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Parameters Correction of an Elliptical Equation Using an *a Posteriori* Estimate

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ABSTRACT: In nature several phenomena touch humanity to act and control these phenomena we try to model their evolution. To simplify the study of the equations obtained, most of the time we try to use linear forms and in this way modelling errors are made which can influence the correct analysis of these phenomena. In this work we focus on the Richards' equation, where the coefficients changes their forms from a given value h_s. This value is unknown then the coefficients are approximated. We prove an *a priori* and an *a posteriori* estimates on the modelling error. This estimates allows us, using local indicators, to build an adaptive algorithm to control the modelling error and automatically determine the "best" approximation of h_s. Numerical results confirm the convergence of this procedure and the interest of this approach.

Key Words: Richard's equation, modelling error, *A priori* estimates, *a posteriori* estimates, modelling indicators.

Contents

1. Introduction

Richards' equation which models the flow of water in a partially saturated porous medium, is given by

$$\frac{\partial\theta(h)}{\partial t} + S_e(\theta)\frac{\partial h}{\partial t} - div(K(h)(\nabla(h+z))) = f, \tag{1.1}$$

where $\theta(h)$ is the soil water retention, $K(h)$ the hydraulic conductivity, h the pression and f the source terme. The empirical expressions of $K(h)$ and $\theta(h)$ were introduced by Brooks and Correy [4] in 1964 and modified by Van Genuchten [9] in 1980. The shape of the soil water retention curve near saturation plays an important role in modelling runoff in the unsaturated saturated zone. Small changes in $\theta(h)$ can significantly affect $K(h)$ values, especially for fine-textured soils which may exhibit extreme non-linearity in $K(h)$ close to saturation. Vogel and co-authors [10] in 2001 suggested a modified Van Genuchten expression by introducing a new parameter h_s as the minimum capillary height instead of zero. Schaap and his co-authors [7] in 2006 gave a new expression and suggested three different forms of

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these functions related respectively to the saturated zone, the humid zone, called the capillary fringe, and the unsaturated zone. In (1.1) if we approximate the time derivative with the backward Euler scheme. The equation obtained is an elliptical partial differential equation which consists to find $\mathbf{c} \in H_0^1(\Omega)$ such that

$$-div(D(\mathbf{c})\nabla \mathbf{c}) + \mu(\mathbf{c})\mathbf{c} = f, \qquad \text{in } \Omega \tag{1.2}$$

where Ω is an open bounded set of $\mathbb{R}^d$, $d = 1, 2, 3$, with Lipschitz boundary, $f \in L^2(\Omega)$ and μ and D are functions which depend respectively on the functions θ and K and having a behaviour that changes depending on some parameters related to values of $\mathbf{c}$ solution of (1.2). These parameters are sometimes difficult to obtain exactly, users solve this problem with approximations $\tilde{\mu}$ and $\widetilde{D}$ of these functions.

In the context of the modelling error, we distinguish at least two types of approaches. The first one is when the model is simplified by neglecting some terms of the equation which are difficult to study. In this context we cite the work done by Stein and Ohnimus [8] in 1999. In [3] Braak and Ern in 2002 investigated the concept of dual-weight residuals to develop an *a posteriori* estimates for a non-linear elliptic problems, to simplify the study they neglected the non-linear terms then they studied the influence of this neglect on the solution. This analysis was extended by Perotto [6] for steady equation to the case of generic time-dependent problem. The second approach consists in simplifying the model by modifying the coefficients of the equation in order to omit some dependence relatively to unknown [2], [1] or to simplify the expression which is the situation of this work. The aim of our paper is to correct the model using model indicators resulting by developing an *a priori* and an *a posteriori* analysis of modelling error committed when the problem (1.2) is solved with the approximated coefficients $\widetilde{\mu}$ and $\widetilde{D}$. The outline of the paper is as follows.

- Section 2 : In this section we give the weak formulation of the problem and under some hypothesis, we show its well posedness by showing existence and uniqueness of the solution.
- Section 3 : In this section we derive an *a priori* and an *a posteriori* estimates of modelling error.
- In Section 4 : In this section we describe an adaptation strategy to approximate the coefficients μ and D, as an application of the indicators developed.
- In Section 5 : In this section we give two manufactured examples to validate the convergence of the algorithm introduced for adaptation.

2. Problem setting

Let $H_0^1(\Omega)$ be the usual Sobolev space of the first order defined by

$$H_0^1(\Omega) := \left\{ v \in L^2(\Omega) \,/\, \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1, \ldots, n, \text{ and } v|_{\partial\Omega} = 0 \right\},$$

equipped with the standard norm, $H^{-1}(\Omega)$ be its topological dual.
In the following we assume that, for all $(\mathbf{c}, v) \in (H_0^1(\Omega))^2$, functions $\mu(\cdot)$, $\widetilde{\mu}(\cdot)$, $D(\cdot)$ and $\widetilde{D}(\cdot)$ satisfying the following properties (we use the notation β for $\mu, \widetilde{\mu}, D$ or $\widetilde{D}$):

(H$_1$) $\exists \gamma > 0$ such that $\displaystyle\int_\Omega (\mu(\mathbf{c})\mathbf{c} - \mu(v)v)(\mathbf{c} - v)dx \geq \gamma \parallel \mathbf{c} - v \parallel_{0,\Omega}^2$,

(H$_2$) $\exists \gamma_1 > 0$ such that

$$\int_\Omega \Big(D(\mathbf{c})\nabla \mathbf{c} - D(v)\nabla v \Big) \cdot \nabla(\mathbf{c} - v) \geq \gamma_1 \parallel \nabla(\mathbf{c} - v) \parallel_{0,\Omega}^2,$$

(H$_3$) $\exists (\beta_1, \beta_2) \in \mathbb{R}^2$ such that $0 < \beta_1 \leq \beta(v) \leq \beta_2$,

(H$_4$) $\exists C_\beta > 0$ such that $\parallel \beta(\mathbf{c}) - \beta(v) \parallel_\infty \leq C_\beta \parallel \mathbf{c} - v \parallel_{1,\Omega}$.

We denote by $\mathbf{c}$ and $\widetilde{\mathbf{c}}$ the solutions of the following problems respectively

$$\left\{\begin{array}{l}\text{find } \mathbf{c} \in H_0^1(\Omega) \ \text{ such that} \\ a(\mu, D, \mathbf{c}, \mathbf{c}, v) = L(v), \qquad \forall v \in H_0^1(\Omega),\end{array}\right. \tag{2.1}$$

$$\left\{\begin{array}{l}\text{find } \widetilde{\mathbf{c}} \in H_0^1(\Omega) \ \text{ such that} \\ a(\widetilde{\mu}, \widetilde{D}, \widetilde{\mathbf{c}}, \widetilde{\mathbf{c}}, v) = L(v), \qquad \forall v \in H_0^1(\Omega),\end{array}\right. \tag{2.2}$$

where

$$a(\mu, D, w, \mathbf{c}, v) = \int_\Omega D(w)\nabla \mathbf{c} \cdot \nabla v dx + \int_\Omega \mu(w)\mathbf{c} v dx,$$

$$a(\widetilde{\mu}, \widetilde{D}, \widetilde{w}, \widetilde{\mathbf{c}}, v) = \int_\Omega \widetilde{D}(\widetilde{w})\nabla \widetilde{\mathbf{c}} \cdot \nabla v dx + \int_\Omega \widetilde{\mu}(\widetilde{w})\widetilde{\mathbf{c}} v dx,$$

$$L(v) = \int_\Omega f v dx.$$

Proposition 2.1. *Under hypothesis* $(\boldsymbol{H}_1) - (\boldsymbol{H}_4)$, *the problems (2.1) and (2.2) admit unique solutions,* $\mathbf{c} \in H_0^1(\Omega)$ *and* $\widetilde{\mathbf{c}} \in H_0^1(\Omega)$ *respectively, and we have the following estimates*

$$\|\mathbf{c}\|_{1,\Omega} \ \leq \ \delta(\mu_1, D_1, L), \qquad \|\widetilde{\mathbf{c}}\|_{1,\Omega} \ \leq \ \delta(\widetilde{\mu}_1, \widetilde{D}_1, L), \tag{2.3}$$

where $\delta(\mu_1, D_1, L) = \dfrac{\|L\|_{H^{-1}(\Omega)}}{\inf(\mu_1, D_1)}$, $\delta(\widetilde{\mu}_1, \widetilde{D}_1, L) = \dfrac{\|L\|_{H^{-1}(\Omega)}}{\inf(\widetilde{\mu}_1, \widetilde{D}_1)}$ *and the constants* μ_1, $\widetilde{\mu}_1$, D_1 *and* $\widetilde{D}_1$ *are given by the hypothesis* $(\boldsymbol{H}_3)$ *for* $\beta = \mu, \widetilde{\mu}, D, \widetilde{D}$.

Proof. The proof of the existence is classical and is based on the fixed point theorem of Schauder-Tychonoff [11], applied to the operator

$$T : z \in H_0^1(\Omega) \longrightarrow \mathbf{c}_z \in H_0^1(\Omega),$$

where $\mathbf{c}_z$ is the solution of the following linear problem

$$\left\{\begin{array}{l}\text{Find } \mathbf{c}_z \in H_0^1(\Omega) \text{ solution of} \\ a_z(\mu, D, \mathbf{c}_z, v) = L(v), \quad \forall v \in H_0^1(\Omega),\end{array}\right. \tag{2.4}$$

with $a_z(\mu, D, \mathbf{c}_z, v) = \displaystyle\int_\Omega D(z)\nabla \mathbf{c}_z \cdot \nabla v dx + \int_\Omega \mu(z)\mathbf{c}_z v dx.$

Using $(\mathbf{H}_3)$, the compactness of the injection of $H^1(\Omega)$ into $L^2(\Omega)$ and $(\mathbf{H}_4)$, we prove that T admits a fixed point, solution of the problem (2.1), the uniqueness is guaranteed by assumptions $(\mathbf{H}_1)$ $(\mathbf{H}_2)$. To have the estimate (2.3) we take $v = \widetilde{\mathbf{c}}$ in (2.2), $v = \mathbf{c}$ in (2.1) and use the hypothesis $(\mathbf{H}_3)$ □

3. Analysis and Approximation

In this section we derive an *a priori* and an *a posteriori* analysis of modelling error, when the problem (2.1) is replaced by the problem (2.2). The *a priori* estimate obtained is achieved without additional regularity assumption on $\mathbf{c}$ and gives a proof of the convergence of $\widetilde{\mathbf{c}}$ towards $\mathbf{c}$ when $\widetilde{\mu}$ tends to μ and $\widetilde{D}$ tends to D. We also obtain an *a posteriori* error estimate by proving an upper and lower bound of the modelling error by some explicit indicators that we use to describe a strategy to correct the model, and provides a way of getting an accurate solution.

3.1. The *a priori* analysis

Theorem 3.1. *Let $\mathbf{c}$ and $\widetilde{\mathbf{c}}$ be the solutions of the problems (2.1) and (2.2) respectively, then we have*

$$\|\mathbf{c}-\widetilde{\mathbf{c}}\|_{1,\Omega} \leq \frac{\delta(\widetilde{\mu}_1,\widetilde{D}_1,L)}{\inf(\gamma;\gamma_1)} \sup\left(\sup_{v\in H_0^1(\Omega)} \|(\widetilde{\mu}-\mu)(v)\|_{L^\infty(\Omega)};\ \sup_{v\in H_0^1(\Omega)} \|(\widetilde{D}-D)(v)\|_{L^\infty(\Omega)}\right), \tag{3.1}$$

where γ and γ_1 are the constants given by the assumptions $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_2)$ and $\delta(\widetilde{\mu}_1,\widetilde{D}_1,L)$ is given in Proposition 2.1.

Proof. Let $R \in H^{-1}(\Omega)$ be the residual functional associated to the problem (2.1) and its approximation (2.2), and defined, for all $v \in H_0^1(\Omega)$, by

$$< R, v >= L(v) - a(\mu, D, \widetilde{\mathbf{c}}, \widetilde{\mathbf{c}}, v). \tag{3.2}$$

Since $\mathbf{c}$ is solution of (2.1) we have

$$\begin{aligned} < R, v > &= a(\mu, D, \mathbf{c}, \mathbf{c}, v) - a(\mu, D, \widetilde{\mathbf{c}}, \widetilde{\mathbf{c}}, v) \\ &= \int_\Omega \Big(\mu(\mathbf{c})\mathbf{c} - \mu(\widetilde{\mathbf{c}})\widetilde{\mathbf{c}}\Big) v dx + \int_\Omega \Big(D(\mathbf{c})\nabla\mathbf{c} - D(\widetilde{\mathbf{c}})\nabla\widetilde{\mathbf{c}}\Big)\cdot\nabla v dx. \end{aligned}$$

For $v = \mathbf{c}-\widetilde{\mathbf{c}}$, and according to assumptions $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$ we have

$$< R, \mathbf{c}-\widetilde{\mathbf{c}} > \ \geq \gamma \int_\Omega (\mathbf{c}-\widetilde{\mathbf{c}})^2 dx + \gamma_1 \int_\Omega \mid \nabla(\mathbf{c}-\widetilde{\mathbf{c}}) \mid^2 dx. \tag{3.3}$$

In the other hand, since $\widetilde{\mathbf{c}}$ is solution of (2.2) we have

$$\begin{aligned} < R, \mathbf{c}-\widetilde{\mathbf{c}} > &= a(\widetilde{\mu}, \widetilde{D}, \widetilde{\mathbf{c}}, \widetilde{\mathbf{c}}, \mathbf{c}-\widetilde{\mathbf{c}}) - a(\mu, D, \widetilde{\mathbf{c}}, \widetilde{\mathbf{c}}, \mathbf{c}-\widetilde{\mathbf{c}}) \\ &= \int_\Omega (\widetilde{\mu}(\widetilde{\mathbf{c}}) - \mu(\widetilde{\mathbf{c}}))\, \widetilde{\mathbf{c}}(\mathbf{c}-\widetilde{\mathbf{c}}) dx + \int_\Omega \Big(\widetilde{D}(\widetilde{\mathbf{c}}) - D(\widetilde{\mathbf{c}})\Big)\nabla\widetilde{\mathbf{c}}\cdot\nabla(\mathbf{c}-\widetilde{\mathbf{c}}) dx. \end{aligned}$$

To simplify, for the following we denote by F the right hand side of (3.3), then the inequality becomes

$$F \leq \int_\Omega (\widetilde{\mu}(\widetilde{\mathbf{c}}) - \mu(\widetilde{\mathbf{c}}))\widetilde{\mathbf{c}}(\mathbf{c}-\widetilde{\mathbf{c}})(x) dx + \int_\Omega (\widetilde{D}(\widetilde{\mathbf{c}}) - D(\widetilde{\mathbf{c}}))\nabla\widetilde{\mathbf{c}}\cdot\nabla(\mathbf{c}-\widetilde{\mathbf{c}})(x) dx,$$

using Cauchy Schwartz inequality we get

$$\begin{aligned} \inf(\gamma;\gamma_1)\|\mathbf{c}-\widetilde{\mathbf{c}}\|_{1,\Omega}^2 &\leq \sup_{v\in H_0^1(\Omega)} \|(\widetilde{\mu}-\mu)(v)\|_{L^\infty(\Omega)} \|\widetilde{\mathbf{c}}\|_{0,\Omega}\|\mathbf{c}-\widetilde{\mathbf{c}}\|_{0,\Omega} \\ &+ \sup_{v\in H_0^1(\Omega)} \|(\widetilde{D}-D)(v)\|_{L^\infty(\Omega)} \|\nabla\widetilde{\mathbf{c}}\|_{0,\Omega}\|\nabla(\mathbf{c}-\widetilde{\mathbf{c}})\|_{0,\Omega}. \end{aligned}$$

Using (2.3) and simplifying by $\|\mathbf{c}-\widetilde{\mathbf{c}}\|_{1,\Omega}$ we obtain (3.1). □

3.2. The *a posteriori* error analysis

To have available computable quantities, we introduce the approximation of (2.2) by a finite element method (FEM).

Let $\mathcal{T}_h = \cup T$ be a regular triangulation of Ω where T is a triangle and h_T its diameter, we denote $h = \max_{T\in\mathcal{T}_h} h_T$ and $P_1(\mathcal{T}_h)$ defined by

$$P_1(\mathcal{T}_h) = \left\{ v \in H^1_0(\Omega) \mid v\,|_T \in P_1(T),\ \forall T \in \mathcal{T}_h \right\},$$

where, for each $T \in \mathcal{T}_h$, $P_1(T)$ stands for the space of restriction to T of polynomials of degree 1. The discrete problem with finite elements method of degree 1 associated to (2.2) is given by

$$\begin{cases} \text{find } \widetilde{\mathbf{c}}_h \in P_1(\mathcal{T}_h) \text{ such that} \\ a(\widetilde{\mu}, \widetilde{D}, \widetilde{\mathbf{c}}_h, \widetilde{\mathbf{c}}_h, v) = L(v), \quad \forall v \in P_1(\mathcal{T}_h). \end{cases} \tag{3.4}$$

The existence and uniqueness of the solution of the problem (3.4) can be established in the same way as for the problem (2.2).

Now, in order to decouple the modelling error from that of discretization error we use the triangle inequality

$$\|\mathbf{c} - \widetilde{\mathbf{c}}_h\|_{1,\Omega} \leq \|\mathbf{c} - \widetilde{\mathbf{c}}\|_{1,\Omega} + \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega}.$$

The term $\|\mathbf{c} - \widetilde{\mathbf{c}}\|_{1,\Omega}$ represents the modelling error and the term $\|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega}$ is the discretization one. Since $\widetilde{\mathbf{c}}$ is not computable, we give an estimate on $\|\mathbf{c} - \widetilde{\mathbf{c}}\|$ in terms of $\widetilde{\mathbf{c}}_h$ which is computable, and $\|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega}$. *A posteriori* error estimates of this last term can be obtained in a classical way using the error discretization indicators. In the following, to give an efficient and reliable estimate on $\|\mathbf{c} - \widetilde{\mathbf{c}}\|_{1,\Omega}$, we develop the upper and lower bounds of this term.

3.2.1. The upper bound.

Theorem 3.2.
let $\mathbf{c}$, $\widetilde{\mathbf{c}}$ and $\widetilde{\mathbf{c}}_h$ the solutions of the problems (2.1), (2.2) and (3.4) respectively.
There is a constant C^* independent of h such that

$$\|\mathbf{c} - \widetilde{\mathbf{c}}_h\|_{1,\Omega} \leq \frac{1}{\inf(\gamma;\gamma_1)} \left(\left(\sum_{T\in\mathcal{T}_h} (\eta_T^{\mu})^2 \right)^{\frac{1}{2}} + \left(\sum_{T\in\mathcal{T}_h} (\eta_T^{D})^2 \right)^{\frac{1}{2}} \right) + C^* \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega}, \tag{3.5}$$

where

$$\eta_T^{\mu} = \left\| \left[\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu(\widetilde{\mathbf{c}}_h)\right] \widetilde{\mathbf{c}}_h \right\|_{0,T}, \qquad \eta_T^{D} = \left\| \left[\widetilde{D}(\widetilde{\mathbf{c}}_h) - D(\widetilde{\mathbf{c}}_h)\right] \nabla\widetilde{\mathbf{c}}_h \right\|_{0,T}. \tag{3.6}$$

Proof. Let R be the residual functional defined by (3.2). With the help of the problem (2.2) and by introducing $\widetilde{\mathbf{c}}_h$, $\mu(\widetilde{\mathbf{c}}_h)$ and $\widetilde{\mu}(\widetilde{\mathbf{c}}_h)$, we have

$$\begin{aligned} < R, v > &= \int_\Omega (\widetilde{\mu}(\widetilde{\mathbf{c}}) - \mu(\widetilde{\mathbf{c}}))\widetilde{\mathbf{c}} v dx + \int_\Omega (\widetilde{D}(\widetilde{\mathbf{c}}) - D(\widetilde{\mathbf{c}}))\nabla\widetilde{\mathbf{c}} \cdot \nabla v dx \\ &= \int_\Omega (\widetilde{\mu}(\widetilde{\mathbf{c}}) - \widetilde{\mu}(\widetilde{\mathbf{c}}_h))\widetilde{\mathbf{c}} v dx + \int_\Omega (\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu(\widetilde{\mathbf{c}}_h))\widetilde{\mathbf{c}}_h v dx \\ &\quad + \int_\Omega (\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu(\widetilde{\mathbf{c}}_h))(\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h) v dx - \int_\Omega (\mu(\widetilde{\mathbf{c}}) - \mu(\widetilde{\mathbf{c}}_h))\widetilde{\mathbf{c}} v dx \\ &\quad\quad + \int_\Omega (\widetilde{D}(\widetilde{\mathbf{c}}) - \widetilde{D}(\widetilde{\mathbf{c}}_h))\nabla\widetilde{\mathbf{c}} \cdot \nabla v dx + \int_\Omega (\widetilde{D}(\widetilde{\mathbf{c}}_h) - D(\widetilde{\mathbf{c}}_h))\nabla\widetilde{\mathbf{c}}_h \cdot \nabla v dx \\ &\quad\quad\quad + \int_\Omega (\widetilde{D}(\widetilde{\mathbf{c}}_h) - D(\widetilde{\mathbf{c}}_h))\nabla(\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h) \cdot \nabla v dx - \int_\Omega (D(\widetilde{\mathbf{c}}) - D(\widetilde{\mathbf{c}}_h))\nabla\widetilde{\mathbf{c}} \cdot \nabla v dx. \end{aligned}$$

Using the Cauchy-Schwartz inequality and the assumptions ($\mathbf{H}_3$) and ($\mathbf{H}_4$) we obtain

$$\begin{aligned} | < R, v > | \quad \leq \quad & \Big(C_{\widetilde{\mu}} \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega} \|\widetilde{\mathbf{c}}\|_{0,\Omega} + \|(\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu(\widetilde{\mathbf{c}}_h))\widetilde{\mathbf{c}}_h\|_{0,\Omega} \\ & + \sup_{v \in H_0^1} \|\widetilde{\mu}(v) - \mu(v)\|_{L^\infty(\Omega)} \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{0,\Omega} + C_\mu \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega} \|\widetilde{\mathbf{c}}\|_{0,\Omega} \\ & + C_{\widetilde{D}} \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega} \|\nabla \widetilde{\mathbf{c}}\|_{0,\Omega} + \| (\widetilde{D}(\widetilde{\mathbf{c}}_h) - D(\widetilde{\mathbf{c}}_h)) \nabla \widetilde{\mathbf{c}}_h \|_{0,\Omega} \\ & + \sup_{v \in H_0^1} \|\widetilde{D}(v) - D(v)\|_{L^\infty(\Omega)} \|\nabla(\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h)\|_{0,\Omega} \\ & + C_D \|\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h\|_{1,\Omega} \|\nabla \widetilde{\mathbf{c}}\|_{0,\Omega} \Big) \| v \|_{1,\Omega}, \end{aligned}$$

by using the estimate (2.3)

$$\begin{aligned} \| R \|_{H^{-1}} \quad \leq \quad & \delta(\mu_1, D_1, L) \Big[C_{\widetilde{\mu}} + C_\mu + + C_{\widetilde{D}} + C_D \Big] \| \widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h \|_{0,\Omega} \\ & + \| (\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu(\widetilde{\mathbf{c}}_h)) \widetilde{\mathbf{c}}_h \|_{0,\Omega} \\ & + \sup_{v \in H_0^1} \| \widetilde{\mu}(v) - \mu(v) \|_{L^\infty(\Omega)} \| \widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h \|_{0,\Omega} \\ & + \| (\widetilde{D}(\widetilde{\mathbf{c}}_h) - D(\widetilde{\mathbf{c}}_h)) \nabla \widetilde{\mathbf{c}}_h \|_{0,\Omega} \\ & + \sup_{v \in H_0^1} \| \widetilde{D}(v) - D(v) \|_{L^\infty(\Omega)} \| \nabla(\widetilde{\mathbf{c}} - \widetilde{\mathbf{c}}_h) \|_{0,\Omega} . \end{aligned}$$

By breaking $\|(\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu(\widetilde{\mathbf{c}}_h))\widetilde{\mathbf{c}}_h\|_{0,\Omega}$ and $\| (\widetilde{D}(\widetilde{\mathbf{c}}_h) - D(\widetilde{\mathbf{c}}_h)) \nabla \widetilde{\mathbf{c}}_h \|_{0,\Omega}$ on each triangle we can write

$$\|R\|_{H^{-1}(\Omega)} \leq \left(\sum_{T \in \mathfrak{T}_h} (\eta_T^\mu)^2 \right)^{\frac{1}{2}} + \left(\sum_{T \in \mathfrak{T}_h} (\eta_T^K)^2 \right)^{\frac{1}{2}} + C_1 \|\widetilde{\mathbf{c}}_h - \widetilde{\mathbf{c}}\|_{1,\Omega}, \tag{3.7}$$

where η_T^μ and η_T^D are defined by (3.6) and

$$\begin{aligned} C_1 \quad = \quad & \sup \Big(\delta(\mu_1, D_1, L) \Big[C_{\widetilde{\mu}} + C_\mu + C_{\widetilde{D}} + C_D \Big], \\ & \sup_{v \in H_0^1(\omega)} \| \widetilde{\mu}(v) - \mu(v) \|_{L^\infty(\Omega)}, \\ & \sup_{v \in H_0^1(\omega)} \| \widetilde{D}(v) - D(v) \|_{L^\infty(\Omega)} \Big). \end{aligned}$$

In this expression the last two terms can be replaced by $\mu_2 + \widetilde{\mu}_2$ and $D_2 + \widetilde{D}_2$ respectively (see ($\mathbf{H}_3$)). Moreover, by the definition of R given by (3.2), the problem (2.1), the hypothesis ($\mathbf{H}_1$) and ($\mathbf{H}_2$) and then taking $v = \mathbf{c} - \widetilde{\mathbf{c}}$ we have

$$\begin{aligned} < R, u - \widetilde{u} > \quad = \quad & \int_\Omega (\mu(\mathbf{c})\mathbf{c} - \mu(\widetilde{\mathbf{c}})\widetilde{\mathbf{c}})(\mathbf{c} - \widetilde{\mathbf{c}}) dx + \int_\Omega (D(\mathbf{c}) \nabla \mathbf{c} - D(\widetilde{\mathbf{c}}) \nabla \widetilde{\mathbf{c}}) \nabla (\mathbf{c} - \widetilde{\mathbf{c}}) dx \\ \geq \quad & \inf(\gamma, \gamma_1) \| \mathbf{c} - \widetilde{\mathbf{c}} \|_{1,\Omega}^2, \end{aligned}$$

which gives

$$\| \mathbf{c} - \widetilde{\mathbf{c}} \|_{1,\Omega} \leq \frac{1}{\inf(\gamma, \gamma_1)} \| R \|_{H^{-1}(\Omega)} . \tag{3.8}$$

Combining the inequalities (3.8) and (3.7) we obtain the *a posteriori* error estimate (3.5) with $C^* = \dfrac{C_1}{\inf(\gamma, \gamma_1)} + 1$. □

3.2.2. THE LOWER BOUND.

Theorem 3.3. *Let* $\mathbf{c}$ *and* $\widetilde{\mathbf{c}}_h$ *the respective solutions of problems (2.1)and (3.4), and* η_T^{μ}, η_T^{D} *defined by (3.6), then we have*

$$\eta_T^{\mu} \leq \delta(\mu_1, D_1, L)\Big(\xi_1\|\mathbf{c}-\widetilde{\mathbf{c}}_h\|_{1,T}+\xi_2\Big), \tag{3.9}$$

$$\eta_T^{D} \leq \delta(\mu_1, D_1, L)\Big(\xi_3\|\mathbf{c}-\widetilde{\mathbf{c}}_h\|_{1,T}+\xi_4\Big), \tag{3.10}$$

where $\quad \xi_1 = C_{\widetilde{\mu}} + C_\mu \quad$ *and* $\quad \xi_2 = \sup_{v\in H_0^1(\Omega)} \|(\widetilde{\mu}-\mu)(v)\|_{L^\infty(\Omega)},$

$\xi_3 = C_{\widetilde{D}} + C_D \quad$ *and* $\quad \xi_4 = \sup_{v\in H_0^1(\Omega)} \|\,(\widetilde{D}-D)(v)\,\|_{L^\infty(\Omega)}\,.$

Proof. Introducing $\widetilde{\mu}(\mathbf{c})$ in the expression of (η_T^{μ}), using the hypothesis $(\mathbf{H}_3)$ and $(\mathbf{H}_4)$ and then the inequality (2.3) we get

$$\begin{aligned}
(\eta_T^{\mu})^2 &= \int_T (\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))^2\,\widetilde{\mathbf{c}}_h^2 dx = \int_T (\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h dx\\
&= \int_T (\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\widetilde{\mu}(\mathbf{c}))\widetilde{\mathbf{c}}_h\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h dx\\
&\qquad + \int_T (\widetilde{\mu}(\mathbf{c})-\mu(\mathbf{c}))\widetilde{\mathbf{c}}_h\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h dx\\
&\qquad\quad + \int_T (\mu(\mathbf{c})-\mu(\widetilde{\mathbf{c}}_h))\widetilde{\mathbf{c}}_h\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h dx\\
&\leq C_{\widetilde{\mu}}\|\widetilde{\mathbf{c}}_h-\mathbf{c}\|_{1,T}\|\widetilde{\mathbf{c}}_h\|_{0,T}\|\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h\|_{0,T}\\
&\qquad + \sup_{v\in H_0^1(\Omega)} \|(\widetilde{\mu}-\mu)(v)\|_{L^\infty(\Omega)}\|\widetilde{\mathbf{c}}_h\|_{0,T}\|\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h\|_{0,T}\\
&\qquad\quad + C_\mu\|\widetilde{\mathbf{c}}_h-\mathbf{c}\|_{1,T}\|\widetilde{\mathbf{c}}_h\|_{0,T}\|\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h\|_{0,T}\\
&\leq \delta(\mu_1, D_1, L)\Bigg((C_{\widetilde{\mu}}+C_\mu)\|\widetilde{\mathbf{c}}_h-\mathbf{c}\|_{1,T}\\
&\qquad + \sup_{v\in H_0^1(\Omega)} \|(\widetilde{\mu}-\mu)(v)\|_{L^\infty(\Omega)}\Bigg)\|\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h\|_{0,T}.
\end{aligned}$$

Simplify by $\|\,(\widetilde{\mu}(\widetilde{\mathbf{c}}_h)-\mu(\widetilde{\mathbf{c}}_h))\,\widetilde{\mathbf{c}}_h\|_{0,T}$, we obtain (3.9).

The estimate (3.10), on the second indicator (η_T^{D}), is obtained in the same way. □

4. Application

We consider the problem (1.2) with the functions $\mu(\cdot)$ and $D(\cdot)$ given by

$$\mu(\mathbf{c}) = \begin{cases} \mu_{f1}(\mathbf{c}) & \text{if } \mathbf{c} < h_s \\ \mu_{f2}(\mathbf{c}) & \text{if } h_s \leq \mathbf{c} < 0 \\ \mu_s & \text{if } \mathbf{c} \geq 0 \end{cases} \tag{4.1}$$

$$D(\mathbf{c}) = \begin{cases} D_{f1}(\mathbf{c}) & \text{if } \mathbf{c} < h_s \\ D_{f2}(\mathbf{c}) & \text{if } h_s \leq \mathbf{c} < 0. \\ D_s & \text{if } \mathbf{c} \geq 0 \end{cases} \tag{4.2}$$

where μ_{f1}, μ_{f2}, D_{f1} and D_{f2} are non-linear functions of $\mathbf{c}$, and μ_s and D_s are constants.
In the context of porous media for example, h_s is a small negative value which depends on the nature of

the soil, called the minimum capillary height, but it depends on the soil nature so it is not well known generally. Because of the complexity of these expressions, we use usually the following approximations of $\mu(\cdot)$ and $D(\cdot)$

$$\mu^*(u) = \begin{cases} \mu_{f1}(\mathbf{c}) & \text{if } \mathbf{c} < 0 \\ \mu_s & \text{if } \mathbf{c} \geq 0 \end{cases} \tag{4.3}$$

$$D^*(u) = \begin{cases} D_{f1}(\mathbf{c}) & \text{if } \mathbf{c} < 0 \\ D_s & \text{if } \mathbf{c} \geq 0. \end{cases} \tag{4.4}$$

The aim of this section is to use the results of Theorems 3.2 and the indicators (3.6) to determine an approximation of the value of h_s and so on for the functions μ and D which will be as close as possible to the exact functions.

4.1. The adaptive procedure

We want to build an algorithm, based on the modelling indicators given in (3.6), such that, starting with the initial expression of μ and D given by (4.3), we construct sequences $(\mu^{(k)})_k$ and $(D^{(k)})_k$ converging to the exact expression of μ and D.
Computing the indicator (3.6) requires the exact expression of μ and D which is unknown since h_s is unknown. We will use a like-fixed point iteration to get a value h_s^* as close as possible to the exact value h_s.

1. Let the first approximations be given by

$$\widetilde{\mu}(y) = \mu_s, \quad \forall y \qquad \text{and} \qquad \widetilde{D}(y) = D_s, \quad \forall y. \tag{4.5}$$

   In this first iteration, μ^* and D^* will play the role of the exact expression, when computing the indicators, and will be updated by the iterations.

2. In the second step, we solve the problem (3.4) with the approximations $\widetilde{\mu}$ and $\widetilde{D}$, which gives the approximated solution $\widetilde{\mathbf{c}}_h$, and we compute the indicators, for all $T \in \mathfrak{T}_h$

$$\eta_T = \left\| \left[\widetilde{\mu}(\widetilde{\mathbf{c}}_h) - \mu^*(\widetilde{\mathbf{c}}_h)\right]\widetilde{\mathbf{c}}_h \right\|_{0,T} + \left\| \left[\widetilde{D}(\widetilde{\mathbf{c}}_h) - D^*(\widetilde{\mathbf{c}}_h)\right] \nabla \widetilde{\mathbf{c}}_h \right\|_{0,T},$$

   and their main value $\bar{\eta}$.

3. For all $T \in \mathfrak{T}_h$, the modelling indicator η_T is an estimation of the error between the model with the functions μ^* and D^* and the model with the functions $\widetilde{\mu}$ and $\widetilde{D}$. This allows us to determine three zones: the one where this indicator vanishes, the one where it is large and the one where it is small enough.
   The zone of the domain where the indicator vanishes is given by

$$\widetilde{\Omega}_s := \{x \in \Omega \,/\, \widetilde{\mathbf{c}}_h(x) \geq 0\}, \tag{4.6}$$

   and the zone where the indicator is larger than the main value is given by

$$A := \{x \in T \in \mathfrak{T}_h \,/\, \eta_T > \bar{\eta}\} \setminus \widetilde{\Omega}_s. \tag{4.7}$$

   The fact that the indicators on $T \in \mathfrak{T}_h$, related to A, are larger than the main value, mains that the model defined by the functions $\widetilde{\mu}$ and $\widetilde{D}$ is a bad approximation in this zone.
   We notice that, for $x \in A$, the values of $\widetilde{\mathbf{c}}_h$ are negative, and define

$$h_s := \max_{x \in A} \widetilde{\mathbf{c}}_h(x), \tag{4.8}$$

   and

$$\widetilde{\Omega}_{f1} \quad := \quad \{x \in \Omega \,/\, \widetilde{\mathbf{c}}_h < h_s\} \tag{4.9}$$

$$\widetilde{\Omega}_{f2} \quad := \quad \{x \in \Omega \,/\, h_s \leq \widetilde{\mathbf{c}}_h < 0\}. \tag{4.10}$$

In order to update the expression of μ^* and D^* we put

$$\mu^{(1)}(\mathbf{c}) = \begin{cases} \mu_{f1}^{(1)}(\mathbf{c}) & \text{if } \mathbf{c} < h_s \\ \mu_{f2}(\mathbf{c}) & \text{if } h_s \leq \mathbf{c} < 0 \\ \mu_s & \text{if } \mathbf{c} \geq 0. \end{cases} \qquad D^{(1)}(\mathbf{c}) = \begin{cases} D_{f1}^{(1)}(\mathbf{c}) & \text{if } \mathbf{c} < h_s \\ D_{f2}(\mathbf{c}) & \text{if } h_s \leq \mathbf{c} < 0 \\ D_s & \text{if } \mathbf{c} \geq 0. \end{cases} \tag{4.11}$$

The functions $\mu_{f1}^{(1)}(\mathbf{c})$ and $D_{f1}^{(1)}(\mathbf{c})$ are obtained by multiplying the functions $\mu_{f1}(\mathbf{c})$ and $D_{f1}(\mathbf{c})$ by a constant respectively to ensure continuity of $\mu^{(1)}$ and $D^{(1)}$.

4. To go to the next iteration we put

$$\widetilde{\mu} = \alpha^*, \qquad \text{and} \qquad \widetilde{D} = K^*$$

then

$$\mu^* = \mu^{(1)} \qquad \text{and} \qquad D^* = D^{(1)},$$

and go to the step 2, until the chosen stop criterion is fulfilled.

The algorithm can be summarized as following.

ALGORITHM

INPUT

Expression of functions: μ_{f1}, μ_{f2}, μ_s, D_{f1}, D_{f2}, D_s,
Tolerance ε.

STEP 1 (*Initialisation*)

$h_s^{(0)} = 0$
μ^* and D^* defined by (4.3),
$\widetilde{\mu} = \mu_s$, $\widetilde{D} = D_s$.

STEP 2 (*The indicators*)

Solve the problem (3.4) with $\widetilde{\mu}$, and $\widetilde{D}$
Compute η_T for all T and $\bar{\eta}$.

STEP 3 (*Determination of the zones*)

Determine the sets Ω_s, by (4.6), A by (4.7)
Compute the new value of h_s by (4.8)
Determine the zones Ω_{f1} by (4.9) and Ω_{f2} by (4.10)
Define $\mu^{(1)}$ and $D^{(1)}$, the new expressions of μ and D by (4.11).

STEP 4 (*Update*)

If $|h_s^{(0)} - h_s| \geq \varepsilon$ then
  put $\widetilde{\mu} = \mu^*$ and $\widetilde{D} = D^*$
  put $\mu^* = \mu^{(1)}$ and $D^* = D^{(1)}$
  put $h_s^{(0)} = h_s$
  go to STEP2.
Otherwise take h_s, $\mu^{(1)}$ and $D^{(1)}$ as the best approximations.
STOP.

In the following we will present two manufactured examples to confirm the convergence of the strategy described before. Calculations are done by the software FreeFem++ [5].

4.2. Numerical tests

Let $\Omega =]0,1[\times]0,1[$ and $\mathbf{c}$ the solution of the problem (1.2) with the coefficients $\mu(\cdot)$ and $D(\cdot)$ having the following expressions

$$\mu(z) = \begin{cases} \left(\dfrac{\exp(-0,3)}{1,09}\right)(z^2+1) & \text{if } z < h_s \\ \exp(z) & \text{if } h_s \leq z < 0 \\ 1 & \text{if } z \geq 0, \end{cases}$$

and

$$D(z) = \begin{cases} z^2+1 & \text{if } z < h_s \\ \left(\dfrac{1}{0,09}\right) z^4 + 1 & \text{if } h_s \leq z < 0 \\ 1 & \text{if } z \geq 0. \end{cases}$$

We have to find an approximation of $\mathbf{c}$ and the value h_s using the algorithm defined above. We consider two manufactured solutions where $h_s = -0.3$. We take the parameter of stopping criterion $\varepsilon = 10^{-4}$.

1. In the first example the exact solution is defined by

$$\mathbf{c}_e(x,y) = 2,2\ \sin(\pi x)\ (0.3 - y).$$

   For the first iteration μ and D equal to the constant 1, and use the error indicators according to the algorithm defined above. The results are given in Tables 1, 2 and 3. We observe, in Tables 1, 2, 3, for a different mesh size h, we need only four or five iterations to rich an acceptable value of $h_s^{(k)}$ and enough small main values of indicators. The L^2-norm of the error between the exact solution and the approximated solution is also acceptable but the H^1-norm remains relatively large. We can explain this by the fact that we do not use the indicators of discretization to adapt the mesh because we notice that when we refine uniformly the mesh we obtain better approximations of $h_s^{(k)}$ and the solution, which confirm the convergence of the algorithm with the iterations and when h gets quite small.

2. In the second example the exact solution is given by

$$\mathbf{c}_e = 17(x^2 - x^3)(0,35 - y).$$

   The results, with the same algorithm, are given in Tables 4, 5, 6, and we have the same observations as the first example.

In the following we put, $h_s^{(k)}, \bar{\eta}, \|\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\|_{0,\Omega}$ and $\|\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\|_{1,\Omega}$ are respectively the approximated value of h_s, the average of the modelling indicators, the L^2-norm and H^1-norm of the error between the exact solution and the approximated solution at the iteration k.

Table 1: \ **Results for the mesh size $h = 1/20$.**

k	$h_s^{(k)}$	$\bar{\eta}$	$\|\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\|_{0,\Omega}$	$\|\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\|_{1,\Omega}$
1	$-0,421222$	$0,0256837$	$0,00432329$	$0,139762$
2	$-0,353366$	$0,0239808$	$0,00158224$	$0,132176$
3	$-0,333854$	$0,0232097$	$0,00117041$	$0,131287$
4	$-0,331336$	$0,0232537$	$0,00134389$	$0,131287$
5	$-0,331049$	$0,0232532$	$0,00134322$	$0,131254$

Table 2: \ **Results for the mesh size** $h = 1/50$**.**

k	$h_s^{(k)}$	$\bar{\eta}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{0,\Omega}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{1,\Omega}$
1	$-0,378704$	$0,00894813$	$0,00405838$	$0,0645721$
2	$-0,341937$	$0,00838358$	$0,0014325$	$0,0545204$
3	$-0,317007$	$0,00817934$	$0,0003697$	$0,0522392$
4	$-0,308856$	$0,00812455$	$0,000200004$	$0,0519712$
5	$-0,308249$	$0,0081272$	$0,000193734$	$0,0519631$

Table 3: \ **Results for the mesh size** $h = 1/80$**.**

k	$h_s^{(k)}$	$\bar{\eta}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{0,\Omega}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{1,\Omega}$
1	$-0,368647$	$0,0054973$	$0,00389191$	$0,0475538$
2	$-0,340095$	$0,00514744$	$0,00141992$	$0,035589$
3	$-0,305783$	$0,00496892$	$0,000214103$	$0,0323952$
4	$-0,303161$	$0,00496897$	$6,8361410^{-5}$	$0,0321963$
5	$-0,302731$	$0,00496702$	$6,7335210^{-5}$	$0,0221944$

Table 5: \ **Results for the mesh size** $h = 1/50$**.**

k	$h_s^{(k)}$	$\bar{\eta}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{0,\Omega}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{1,\Omega}$
1	$-0,381903$	$0,0116102$	$0,00399798$	$0,0921355$
2	$-0,334086$	$0,0105596$	$0,00119841$	$0,0816705$
3	$-0,331787$	$0,0105538$	$0,000544999$	$0,0804815$
4	$-0,330134$	$0,0105704$	$0,000570222$	$0,0805092$

Table 6: \ **Results for the mesh size** $h = 1/100$**.**

k	$h_s^{(k)}$	$\bar{\eta}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{0,\Omega}$	$\Vert\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\Vert_{1,\Omega}$
1	$-0,366074$	$0,00570192$	$0,00383143$	$0,056944$
2	$-0,330018$	$0,00528268$	$0,00114668$	$0,0423303$
3	$-0,309608$	$0,00517833$	$0,000213726$	$0,040044$
4	$-0,304221$	$0,0051748$	$9,547610^{-5}$	$0,0398681$
5	$-0,303829$	$0,00517387$	$9,6292510^{-5}$	$0,0398688$

5. Conclusion and perspectives.

In this work we developed two estimates on the modelling error, for an elliptic non-linear boundary value problem, when the coefficients of the equation are modified. The first one is an *a priori* error estimate and is achieved without additional regularity assumption on the solution. The second one is an *a posteriori* error estimate where we developed an upper and a lower bounds of the error by computable indicators and hence proved that the estimate is efficient and reliable. We have presented an adaptive modelling strategy based on explicit evaluation of the modelling error, via the indicators, as an application, when the error is caused by the incomplete knowledge of the coefficients. The numerical tests validate this strategy.

In future works, several issues are left to be investigated. The first one consists to combine the mesh indicator with the modelling indicator to balancing modelling and discretization errors in order to increase accuracy in an economical way. The second is to extend this estimate and this strategy to the more realistic problem of the vadose zone in the context of the infiltration in partially saturated porous media.

Table 4: \ **Results for the mesh size $h = 1/20$.**

k	$h_s^{(k)}$	$\bar{\eta}$	$\|\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\|_{0,\Omega}$	$\|\widetilde{\mathbf{c}}_h^{(k)} - \mathbf{c}_e\|_{1,\Omega}$
1	$-0,357726$	$0,0270085$	$0,00305144$	$0,201923$
2	$-0,347771$	$0,0257244$	$0,00238952$	$0,199462$
3	$-0,346233$	$0,0257678$	$0,00221624$	$0,19915$
4	$-0,345044$	$0,02558093$	$0,00219454$	$0,19913$

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