (q is odd), $\mathbb{Z}_4 \times \mathbb{Z}_2$, D_8, Q_8 or $\langle x, y : x^4 = y^4 = 1,\ y^{-1}xy = x^{-1}\rangle$.

If $G \cong \mathbb{Z}_q \times \mathbb{Z}_q$ then, by Lemma 2.5, we have $\Pr(G, \operatorname{Aut}(G)) = \frac{2}{q^2}$. We also have $|\operatorname{Aut}(\mathbb{Z}_q \times \mathbb{Z}_q)| = q^2(q-1)^2$. Therefore $p = 2$. Thus

$$\frac{2}{q^2} = \Pr(G, \operatorname{Aut}(G)) = \frac{q^2+p-1}{pq^2} = \frac{q^2+1}{2q^2} \tag{2.1}$$

which gives $q^2 = 3$, a contradiction since q is a prime integer.

If $G \cong \mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_2$ (q is odd) then, by Lemma 2.6, we have

$$\Pr(G, \operatorname{Aut}(G)) = \Pr(\mathbb{Z}_q \times \mathbb{Z}_q, \operatorname{Aut}(\mathbb{Z}_q \times \mathbb{Z}_q)) \Pr(\mathbb{Z}_2, \operatorname{Aut}(\mathbb{Z}_2)) = \frac{2}{q^2}.$$

By Lemma 2.7, we have

$$|\operatorname{Aut}(\mathbb{Z}_q \times \mathbb{Z}_q \times \mathbb{Z}_2)| = |\operatorname{Aut}(\mathbb{Z}_q \times \mathbb{Z}_q)||\operatorname{Aut}(\mathbb{Z}_2)| = q^2(q-1)^2.$$

Therefore $p = 2$. Thus we get (2.1) and eventually get a contradiction.

If $G \cong \mathbb{Z}_4 \times \mathbb{Z}_2$ then, by [1, Table 1], we have $\Pr(G, \operatorname{Aut}(G)) = \frac{1}{2}$. Thus

$$\frac{1}{2} = \Pr(G, \operatorname{Aut}(G)) = \frac{q^2+p-1}{pq^2} = \frac{3+p}{4p}. \tag{2.2}$$

Therefore $p = 3$, which is a contradiction since $|\operatorname{Aut}(\mathbb{Z}_4 \times \mathbb{Z}_2)| = 8$.

If $G \cong D_8$ then $q = 2$ and by [1, Example 2], we have $\Pr(G, \operatorname{Aut}(G)) = \frac{1}{2}$. Thus we get (2.2), which gives $p = 3$; a contradiction since $|\operatorname{Aut}(D_8)| = 8$.

If $G \cong Q_8$ then $q = 2$ and by [1, Example 3], we have

$$\frac{3}{8} = \Pr(G, \operatorname{Aut}(G)) = \frac{3+p}{4p}. \tag{2.3}$$

Therefore $p = 6$, which is a contradiction.

Let $G \cong \langle x, y : x^4 = y^4 = 1,\ y^{-1}xy = x^{-1}\rangle$. Then the automorphism α_i's of G are given by

$$\begin{array}{llll}
\alpha_0 : a \mapsto a & \alpha_1 : a \mapsto ab^2 & \alpha_2 : a \mapsto a^3 & \alpha_3 : a \mapsto a^3y^2 \\
\quad b \mapsto b, & \quad b \mapsto b, & \quad b \mapsto b, & \quad b \mapsto b, \\
\alpha_4 : a \mapsto a & \alpha_5 : a \mapsto ab^2 & \alpha_6 : a \mapsto a^3 & \alpha_7 : a \mapsto a^3b^2 \\
\quad b \mapsto b^3, & \quad b \mapsto b^3, & \quad b \mapsto b^3, & \quad b \mapsto b^3, \\
\alpha_8 : a \mapsto a & \alpha_9 : a \mapsto ab^2 & \alpha_{10} : a \mapsto a^3 & \alpha_{11} : a \mapsto a^3b^2 \\
\quad b \mapsto ab, & \quad b \mapsto ab, & \quad b \mapsto ab, & \quad b \mapsto ab, \\
\alpha_{12} : a \mapsto a & \alpha_{13} : a \mapsto ab^2 & \alpha_{14} : a \mapsto a^3 & \alpha_{15} : a \mapsto a^3b^2 \\
\quad b \mapsto ab^3, & \quad b \mapsto ab^3, & \quad b \mapsto ab^3, & \quad b \mapsto ab^3, \\
\alpha_{16} : a \mapsto a & \alpha_{17} : a \mapsto ab^2 & \alpha_{18} : a \mapsto a^3 & \alpha_{19} : a \mapsto a^3b^2 \\
\quad b \mapsto a^2b, & \quad b \mapsto a^2b, & \quad b \mapsto a^2b, & \quad b \mapsto a^2b,
\end{array}$$

$$\begin{array}{llll}
\alpha_{20} : a \mapsto a & \alpha_{21} : a \mapsto ab^2 & \alpha_{22} : a \mapsto a^3 & \alpha_{23} : a \mapsto a^3b^2 \\
\quad b \mapsto a^2b^3, & \quad b \mapsto a^2b^3, & \quad b \mapsto a^2b^3, & \quad b \mapsto a^2b^3. \\
\alpha_{24} : a \mapsto a & \alpha_{25} : a \mapsto ab^2 & \alpha_{26} : a \mapsto a^3 & \alpha_{27} : a \mapsto a^3b^2 \\
\quad b \mapsto a^3, & \quad b \mapsto a^3b, & \quad b \mapsto a^3b, & \quad b \mapsto a^3b. \\
\alpha_{28} : a \mapsto a & \alpha_{29} : a \mapsto ab^2 & \alpha_{30} : a \mapsto a^3 & \alpha_{31} : a \mapsto a^3b^2 \\
\quad b \mapsto a^3b^3, & \quad b \mapsto a^3b^3, & \quad b \mapsto a^3b^3, & \quad b \mapsto a^3b^3.
\end{array}$$

It can be seen that

$$\begin{aligned}
\operatorname{orb}_G(1) &= \{1\}, \\
\operatorname{orb}_G(x) &= \{x, x^3, xy^2, x^3y^2\}, \\
\operatorname{orb}_G(x^2) &= \{x^2\}, \\
\operatorname{orb}_G(y) &= \{y, y^3, xy, xy^3, x^2y, x^2y^3, x^3y, x^3y^3\}, \\
\operatorname{orb}_G(y^2) &= \{y^2\} \text{ and} \\
\operatorname{orb}_G(x^2y^2) &= \{x^2y^2\}
\end{aligned}$$

are the distinct orbits of G. Therefore $|\operatorname{orb}_G(G)| = 6$. Hence, from (1.1), $\Pr(G, \operatorname{Aut}(G)) = \frac{3}{8}$. Since $q = 2$ we get (2.3) which eventually gives a contradiction. This completes the proof. $\square$

Acknowledgment

The authors would like to thank Prof. P. Hegarty for his suggestions in proving the results of this paper. The first author is grateful to the Department of Mathematical Sciences of Tezpur University for its support while this investigation was carried out as a part of his Ph. D. Thesis.

References

1. Arora, H. and Karan, R. *What is the probability an automorphism fixes a group element?*. Commun. Algebra 45(3), 1141-1150, (2017).
2. Chaboksavar, M., Farrokhi, Derakhshandeh, Ghouchan, M., and Saeedi, F, *Finite groups with a given absolute central factor group*. Arch. Math. 102, 401-409, (2014).
3. Dutta, P. and Nath, R. K, *Autocommuting probabilty of a finite group*. Commun. Algebra 46(3), 961-969, (2018).

4. Dutta, P. and Nath, R. K, *On generalized autocommutativity degree of finite groups.* Hacet. J. Math. Stat. 48(2), 472-478, (2019).

5. Dutta, P. and Nath, R. K, *Generalized autocommuting probability of a finite group relative to its subgroups.* Hacet. J. Math. Stat., 49(1), 389-398, (2020).

6. Goyal, A., Kalra, H. and Gumber, D, *On the probability that an automorphism fixes a group element.* Am. Math. Mon. 126(8), 748-753, (2019).

7. Goyal, A., Kalra, H. and Gumber, D, *On the probability that an automorphism of a group fixes an element of the group.* J. Algebra Appl. 19(10), 2050198, (2020).

8. Hegarty, P, *The absolute centre of a group.* J. Algebra 169, 929-935, (1994).

9. Hillar, C. J. and Rhea, D. L, *Automorphism of finite abelian groups.* Am. Math. Mon. 114(10), 917-923, (2007).

10. Sherman, G. J, *What is the probability an automorphism fixes a group element?.* Am. Math. Mon. 82, 261-264, (1975).

Parama Dutta,
Department of Mathematics,
Lakhimpur Girls' College,
India.
E-mail address: parama@gonitsora.com

and

Rajat Kanti Nath,
Department of Mathematical Sciences,
Tezpur University,
India.
E-mail address: rajatkantinath@yahoo.com