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On the well-posedness and stability of retarded impulsive wave equations with a frictional damping

Aissa Guesmia* and Nasser-eddine Tatar

ABSTRACT: Of concern is, first, a wave equation subject to a discrete time delay in the state itself (and not in its time derivative). It is also under the action of impulses and a control given in the form of a frictional damping. We prove the well-posedness and exponential stability of our system. Second, we give an extension to the case where the time delay occurs in both, the state and its time derivative, in addition to impulsive boundary conditions in the delays. Our stability results show that the dissipation generated by the frictional damping is not destroyed neither by the impulses nor by the delays even when acting simultaneously.

Key Words: Impulsive wave equation, Discrete delay, Well-posedness, Stability, Semigroup approach, Energy method, Differential inequality.

Contents

1. Introduction

We are witnessing nowadays an increasing interest in the study of the influence of impulsive effects on many phenomena. This is due to the mathematical challenges they bring and to the extensive applications in nature, science and technology [2,3,12,18]. Mechanisms that are subject to short changes in their states may be modelled by Impulsive Differential Equations (IDEs). The short changes are either natural or imposed by controls and they cause discontinuities in the trajectories. In general, the model comprises a continuous differential equation (or more) acting between the impulses, impulse equations defined at the moments of changes and a definition of the set of jump points. It has been observed that impulsive actions may be the cause of drastic changes in the behavior of solutions. Therefore, it is important to pay considerable attention to such systems.

The investigation of the fallouts of impulses on processes has attracted the attention of a considerable number of scholars owing to the countless number of solicitations in practical problems [2,3,12,18]. Indeed, actions taking place in footling moments may be handled correctly as impulses and treated adequately in the IDEs framework. The effects of these changes are unpredictible. They can stabilize an unstable system and vice-versa, they can destabilize an initially stable system, thereby justifying the great attention paid to these phenomena.

* Corresponding author

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Let $N \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary Γ and a closure $\overline{\Omega}$, $a_0,\ a_1,\ \tau > 0,\ a_2 \in \mathbb{R}$, $\{t_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}_+ := [0, +\infty)$ such that

$$t_0 := 0 < t_1 < \cdots < t_k < \cdots, \quad \inf_{k\in\mathbb{N}^*} \{t_{k+1} - t_k\} > \tau \quad \text{and} \quad \lim_{k\to+\infty} t_k = +\infty. \tag{1.1}$$

Moreover, we assume that the functions $g_k,\ f_k : \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N}^*$, satisfy, for some $\xi_k,\ \tilde{\xi}_k > 0$,

$$g_k \in C^1(\mathbb{R}_+), \quad g_k(0) = 0, \quad \sup_{k\in\mathbb{N}^*} \xi_k < +\infty \quad \text{and} \quad |g_k'| \le \xi_k, \tag{1.2}$$

and

$$f_k \in C(\mathbb{R}_+), \quad \sup_{k\in\mathbb{N}^*} \tilde{\xi}_k < +\infty \quad \text{and} \quad |f_k(s)| \le \tilde{\xi}_k|s|, \quad s \in \mathbb{R}. \tag{1.3}$$

We consider the following system:

$$\begin{cases} u_{tt}(x,t) - a_0\Delta u(x,t) + a_1 u_t(x,t) + a_2 u(x,t-\tau) = 0, & x \in \Omega,\ t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}}, \\ u(x,t) = 0, & x \in \Gamma,\ t > 0, \\ u(x,s) = u_0(x,s),\ u_t(x,0) = u_1(x), & x \in \overline{\Omega},\ s \in [-\tau, 0], \\ u(x,t_k) = g_k(u(x,t_k^-)),\ u_t(x,t_k) = f_k(u_t(x,t_k^-)), & x \in \overline{\Omega},\ k \in \mathbb{N}^*, \end{cases} \tag{1.4}$$

where $u(x,t)$ is the unknown of (1.4), u_0 and u_1 are given functions (initial data), Δ is the classical Laplacian operator, the subscript t denotes the derivative with respect to t, and t_k^- and t_k^+ denote, respectively, the limit when t converges to t_k from the left and from the right. The term $a_1 u_t$ is the frictional damping, which plays the role of control for (1.4). The constant τ represents the considered discrete time delay with amplitude a_2. It is clear that in the particular case $a_2 = 0$, there will be no time delay. The impulses are taken in consideration through the functions g_k and f_k. In case of continuity and $g_k(s) = f_k(s) = s$, there will be no impulses.

To the best of our knowledge, this kind of problems in this form and with impulsive conditions has not been considered so far in the literature. Therefore, we shall survey briefly some related works without impulses and with a discrete delay in the velocity of the state $u_t(x, t-\tau)$ rather than the state itself $u(x, t-\tau)$. This is quite a different situation as it offers the possibility of contrasting both velocities (with and without delay) together.

We can find a good number of papers by Nicaise and collaborators on this subject (without impulses and without time delay in the state). Several results have been achieved when both the (frictional damping) term $a_1 u_t(x,t)$ and the (delayed velocity) term $a_3 u_t(x, t-\tau_2)$ act internally. Moreover, we can find papers dealing with the cases where one of them is effective internally and the other one at the boundary. In addition, the time-varying delay [15,16] and the distributed delay cases have been investigated. We will mention here only works on the former case.

We start by the work of Datko *et al.* [7] where the authors established an exponential stability result in the 1-d case with the internal terms

$$2bu_t(x,t) + b^2 u(x,t)$$

and the boundary delayed control

$$u_x(1,t) + qu_t(1, t-\tau) = 0$$

provided that q is small enough. There are other works where the delayed velocity and the undelayed velocity act one internally and one in the boundary (we do not report here). In [14], Nicaise and Pignotti considered the problem

$$\begin{cases} w_{tt}(x,t) - \Delta w(x,t) + \eta(x)\left[a_1 w_t(x,t) + a_2 w_t(x,t-\tau)\right] = 0, & x \in \Omega,\ t > 0, \\ w(x,t) = 0, & x \in \Gamma_D,\ t > 0, \\ \frac{\partial w}{\partial \nu}(x,t) = 0, & x \in \Gamma_N,\ t > 0, \\ w(x,0) = w_0(x),\ w_t(x,0) = w_1(x), & x \in \Omega, \\ w_t(x, t-\tau) = w_2(x, t-\tau), & x \in \Omega,\ t \in (0,\tau) \end{cases}$$

and also the case where the control acts on the boundary

$$\frac{\partial w}{\partial \nu}(x,t) + a_1 w_t(x,t) + a_2 w(x,t-\tau) = 0.$$

They extended the work of Xu *et al.* [20] from 1-d to any dimension. Namely, they proved that the system is exponentially stable if $a_2 < a_1$ and unstable otherwise. This is in spite of the destructive nature of delays as demonstrated in [5,6,7,13]. Needless to say, the exponential stability takes place in the absence of the delay term. The case of time-varying delays is treated in [15,16].

In this work, for problem (1.4), we assume (1.1) and the growth conditions (1.2) and (1.3) on the impulses g_k and f_k, respectively. Well-posedness and exponential stability results will be proved for a certain range of values of a_2. This means that the dissipation effect of the frictional damping resists to both impulses as well as the delay occuring in the state. To this end, we shall apply the semigroup approach and combine the multiplier technique (to get some appropriate estimates) in a crucial manner with an impulsive version of the Halanay inequality (see Lemma 3.3 below). Moreover, we present an extension of these results to an impulsive differential problem with delays even in the impulsive conditions. This shows that the frictional damping is able to stabilize the system despite the destructive nature of the delays and the impulses.

The paper is organized as follows: in the next section, we establish existence, uniqueness and smoothness results for our problem (1.4). Section 3 contains the statement and proof of our stability result. The case of time delays in state, its time derivative and impulses will be treated in Section 4. We end the paper by a section containing several remarks, possible extensions and generalizations.

2. Well-posedness

In the sequel, and in order to simplify the exposition, the variables x, t and s are indicated only when necessary to avoid ambiguity. We use $\|\cdot\|$ to denote both $L^2(\Omega)$ and $\left(L^2(\Omega)\right)^N$ norms, where the related scalar product is designated by $\langle\cdot,\cdot\rangle$.

In this section, we will discuss the existence, uniqueness and smoothness of a solution of (1.4) in an appropriate underlying space. To this end, we will distinguish the two cases of absence and presence of impulses, and apply the semigroup approach.

2.1. Absence of impulses

We consider the following delayed wave problem without impulses:

$$\begin{cases} w_{tt}(x,t) - a_0\Delta w(x,t) + a_1 w_t(x,t) + a_2 w(x,t-\tau) = 0, & x\in\Omega,\ t>0,\\ w(x,t) = 0, & x\in\Gamma,\ t>0,\\ w(x,s) = u_0(x,s),\ \ w_t(x,0) = u_1(x), & x\in\overline{\Omega},\ s\in[-\tau,0]. \end{cases} \tag{2.1}$$

We shall recast this problem (2.1) in a first order abstract form. Let $\mathcal{W} = (w, w_t, z)^T$,

$$\mathcal{W}_0 = (u_0(\cdot,0), u_1, z_0)^T \in \mathcal{H},$$

$$\begin{cases} z(x,t,p) = w(x,t-\tau p), & x\in\Omega,\ t>0,\ \ p\in(0,1),\\ z_0(x,p) = z(x,0,p) = u_0(x,-\tau p), & x\in\Omega,\ p\in(0,1), \end{cases} \tag{2.2}$$

$$\mathcal{H} = H_0^1(\Omega)\times L^2(\Omega)\times\mathcal{L}$$

and

$$\mathcal{L} = L^2\left((0,1), L^2(\Omega)\right) = \left\{ v : (0,1)\to L^2(\Omega) : \int_0^1 \|v(p)\|^2\, dp < +\infty \right\}. \tag{2.3}$$

The space $\mathcal{H}$ is endowed with the scalar product, for $V = (v_1, v_2, v_3)^T$ and $\tilde{V} = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T$,

$$\left\langle V, \tilde{V}\right\rangle_{\mathcal{H}} = a_0\left\langle \nabla v_1, \nabla\tilde{v}_1\right\rangle + \left\langle v_2, \tilde{v}_2\right\rangle + \tau|a_2|\left\langle v_3, \tilde{v}_3\right\rangle_{\mathcal{L}}.$$

It is understood that the space $\mathcal{L}$ is endowed with the scalar product

$$\langle v, \tilde{v}\rangle_{\mathcal{L}} = \int_0^1 \langle v, \tilde{v}\rangle \, dp,$$

which makes $\mathcal{H}$ a Hilbert space. Notice that the function z defined in (2.2) satisfies

$$\begin{cases} \tau z_t(x,t,p) + z_p(x,t,p) = 0, \\ z(x,t,0) = w(x,t). \end{cases} \tag{2.4}$$

We define the linear operators

$$\mathcal{A}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} v_2 - \frac{|a_2|c_0}{2a_0} v_1 \\ a_0\Delta v_1 - (a_1 + |a_2|)\, v_2 - a_2 v_3(1) \\ -\frac{1}{\tau}\frac{\partial v_3}{\partial p} \end{pmatrix}$$

and

$$\mathcal{B}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \frac{|a_2|c_0}{2a_0} v_1 \\ |a_2| v_2 \\ 0 \end{pmatrix},$$

where $c_0 > 0$ is Poincaré's constant defined by

$$\|v\|^2 \leq c_0 \|\nabla v\|^2, \quad v \in H_0^1(\Omega). \tag{2.5}$$

The domains of $\mathcal{B}$ and $\mathcal{A}$ are given by $D(\mathcal{B}) = \mathcal{H}$ and

$$D(\mathcal{A}) = \left\{ V \in \mathcal{H} : v_1 \in H^2(\Omega),\, v_2 \in H_0^1(\Omega),\, \frac{\partial v_3}{\partial p} \in \mathcal{L},\, v_3(0) = v_1 \right\}.$$

Clearly, $D(\mathcal{A})$ is densely embedded in $\mathcal{H}$.

Now, we see that (2.1) is equivalent to

$$\begin{cases} \mathcal{W}_t(t) = (\mathcal{A} + \mathcal{B})\, \mathcal{W}(t),\ t > 0, \\ \mathcal{W}(0) = \mathcal{W}_0. \end{cases} \tag{2.6}$$

The existence, uniqueness and smoothness of solutions for (2.1) are stated in the next theorem.

Theorem 2.1 *For any $\mathcal{W}_0 \in \mathcal{H}$, the problem (2.6) admits a unique weak solution*

$$\mathcal{W} \in C\left(\mathbb{R}_+, \mathcal{H}\right),$$

so, for any

$$(u_0(\cdot, 0), u_1, z_0) \in H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{L}$$

(z_0 is defined in (2.2)), (2.1) admits a unique weak solution

$$w \in C\left(\mathbb{R}_+, H_0^1(\Omega)\right) \cap C^1\left(\mathbb{R}_+, L^2(\Omega)\right) \quad \text{and} \quad z \in C\left(\mathbb{R}_+, \mathcal{L}\right). \tag{2.7}$$

If $\mathcal{W}_0 \in D(\mathcal{A})$, then the solution of (2.6) is classical

$$\mathcal{W} \in C^1\left(\mathbb{R}_+, \mathcal{H}\right) \cap C\left(\mathbb{R}_+, D(\mathcal{A})\right),$$

so, if

$$(u_0(\cdot, 0), u_1, z_0) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega) \times \mathcal{L}$$

such that $\frac{\partial z_0}{\partial p} \in \mathcal{L}$, then the solution of (2.1) is classical

$$w \in C\left(\mathbb{R}_+, H^2(\Omega)\right) \cap C^1\left(\mathbb{R}_+, H_0^1(\Omega)\right) \cap C^2\left(\mathbb{R}_+, L^2(\Omega)\right) \quad \text{and} \quad z \in C^1\left(\mathbb{R}_+, \mathcal{L}\right). \tag{2.8}$$

Here, $D(\mathcal{A})$ is endowed with the graph norm

$$\|\mathcal{W}\|_{D(\mathcal{A})} = \|\mathcal{W}\|_{\mathcal{H}} + \|\mathcal{A}\mathcal{W}\|_{\mathcal{H}}.$$

Proof: As $\mathcal{B}$ is Lipschitz continuous, the sum $\mathcal{A}+\mathcal{B}$ generates a linear C_0-semigroup on $\mathcal{H}$ if we can prove that $\mathcal{A}$ is a maximal monotone operator (see [11]). This suffices to deduce the well-posedness results stated in Theorem 2.1 (see [4,11,17]).

Let us show, first, that the operator $\mathcal{A}$ is dissipative. Clearly, we have, for any $V \in D(\mathcal{A})$,

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} = a_0 \left\langle \nabla v_2 - \frac{|a_2|\, c_0}{2a_0}\nabla v_1, \nabla v_1 \right\rangle + \langle a_0 \Delta v_1 - (a_1 + |a_2|)\, v_2 - a_2 v_3(1), v_2\rangle - |a_2| \left\langle \frac{\partial v_3}{\partial p}, v_3 \right\rangle_{\mathcal{L}}$$

$$= -\frac{|a_2|\, c_0}{2}\|\nabla v_1\|^2 - (a_1 + |a_2|)\, \|v_2\|^2 - a_2 \langle v_3(1), v_2\rangle - |a_2| \left\langle \frac{\partial v_3}{\partial p}, v_3 \right\rangle_{\mathcal{L}},$$

and using Cauchy-Schwartz inequality, integrating with respect to p and exploiting the equality $v_3(0) = v_1$ (according to the definition of $D(\mathcal{A})$) allow us to write

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} \le -\frac{|a_2|\, c_0}{2}\|\nabla v_1\|^2 - (a_1 + |a_2|)\, \|v_2\|^2 + \frac{|a_2|}{2}\|v_3(1)\|^2 + \frac{|a_2|}{2}\|v_2\|^2 + \frac{|a_2|}{2}\left(\|v_1\|^2 - \|v_3(1)\|^2\right).$$

We conclude, by using Poincaré's inequality, that

$$\langle \mathcal{A}V, V\rangle_{\mathcal{H}} \le -a_1 \|v_2\|^2 \le 0. \tag{2.9}$$

Next, we claim that $Id - \mathcal{A}$ is surjective from $D(\mathcal{A})$ to $\mathcal{H}$ (Id denotes the identity operator); that is, for any $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, there exists a $V \in D(\mathcal{A})$ such that

$$(Id - \mathcal{A})V = F. \tag{2.10}$$

We need to show that

$$\begin{cases} v_1 - v_2 + \frac{|a_2| c_0}{2a_0} v_1 = f_1, \\ v_2 - a_0 \Delta v_1 + (a_1 + |a_2|)\, v_2 + a_2 v_3(1) = f_2, \\ v_3 + \frac{1}{\tau}\frac{\partial v_3}{\partial p} = f_3 \end{cases}$$

is solvable in $D(\mathcal{A})$. Clearly, the first equation is equivalent to

$$v_2 = \left(1 + \frac{|a_2| c_0}{2a_0}\right) v_1 - f_1. \tag{2.11}$$

Moreover, $v_3(0) = v_1$ and the third equation in the previous system give

$$v_3 = e^{-p\tau}\left(v_1 + \tau \int_0^p f_3(y) e^{\tau y} dy\right). \tag{2.12}$$

Therefore, the second equation becomes

$$-a_0 \Delta v_1 + \left[(1 + a_1 + |a_2|)\left(1 + \frac{|a_2| c_0}{2a_0}\right) - a_2 e^{-\tau}\right] v_1 = (1 + a_1 + |a_2|)\, f_1 \tag{2.13}$$

$$+ f_2 - \tau a_2 e^{-\tau} \int_0^1 f_3(y) e^{\tau y} dy.$$

Now, we prove the existence of a unique solution

$$v_1 \in H^2(\Omega) \cap H_0^1(\Omega) \tag{2.14}$$

of (2.13). First, observing that

$$\tilde{f} := (1 + a_1 + |a_2|)\, f_1 + f_2 - \tau a_2 e^{-\tau} \int_0^1 f_3(y) e^{\tau y} dy \in L^2(\Omega) \tag{2.15}$$

and

$$\tilde{a} := (1 + a_1 + |a_2|)\left(1 + \frac{|a_2|c_0}{2a_0}\right) - a_2 e^{-\tau} \geq (1 + a_1)\left(1 + \frac{|a_2|c_0}{2a_0}\right) > 0. \tag{2.16}$$

Second, assuming that (2.13) has a solution satisfying (2.14), then, multiplying (2.13) by $\varphi \in H_0^1(\Omega)$, integrating by parts and using the boundary condition $\varphi = 0$ on Γ, we remark that v_1 is necessarily a solution of the variational formulation

$$P(v_1, \varphi) = \tilde{P}(\varphi), \quad \forall \varphi \in H_0^1(\Omega), \tag{2.17}$$

where P is a bilinear form on $H_0^1(\Omega) \times H_0^1(\Omega)$ given by

$$P(v_1, \varphi) = a_0 \langle \nabla v_1, \nabla \varphi \rangle_{(L^2(\Omega))^N} + \tilde{a} \langle v_1, \varphi \rangle$$

and $\tilde{P}$ is a linear form on $H_0^1(\Omega)$ defined by

$$\tilde{P}(\varphi) = \left\langle \tilde{f}, \varphi \right\rangle.$$

According to (2.16) and because $a_0 > 0$, it is clear that P is continuous and coercive, and moreover, thanks to (2.15), $\tilde{P}$ is continuous. Then, the Lax-Milgram theorem implies that (2.17) has a unique solution

$$v_1 \in H_0^1(\Omega). \tag{2.18}$$

By considering in (2.17) the particular test function $\varphi \in C_c^\infty(\Omega)$, integrating by parts and using the density of $C_c^\infty(\Omega)$ in $L^2(\Omega)$, we get (2.13). Therefore, thanks to (2.15) and (2.18), we observe that (2.13) leads to $\Delta v_1 \in L^2(\Omega)$, then $v_1 \in H^2(\Omega)$, and so (2.14) holds. Consequently, by appealing to Lax-Milgram theorem and performing some classical regularity arguments for the variational formulation (2.17), we get the existence of a unique solution $v_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ of (2.13). Therefore, (2.11) and (2.12) imply that $v_2 \in H_0^1(\Omega)$ and $v_3, \frac{\partial v_3}{\partial p} \in \mathcal{L}$. Finally, $V \in D(\mathcal{A})$. □

2.2. Presence of impulses

Let H be a Hilbert space. We introduce the spaces

$$PC(\mathbb{R}_+, H) = \left\{ \begin{array}{c} v \in C(\mathbb{R}_+ \backslash \{t_k\}_{k \in \mathbb{N}^*}, H),\ v(x, t_k^+),\ v(x, t_k^-) \text{ exist} \\ \text{and } v(x, t_k) = v(x, t_k^+) = g_k(v(x, t_k^-)),\ k \in \mathbb{N}^* \end{array} \right\}$$

and

$$PC^1(\mathbb{R}_+, H) = \left\{ \begin{array}{c} v \in C^1(\mathbb{R}_+ \backslash \{t_k\}_{k \in \mathbb{N}^*}, H),\ v(x, t_k^+),\ v(x, t_k^-),\ v_t(x, t_k^+),\ v_t(x, t_k^-) \text{ exist}, \\ v(x, t_k) = v(x, t_k^+) = g_k(v(x, t_k^-)) \text{ and } v_t(x, t_k) = v_t(x, t_k^+) = f_k(v_t(x, t_k^-)),\ k \in \mathbb{N}^* \end{array} \right\}.$$

Theorem 2.2 *For any*

$$(u_0(\cdot, 0), u_1, z_0) \in H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{L}, \tag{2.19}$$

the problem (1.4) *admits a unique weak solution*

$$u \in PC(\mathbb{R}_+, H_0^1(\Omega)) \cap PC^1(\mathbb{R}_+, L^2(\Omega)). \tag{2.20}$$

Proof: Let $J_k = [t_k, t_{k+1})$, for $k \in \mathbb{N}$. Let w_0 be the solution of (2.1) corresponding to the initial data (2.19) and define our solution u on $J_0 = [t_0, t_1) = [0, t_1)$ by $u(x, t) = w_0(x, t) = w_0(x, t - t_0)$. Then, thanks to (2.7),

$$\left(w_0(\cdot, t_1^-), \frac{\partial w_0}{\partial t}(\cdot, t_1^-), w_0(\cdot, t_1^- - \tau p) \right) \in H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{L}$$

(noticing that $w_0(\cdot, t_1^- - \tau p)$ makes a sense because $t_1^- - \tau p \geq t_1 - \tau = t_1 - t_0 - \tau > 0$ according to (1.1)), and therefore, according to (1.2) and (1.3),

$$\left(g_1(w_0(\cdot, t_1^-)), f_1\left(\frac{\partial w_0}{\partial t}(\cdot, t_1^-)\right), z_1 \right) \in H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{L}, \tag{2.21}$$

where $z_1(x,p) = g_1\left(w_0(x, t_1^- - \tau p)\right)$. So let w_1 be the solution of (2.1) corresponding to the initial data (2.21) instead of $(u_0(\cdot,0), u_1, z_0)$ and define our solution u on J_1 by $u(x,t) = w_0(x, t - t_1)$. Then

$$\left(w_1(\cdot, (t_2 - t_1)^-), \frac{\partial w_1}{\partial t}(\cdot, (t_2 - t_1)^-), w_1(\cdot, (t_2 - t_1)^- - \tau p) \right) \in H_0^1(\Omega) \times L^2(\Omega) \times \mathcal{L}.$$

Thus, by induction argument, one can consider w_k, $k \in \mathbb{N}^* \backslash \{1\}$, as the solution of (2.1) corresponding to the initial data

$$\left(g_k(w_{k-1}(x, (t_k - t_{k-1})^-)), f_k\left(\frac{\partial w_{k-1}}{\partial t}(x, (t_k - t_{k-1})^-)\right), z_k \right)$$

instead of $(u_0(\cdot,0), u_1, z_0)$, where

$$z_k(x,p) = g_k\left(w_{k-1}(x, (t_k - t_{k-1})^- - \tau p)\right)$$

(z_k exists because $(t_k - t_{k-1})^- - \tau p > 0$, for $k \in \mathbb{N}^* \backslash \{1\}$, thanks to (1.1)). Hence, the function

$$u(x,t) = \sum_{k \in \mathbb{N}} 1_{J_k}(t) w_k(x, t - t_k) \tag{2.22}$$

satisfies (1.4) and (2.20), where 1_{J_k} denotes the caracteristic function of J_k; that is

$$u(t,x) = \begin{cases} w_0(x,t), & (x,t) \in \Omega \times J_0 = \Omega \times [0, t_1), \\ w_1(x, t - t_1), & (x,t) \in \Omega \times J_1 = \Omega \times [t_1, t_2), \\ . \\ . \\ . \\ w_k(x, t - t_k), & (x,t) \in \Omega \times J_k = \Omega \times [t_k, t_{k+1}), \\ . \\ . \\ . \end{cases}$$

The proof is complete. □

3. Stability

We define the corresponding energy E to (1.4) as follows:

$$E(t) = \frac{a_0}{2}\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{|a_2|}{2}\int_{t-\tau}^{t} \|\nabla u(s)\|^2 ds, \quad t \in \mathbb{R}_+. \tag{3.1}$$

It is clear that we have the equivalence

$$\alpha_2 E_0(t) \leq E(t) \leq \alpha_1 E_0(t), \quad t \in \mathbb{R}_+, \tag{3.2}$$

where $\alpha_1 = \frac{1}{2}\max\{a_0, 1\}$, $\alpha_2 = \frac{1}{2}\min\{a_0, 1\}$ and

$$E_0(t) = \|\nabla u\|^2 + \|u_t\|^2 + |a_2|\int_{t-\tau}^{t} \|\nabla u(s)\|^2 ds, \quad t \in \mathbb{R}_+. \tag{3.3}$$

Furthermore, the next estimate of the derivative of E holds.

Lemma 3.1 *The energy functional E satisfies*

$$E'(t) \leq -\left(a_1 - \frac{c_0|a_2|}{2}\right)\|u_t\|^2 + \frac{|a_2|}{2}\|\nabla u\|^2, \quad t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}^*}. \tag{3.4}$$

Proof: Let $t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}^*}$. It is clear that

$$\frac{|a_2|}{2}\left(\int_{t-\tau}^{t}\|\nabla u(s)\|^2 ds\right)_t = \frac{|a_2|}{2}\left(\|\nabla u\|^2 - \|\nabla u(t-\tau)\|^2\right). \tag{3.5}$$

By differentiating with respect to t, integrating by parts and using $(1.4)_1$ and $(1.4)_2$, we obtain

$$\begin{aligned} \frac{1}{2}\left(a_0\|\nabla u\|^2 + \|u_t\|^2\right)_t &= \int_\Omega (a_0\nabla u\cdot\nabla u_t + u_t u_{tt})\,dx \\ &= \int_\Omega u_t(u_{tt} - a_0\Delta u)dx \\ &= -a_1\|u_t\|^2 - a_2\int_\Omega u_t u(t-\tau)dx. \end{aligned} \tag{3.6}$$

By adding (3.5) and (3.6), we get

$$E'(t) = -a_1\|u_t\|^2 - a_2\int_\Omega u_t u(t-\tau)dx + \frac{|a_2|}{2}\left(\|\nabla u\|^2 - \|\nabla u(t-\tau)\|^2\right),$$

then, applying Young's and Poincaré's inequalities to $u_t u(t-\tau)$ and $\|u(t-\tau)\|^2$, respectively, the desired inequality (3.4) holds. □

The following lemma gives rise to a differential inequality to which an implicit version of the Halanay inequality will be applied.

Lemma 3.2 *Let $\lambda_0 > 0$ and*

$$L(t) = \lambda_0 E(t) + \left(\frac{a_1}{2}\|u\|^2 + \int_\Omega u_t u dx\right), \quad t \in \mathbb{R}_+. \tag{3.7}$$

Then, there exist M_1, M_2, c_1, $c_2 > 0$ (independent of a_2) such that

$$M_1 E(t) \leq L(t) \leq M_2 E(t), \quad t \in \mathbb{R}_+ \tag{3.8}$$

and

$$L'(t) \leq -c_1 L(t) + c_2|a_2| \sup_{t-\tau\leq s\leq t} L(s), \quad t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}^*}. \tag{3.9}$$

Proof: Let $t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}^*}$. By differentiating with respect to t, integrating by parts and using $(1.4)_1$ and $(1.4)_2$, we obtain

$$\begin{aligned} \left(\tfrac{a_1}{2}\|u\|^2 + \int_\Omega u_t u dx\right)_t &= a_1\int_\Omega u u_t dx + \|u_t\|^2 + \int_\Omega u\left(a_0\Delta u - a_1 u_t - a_2 u(t-\tau)\right)dx \\ &= \|u_t\|^2 - a_0\|\nabla u\|^2 - a_2\int_\Omega u(t)u(t-\tau)dx, \end{aligned}$$

then, applying Young's and Poincaré's inequalities to the last integral in the above identity, we find

$$\left(\frac{a_1}{2}\|u\|^2 + \int_\Omega u_t u dx\right)_t \leq -\left(a_0 - \frac{|a_2|}{2}\right)\|\nabla u\|^2 + \|u_t\|^2 + \frac{c_0^2}{2}|a_2|\|\nabla u(t-\tau)\|^2. \tag{3.10}$$

Combining (3.4) and (3.10), we get

$$L'(t) \leq -a_0\|\nabla u\|^2 - (\lambda_0 a_1 - 1)\|u_t\|^2 + \frac{|a_2|}{2}\left[(1+\lambda_0)\|\nabla u\|^2 + \lambda_0 c_0\|u_t\|^2 + c_0^2\|\nabla u(t-\tau)\|^2\right]. \tag{3.11}$$

Then, we see that, for any λ_0 satisfying

$$\lambda_0 > \max\left\{\frac{a_0+1}{a_1}, \frac{c_0(a_1+1)}{2\alpha_2}, \frac{1}{2\alpha_2}\right\}, \tag{3.12}$$

one can conclude from (3.11) that

$$L'(t) \leq -a_0 E_0(t) + \frac{|a_2|}{2}\left[(1+\lambda_0)\|\nabla u\|^2 + \lambda_0 c_0\|u_t\|^2 + c_0^2\|\nabla u(t-\tau)\|^2 + 2a_0\int_{t-\tau}^{t}\|\nabla u(s)\|^2 ds\right].$$

Therefore, using (3.2), we get

$$L'(t) \leq -\frac{a_0}{\alpha_1}E(t) + \frac{|a_2|}{2}\left[(1+\lambda_0)\|\nabla u\|^2 + \lambda_0 c_0\|u_t\|^2 + c_0^2\|\nabla u(t-\tau)\|^2\right] + a_0|a_2|\int_{t-\tau}^{t}\|\nabla u(s)\|^2 ds. \tag{3.13}$$

Because

$$\int_{t-\tau}^{t}\|\nabla u(s)\|^2 ds \leq \tau \sup_{t-\tau\leq s\leq t}\|\nabla u(s)\|^2 \tag{3.14}$$

and employing (3.2) again, we deduce from (3.13) and (3.14) that, for $\lambda_1 = \frac{1}{2\alpha_2}\max\{\lambda_0 c_0, c_0^2, 2\tau a_0, 1+\lambda_0\}$,

$$L'(t) \leq -\frac{a_0}{\alpha_1}E(t) + |a_2|\lambda_1 \sup_{t-\tau\leq s\leq t} E(s), \quad t \in \mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}^*}. \tag{3.15}$$

On the other hand, utilizing (3.2) and applying Young's and Poincaré's inequalities, we see that

$$|L(t) - \lambda_0 E(t)| \leq \frac{c_0}{2}(a_1+1)\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2 \leq \frac{1}{2\alpha_2}\max\{c_0(a_1+1), 1\}E(t), \quad t \in \mathbb{R}_+.$$

Thus, by fixing λ_0 satisfying (3.12), we get (3.8) with

$$M_1 = \lambda_0 - \frac{1}{2\alpha_2}\max\{c_0(a_1+1), 1\} \quad \text{and} \quad M_2 = \lambda_0 + \frac{1}{2\alpha_2}\max\{c_0(a_1+1), 1\}.$$

Consequently, (3.8) and (3.15) lead to (3.9) with

$$c_1 = \frac{a_0}{\alpha_1 M_2} \quad \text{and} \quad c_2 = \frac{\lambda_1}{M_1}$$

where we observe that the constants M_1, M_2, c_1 and c_2 do not depend on a_2 because α_1, α_2 and λ_0 do not depend on a_2. □

Before stating our theorem, let us recall the following impulsive version of the Halanay inequality [8].

Lemma 3.3 *[8] Let $\{t_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}_+$ be such that*

$$t_0 := 0 < t_1 < \cdots < t_k < \cdots \quad \text{and} \quad \lim_{k\to+\infty} t_k = +\infty.$$

Let $\{a_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}_+$, $\{b_k\}_{k\in\mathbb{N}^*} \subset \mathbb{R}_+$, $a > 0$, $b \geq 0$, $\delta > 1$ and $\tau > 0$ be such that

$$a > b \quad \text{and} \quad \inf_{k\in\mathbb{N}^*}\{t_{k+1} - t_k\} > \delta\tau. \tag{3.16}$$

Let η be the unique solution of the equation $\eta = a - be^{\eta\tau}$ and $r_k = \max\{1, a_k + b_k e^{\eta\tau}\}$ such that there exist M, $\rho > 0$ satisfying

$$r_0 r_1 \cdots r_{k+1} e^{k\eta\tau} \leq M e^{\rho(t_k - t_0)}, \quad k \in \mathbb{N}^*. \tag{3.17}$$

Let h be a nonnegative function continuous except at the jump discontinuity points $\{t_k\}_{k\in\mathbb{N}}$ solution of

$$\begin{cases} h'(t) \le -ah(t) + b \sup_{t-\tau\le s\le t} h(s), \quad t\in\mathbb{R}_+\backslash\{t_k\}_{k\in\mathbb{N}^*}, \\ h(t_k) \le a_k h(t_k^-) + b_k \sup_{t_k^- -\tau\le s\le t_k^-} h(s), \quad k\in\mathbb{N}^*. \end{cases}$$

Then

$$h(t) \le M \sup_{t_0-\tau\le s\le t_0} h(s) e^{-(\eta-\rho)(t-t_0)}, \quad t\in\mathbb{R}_+.$$

Moreover, for $\omega = \sup_{k\in\mathbb{N}^*}\{1, a_k + b_k e^{\eta\tau}\}$, we have

$$h(t) \le \omega \sup_{t_0-\tau\le s\le t_0} h(s) e^{-\left(\eta - \frac{\ln[\omega e^{\eta\tau}]}{\delta\tau}\right)(t-t_0)}, \quad t\in\mathbb{R}_+.$$

Theorem 3.1 *Assume that (3.17) holds, where $a = c_1$, $b = c_2|a_2|$, c_1 and c_2 are defined in Lemma 3.2, $a_k = \tilde{C}_k$, $\tilde{C}_k$ is defined in (3.20) below and $b_k = 0$. Assume that*

$$|a_2| < \frac{c_1}{c_2}, \tag{3.18}$$

so (3.16) is satisfied. Then, there exist c_3, $c_4 > 0$ such that

$$E(t) \le c_3 e^{-c_4 t}, \quad t\in\mathbb{R}_+. \tag{3.19}$$

Proof: As we have assumed that the hypotheses of Lemma 3.3 hold, it suffices only to check the second set of conditions in the system in Lemma 3.3. By exploiting (1.2), (1.3) and $(1.4)_4$, we see that, for any $k\in\mathbb{N}^*$,

$$\|\nabla u(t_k)\|^2 = \|\nabla(g_k(u(t_k^-)))\|^2 \le \|g_k'\|_\infty^2 \|\nabla u(t_k^-)\|^2 \le \xi_k^2 \|\nabla u(t_k^-)\|^2,$$

$$\|u_t(t_k)\|^2 = \|f_k(u_t(t_k^-))\|^2 \le \tilde{\xi}_k^2 \|u_t(t_k^-)\|^2$$

and

$$\int_{t_k-\tau}^{t_k} \|\nabla u(s)\|^2 ds = \int_{t_k^- -\tau}^{t_k^-} \|\nabla u(s)\|^2 ds.$$

Then

$$E_0(t_k) \le C_k E_0(t_k^-),$$

for some $C_k > 0$, and, using (3.2) and (3.8), we get

$$L(t_k) \le \tilde{C}_k L(t_k^-), \tag{3.20}$$

for some $\tilde{C}_k > 0$. Consequently, according to Lemma 3.3 (with $h = L$), (3.8), (3.9) and (3.20) lead to (3.19). □

4. Time-delayed impulses

Let $a_3\in\mathbb{R}$ and $\tau_1 \ge \tau_2 > 0$ such that (1.1) holds with $\tau = 2\tau_1$. Let $g_{j,k}$, $f_{j,k} : \mathbb{R}\to\mathbb{R}$, $j = 1, 2$ and $k\in\mathbb{N}^*$, satisfying, for some $\xi_{j,k}$, $\tilde{\xi}_{j,k} > 0$,

$$g_{j,k}\in C^1(\mathbb{R}_+), \quad g_{j,k}(0) = 0, \quad \sup_{k\in\mathbb{N}^*}\xi_{j,k} < +\infty \quad \text{and} \quad |g_{j,k}'| \le \xi_{j,k}, \tag{4.1}$$

and

$$f_{j,k}\in C(\mathbb{R}_+), \quad \sup_{k\in\mathbb{N}^*}\tilde{\xi}_{j,k} < +\infty \quad \text{and} \quad |f_{j,k}(s)| \le \tilde{\xi}_{j,k}|s|, \quad s\in\mathbb{R}. \tag{4.2}$$

We consider the following system:

$$\begin{cases} u_{tt}(x,t) - a_0\Delta u(x,t) + a_1 u_t(x,t) + a_2 u(x,t-\tau_1) \\ \qquad\qquad +a_3 u_t(x,t-\tau_2) = 0, & x\in\Omega,\ t\in\mathbb{R}_+^*\backslash\{t_k\}_{k\in\mathbb{N}^*}, \\ u(x,t)=0, & x\in\Gamma,\ t>0, \\ u(x,s)=u_0(x,s), & x\in\overline{\Omega},\ s\in[-\tau_1,0], \\ u_t(x,s)=u_1(x,s), & x\in\overline{\Omega},\ s\in[-\tau_2,0], \\ u(x,t_k)=g_{1,k}(u(x,t_k^-))+g_{2,k}(u(x,t_k^- -\tau_1)), & x\in\overline{\Omega},\ k\in\mathbb{N}^*, \\ u_t(x,t_k)=f_{1,k}(u_t(x,t_k^-))+f_{2,k}(u_t(x,t_k^- -\tau_2)), & x\in\overline{\Omega},\ k\in\mathbb{N}^*. \end{cases} \tag{4.3}$$

System (4.3) is subject to a frictional damping of coefficient a_1 and two discrete time delays τ_1 and τ_2 in the state and its velocity, respectively. The impulses are considered in both $(u(x,t_k^-),u_t(x,t_k^-))$ and $(u(x,t_k^- -\tau_1),u_t(x,t_k^- -\tau_2))$ thanks to the functions $g_{j,k}$ and $f_{j,k}$.

4.1. Well-posedness

The existence, uniqueness and smoothness of a solution for (4.3) may be proved as for (1.4). We consider the delayed wave equation without impulses related to (4.3)

$$\begin{cases} w_{tt}(x,t) - a_0\Delta w(x,t) + a_1 w_t(x,t) + a_2 w(x,t-\tau_1) \\ \qquad\qquad +a_3 w_t(x,t-\tau_2) = 0, & x\in\Omega,\ t>0, \\ w(x,t)=0, & x\in\Gamma,\ t>0, \\ w(x,s)=u_0(x,s), & x\in\overline{\Omega},\ s\in[-\tau_1,0], \\ w_t(x,s)=u_1(x,s), & x\in\overline{\Omega},\ s\in[-\tau_2,0], \end{cases} \tag{4.4}$$

Let $\mathcal{W}=(w,w_t,z,\eta)^T$, $\mathcal{W}_0=(u_0(\cdot,0),u_1,z_0,\eta_0)^T\in\mathcal{H}$,

$$\begin{cases} z(x,t,p)=w(x,t-\tau_1 p), & x\in\Omega,\ t>0,\ p\in(0,1), \\ \eta(x,t,p)=w_t(x,t-\tau_2 p), & x\in\Omega,\ t>0,\ p\in(0,1), \\ z_0(x,p)=z(x,0,p)=u_0(x,-\tau_1 p), & x\in\Omega,\ p\in(0,1), \\ \eta_0(x,p)=\eta(x,0,p)=u_1(x,-\tau_2 p), & x\in\Omega,\ p\in(0,1), \end{cases} \tag{4.5}$$

$$\mathcal{H}=H_0^1(\Omega)\times L^2(\Omega)\times\mathcal{L}\times\mathcal{L},$$

$\mathcal{L}$ is defined in (2.3) and, for $V=(v_1,v_2,v_3,v_4)^T$ and $\tilde{V}=(\tilde{v}_1,\tilde{v}_2,\tilde{v}_3,\tilde{v}_4)^T$,

$$\left\langle V,\tilde{V}\right\rangle_{\mathcal{H}} = a_0\langle\nabla v_1,\nabla\tilde{v}_1\rangle + \langle v_2,\tilde{v}_2\rangle + \tau_1|a_2|\langle v_3,\tilde{v}_3\rangle_{\mathcal{L}} + \tau_2|a_3|\langle v_4,\tilde{v}_4\rangle_{\mathcal{L}}.$$

The variables z and η defined in (4.5) satisfy

$$\begin{cases} \tau_1 z_t(x,t,p)+z_p(x,t,p)=0, \\ \tau_2\eta_t(x,t,p)+\eta_p(x,t,p)=0, \\ z(x,t,0)=w(x,t), \quad \eta(x,t,0)=w_t(x,t). \end{cases} \tag{4.6}$$

The linear operators $\mathcal{A}$ and $\mathcal{B}$ are defined in this case by

$$\mathcal{A}\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix} = \begin{pmatrix} v_2-\frac{|a_2|c_0}{2a_0}v_1 \\ a_0\Delta v_1-(a_1+|a_2|+|a_3|)\,v_2-a_2v_3(1)-a_3v_4(1) \\ -\frac{1}{\tau_1}\frac{\partial v_3}{\partial p} \\ -\frac{1}{\tau_2}\frac{\partial v_4}{\partial p}\end{pmatrix}$$

and

$$\mathcal{B}\begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\end{pmatrix} = \begin{pmatrix} \frac{|a_2|c_0}{2a_0}v_1 \\ (|a_2|+|a_3|)\,v_2 \\ 0 \\ 0\end{pmatrix}$$

with $D(\mathcal{B})=\mathcal{H}$ and

$$D(\mathcal{A})=\left\{(v_1,v_2,v_3,v_4)^T\in\mathcal{H}: v_1\in H^2(\Omega),\, v_2\in H_0^1(\Omega),\, \frac{\partial v_3}{\partial p},\, \frac{\partial v_4}{\partial p}\in\mathcal{L},\, v_3(0)=v_1,\, v_4(0)=v_2\right\}.$$

Obviously, (4.4) may be rewritten in the abstract form (2.6).

Theorem 4.1 *For any*

$$(u_0(\cdot,0),u_1(\cdot,0),z_0,\eta_0)\in H_0^1(\Omega)\times L^2(\Omega)\times\mathcal{L}\times\mathcal{L}$$

(z_0 and η_0 are defined in (4.5)), (4.4) admits a unique weak solution

$$w\in C\left(\mathbb{R}_+,H_0^1(\Omega)\right)\cap C^1\left(\mathbb{R}_+,L^2(\Omega)\right)\quad \textit{and}\quad z,\,\eta\in C\left(\mathbb{R}_+,\mathcal{L}\right). \tag{4.7}$$

If

$$(u_0(\cdot,0),u_1(\cdot,0),z_0,\eta_0)\in\left(H^2(\Omega)\cap H_0^1(\Omega)\right)\times H_0^1(\Omega)\times\mathcal{L}\times\mathcal{L}$$

such that $\frac{\partial z_0}{\partial p},\frac{\partial \eta_0}{\partial p}\in\mathcal{L}$, then the solution of (4.4) is classical

$$w\in C\left(\mathbb{R}_+,H^2(\Omega)\right)\cap C^1\left(\mathbb{R}_+,H_0^1(\Omega)\right)\cap C^2\left(\mathbb{R}_+,L^2(\Omega)\right)\quad\textit{and}\quad z,\,\eta\in C^1\left(\mathbb{R}_+,\mathcal{L}\right). \tag{4.8}$$

Proof: As for (2.1), it suffices to prove that $\mathcal{A}$ is a maximal monotone operator. We have

$$\langle\mathcal{A}V,V\rangle_{\mathcal{H}}=a_0\left\langle\nabla v_2-\frac{|a_2|\,c_0}{2a_0}\nabla v_1,\nabla v_1\right\rangle+\langle a_0\Delta v_1-(a_1+|a_2|+|a_3|)\,v_2-a_2v_3(1)-a_3v_4(1),v_2\rangle$$
$$+\tau_1|a_2|\left\langle-\frac{1}{\tau_1}\frac{\partial v_3}{\partial p},v_3\right\rangle_{\mathcal{L}}+\tau_2|a_3|\left\langle-\frac{1}{\tau_2}\frac{\partial v_4}{\partial p},v_4\right\rangle_{\mathcal{L}}$$
$$=-\frac{|a_2|\,c_0}{2}\|\nabla v_1\|^2-(a_1+|a_2|+|a_3|)\,\|v_2\|^2-\langle a_2v_3(1)+a_3v_4(1),v_2\rangle-|a_2|\left\langle\frac{\partial v_3}{\partial p},v_3\right\rangle_{\mathcal{L}}-|a_3|\left\langle\frac{\partial v_4}{\partial p},v_4\right\rangle_{\mathcal{L}}.$$

Then, using Cauchy-Schwartz and Poincaré's inequalities, integrating with respect to p and exploiting the equalities

$$v_3(0)=v_1\quad\text{and}\quad v_4(0)=v_2 \tag{4.9}$$

(thanks to the definition of $D(\mathcal{A})$), (2.9) holds. Next, for any $F=(f_1,f_2,f_3,f_4)^T\in\mathcal{H}$, we prove that there exists a $V\in D(\mathcal{A})$ such that (2.10) is satisfied, which is equivalent to

$$\begin{cases} v_1-v_2+\frac{|a_2|c_0}{2a_0}v_1=f_1,\\ v_2-a_0\Delta v_1+(a_1+|a_2|)\,v_2+a_2v_3(1)+a_3v_4(1)=f_2,\\ v_3+\frac{1}{\tau_1}\frac{\partial v_3}{\partial p}=f_3,\\ v_4+\frac{1}{\tau_2}\frac{\partial v_4}{\partial p}=f_4.\end{cases} \tag{4.10}$$

We see that the first equation of (4.10) has the unique solution v_2 defined in (2.11). On the other hand, the unique solutions of the third and fourth equations of (4.10) satisfying (4.9) are

$$v_3=e^{-p\tau_1}\left(v_1+\tau_1\int_0^p f_3(y)e^{\tau_1 y}dy\right)\quad\text{and}\quad v_4=e^{-p\tau_2}\left(v_2+\tau_2\int_0^p f_4(y)e^{\tau_2 y}dy\right). \tag{4.11}$$

Therefore, using (2.11) and (4.11), we observe that the second equation of (4.10) is reduced to

$$-a_0\Delta v_1+\left[\left(1+a_1+|a_2|+|a_3|+a_3e^{-\tau_2}\right)\left(1+\frac{|a_2|c_0}{2a_0}\right)+a_2e^{-\tau_1}\right]v_1=\left(1+a_1+|a_2|+|a_3|+a_3e^{-\tau_2}\right)f_1$$
$$+f_2-\int_0^1\left(\tau_1a_2e^{-\tau_1}f_3(y)e^{\tau_1 y}+\tau_2a_3e^{-\tau_2}f_4(y)e^{\tau_2 y}\right)dy,$$

which is similar to (2.13). Then the proof of Theorem 4.1 may be ended as for Theorem 2.1. □

Now, for a given Hilbert space H, we put

$$PC(\mathbb{R}_+, H) = \left\{ \begin{array}{c} v \in C(\mathbb{R}_+ \backslash \{t_k\}_{k\in\mathbb{N}^*}, H),\ v(x,t_k^+),\ v(x,t_k^-)\text{ exist} \\ \text{and } v(x,t_k) = v(x,t_k^+) = g_{1,k}(v(x,t_k^-)) + g_{2,k}(v(x,t_k^- - \tau_1)),\ k\in\mathbb{N}^* \end{array} \right\}$$

and

$$PC^1(\mathbb{R}_+, H) = \left\{ \begin{array}{c} v \in C^1(\mathbb{R}_+ \backslash \{t_k\}_{k\in\mathbb{N}^*}, H),\ v(x,t_k^+),\ v(x,t_k^-),\ v_t(x,t_k^+),\ v_t(x,t_k^-)\text{ exist,} \\ v(x,t_k) = v(x,t_k^+) = g_{1,k}(v(x,t_k^-)) + g_{2,k}(v(x,t_k^- - \tau_1)) \\ \text{and } v_t(x,t_k) = v_t(x,t_k^+) = f_{1,k}(v_t(x,t_k^-)) + f_{2,k}(v_t(x,t_k^- - \tau_2)),\ k\in\mathbb{N}^* \end{array} \right\}.$$

We get the following well-posedness result for problem (4.3):

Theorem 4.2 *For any*

$$(u_0(\cdot,0), u_1(\cdot,0), z_0, \eta_0) \in H_0^1(\Omega)\times L^2(\Omega)\times\mathcal{L}\times\mathcal{L},$$

the problem (4.3) *admits a unique weak solution satisfying* (2.20).

Proof: Using Theorem 4.1 and proceeding as in Theorem 2.2, one can see that the unique solution of (4.3) satisfying (2.20) is the one defined in (2.22), where

$$g_{1,0}(s) = g_{2,0}(s) = f_{1,0}(s) = f_{2,0}(s) = s, \quad w_{-1}(x) = u_0(x,0), \quad \frac{\partial w_{-1}}{\partial t}(x) = u_1(x,0),$$

and, for $k\in\mathbb{N}$, $J_k = [t_k, t_{k+1})$ and w_k is the solution of (4.4) corresponding to the initial data (u_0, u_1, z_0, η_0) if $k = 0$, and the initial data

$$x \mapsto \Big[g_{1,k}(w_{k-1}(x,(t_k-t_{k-1})^-)) + g_{2,k}(w_{k-1}(x,(t_k-t_{k-1})^- - \tau_1)),$$
$$f_{1,k}\left(\frac{\partial w_{k-1}}{\partial t}(x,(t_k-t_{k-1})^-)\right) + f_{2,k}\left(\frac{\partial w_{k-1}}{\partial t}(x,(t_k-t_{k-1})^- - \tau_2)\right), z_k(x,\cdot), \eta_k(x,\cdot)\Big]$$

if $k\in\mathbb{N}^*$, where

$$z_k(x,p) = g_{1,k}\left(w_{k-1}(x,(t_k-t_{k-1})^- - \tau_1 p)\right) + g_{2,k}\left(w_{k-1}(x,(t_k-t_{k-1})^- - \tau_1 p - \tau_1)\right)$$

and

$$\eta_k(x,p) = f_{1,k}\left(\frac{\partial w_{k-1}}{\partial t}(x,(t_k-t_{k-1})^- - \tau_2 p)\right) + f_{2,k}\left(\frac{\partial w_{k-1}}{\partial t}(x,(t_k-t_{k-1})^- - \tau_2 p - \tau_2)\right),$$

which exist because

$$(t_k-t_{k-1})^- - \tau_j p > 0 \quad\text{and}\quad (t_k-t_{k-1})^- - \tau_j p - \tau_j > 0, \quad j = 1,2,$$

since we are assuming (1.1) with $\tau = 2\tau_1$ and $\tau_1 \geq \tau_2$. □

4.2. Stability

We define the corresponding energy E to (4.3) as follows:

$$E(t) = \frac{1}{2}a_0\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2 + \frac{1}{2}|a_2|\int_{t-\tau_1}^t \|\nabla u(s)\|^2 ds + \frac{1}{2}|a_3|\int_{t-\tau_2}^t \|u_t(s)\|^2 ds, \quad t\in\mathbb{R}_+. \tag{4.12}$$

It is clear that (3.2) holds with the same constants α_1 and α_2 and

$$E_0(t) = \|\nabla u\|^2 + \|u_t\|^2 + |a_2|\int_{t-\tau_1}^t \|\nabla u(s)\|^2 ds + |a_3|\int_{t-\tau_2}^t \|u_t(s)\|^2 ds, \quad t\in\mathbb{R}_+. \tag{4.13}$$

Lemma 4.1 *The energy functional E satisfies*

$$E'(t) \leq -\left[a_1 - \frac{1}{2}\left(c_0|a_2| + 2|a_3|\right)\right]\|u_t\|^2 + \frac{|a_2|}{2}\|\nabla u\|^2, \quad t \in \mathbb{R}_+^* \backslash \{t_k\}_{k\in\mathbb{N}^*}, \tag{4.14}$$

where c_0 is defined in (2.5).

Proof: Let $t \in \mathbb{R}_+ \backslash \{t_k\}_{k\in\mathbb{N}^*}$. As

$$\frac{|a_2|}{2}\frac{d}{dt}\left(\int_{t-\tau_1}^{t}\|\nabla u(s)\|^2 ds\right) = \frac{|a_2|}{2}\left(\|\nabla u\|^2 - \|\nabla u(t-\tau_1)\|^2\right) \tag{4.15}$$

and

$$\frac{|a_3|}{2}\frac{d}{dt}\left(\int_{t-\tau_2}^{t}\|u_t(s)\|^2 ds\right) = \frac{|a_3|}{2}\left(\|u_t\|^2 - \|u_t(t-\tau_2)\|^2\right) \tag{4.16}$$

a differentiation with respect to t, integration by part and use of $(4.3)_1$ and $(4.3)_2$, leads to

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(a_0\|\nabla u\|^2 + \|u_t\|^2\right) &= \int_\Omega \left(a_0 \nabla u \cdot \nabla u_t + u_t u_{tt}\right) dx \\ &= \int_\Omega u_t(u_{tt} - a_0\Delta u) dx \\ &= -a_1\|u_t\|^2 - a_2\int_\Omega u_t u(t-\tau_1)dx - a_3\int_\Omega u_t u_t(t-\tau_2)dx.\end{aligned} \tag{4.17}$$

By adding (4.15)-(4.17), we get

$$\begin{aligned}E'(t) &= -a_1\|u_t\| + \frac{|a_2|}{2}\left(\|\nabla u\|^2 - \|\nabla u(t-\tau_1)\|^2\right) + \frac{|a_3|}{2}\left(\|u_t\|^2 - \|u_t(t-\tau_2)\|^2\right) \\ &\quad -a_2\int_\Omega u_t u(t-\tau_1)dx - a_3\int_\Omega u_t u_t(t-\tau_2)dx,\end{aligned}$$

then, applying Young's and Poincaré's inequalities to the last two integrals, we find (4.14). □

Lemma 4.2 *Let $\lambda_0 > 0$ and L be the function defined in (3.7). Then there exist M_1, M_2, c_1, $c_2 > 0$ (independent of a_2 and a_3) such that (3.8) holds and*

$$L'(t) \leq -c_1 L(t) + c_2(|a_2| + |a_3|)\sup_{t-\tau_1\leq s\leq t} L(s), \quad t \in [\tau_1, +\infty)\backslash\{t_k\}_{k\in\mathbb{N}^*}. \tag{4.18}$$

Proof: Let $t \in [\tau_1, +\infty[\backslash\{t_k\}_{k\in\mathbb{N}^*}$ (notice that $\tau_1 = \max\{\tau_1, \tau_2\}$ by assumption). By differentiating with respect to t, integrating by part and using $(4.3)_1$ and $(4.3)_2$, we obtain

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}a_1\|u\|^2 + \int_\Omega u_t u dx\right) &= a_1\int_\Omega u u_t dx + \|u_t\|^2 + \int_\Omega u\left(a_0\Delta u - a_1 u_t - a_2 u(t-\tau_1) - a_3 u(t-\tau_2)\right) dx \\ &= \|u_t\|^2 - a_0\|\nabla u\|^2 - a_2\int_\Omega u(t)u(t-\tau_1)dx - a_3\int_\Omega u(t)u_t(t-\tau_2)dx,\end{aligned}$$

then, applying Young's and Poincaré's inequalities to the last two integrals in the above identity, we find

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}a_1\|u\|^2 + \int_\Omega u_t u dx\right) &\leq -\left[a_0 - \frac{1}{2}\left(|a_2| + |a_3|\right)\right]\|\nabla u\|^2 + \|u_t\|^2 \\ &\quad + \frac{c_0^2}{2}|a_2|\|\nabla u(t-\tau_1)\|^2 + \frac{c_0}{2}|a_3|\|u_t(t-\tau_2)\|^2.\end{aligned} \tag{4.19}$$

Combining (4.14) and (4.19), we get, for $M = \frac{1}{2}\max\{\lambda_0(c_0+2), c_0, c_0^2, 1\}$,

$$\begin{aligned} L'(t) &\leq -a_0\|\nabla u\|^2 - (\lambda_0 a_1 - 1)\|u_t\|^2 \\ &\quad + M(|a_2|+|a_3|)\left[\|\nabla u\|^2 + \|u_t\|^2 + \|\nabla u(t-\tau_1)\|^2 + \|u_t(t-\tau_2)\|^2\right]. \end{aligned} \tag{4.20}$$

Then we see that, for any λ_0 satisfying (3.12), one can conclude from (4.20) that

$$L'(t) \leq -a_0 E_0(t) + a_0|a_2|\int_{t-\tau_1}^{t} \|\nabla u(s)\|^2 ds + a_0|a_3|\int_{t-\tau_2}^{t} \|u_t(s)\|^2 ds$$

$$+M(|a_2|+|a_3|)\left(\|\nabla u\|^2 + \|u_t\|^2 + \|\nabla u(t-\tau_1)\|^2 + \|u_t(t-\tau_2)\|^2\right).$$

Employing (3.2), we entail

$$\begin{aligned} L'(t) &\leq -\frac{a_0}{\alpha_1}E(t) + a_0|a_2|\int_{t-\tau_1}^{t} \|\nabla u(s)\|^2 ds + a_0|a_3|\int_{t-\tau_2}^{t} \|u_t(s)\|^2 ds \\ &\quad + M(|a_2|+|a_3|)\left[\|\nabla u\|^2 + \|u_t\|^2 + \|\nabla u(t-\tau_1)\|^2 + \|u_t(t-\tau_2)\|^2\right]. \end{aligned} \tag{4.21}$$

Because

$$\int_{t-\tau_1}^{t} \|\nabla u(s)\|^2 ds \leq \tau_1 \sup_{t-\tau_1\leq s\leq t} \|\nabla u(s)\|^2 \leq \tau_1 \sup_{t-\tau_1\leq s\leq t} \|\nabla u(s)\|^2, \tag{4.22}$$

$$\int_{t-\tau_2}^{t} \|u_t(s)\|^2 ds \leq \tau_2 \sup_{t-\tau_2\leq s\leq t} \|u_t(s)\|^2 \leq \tau_2 \sup_{t-\tau_1\leq s\leq t} \|u_t(s)\|^2, \tag{4.23}$$

and using again (3.2), we deduce from (4.21) and (4.22) that

$$L'(t) \leq -\frac{a_0}{\alpha_1}E(t) + \frac{1}{\alpha_2}\left[2M + a_0(\tau_1+\tau_2)\right](|a_2|+|a_3|)\sup_{t-\tau_1\leq s\leq t} E(s), \quad t\in[\tau_1,+\infty)\backslash\{t_k\}_{k\in\mathbb{N}}. \tag{4.24}$$

On the other hand, using (3.2) and applying Young's and Poincaré's inequalities, we see that

$$|L(t) - \lambda_0 E(t)| \leq \frac{c_0}{2}(a_1+1)\|\nabla u\|^2 + \frac{1}{2}\|u_t\|^2 \leq \frac{1}{2\alpha_2}\max\{c_0(a_1+1), 1\}E(t), \quad t\in\mathbb{R}_+,$$

then, by fixing λ_0 satisfying (3.12), we get (3.8) with the same constants M_1 and M_2. Consequently, (3.8) and (4.21) lead to (4.18) with

$$c_1 = \frac{a_0}{\alpha_1 M_2} \quad \text{and} \quad c_2 = \frac{1}{\alpha_2 M_1}\left[2M + a_0(\tau_1+\tau_2)\right], \tag{4.25}$$

where we observe that M_1, M_2, c_1 and c_2 do not depend neither on a_2 nor on a_3 because α_1, α_2, M and λ_0 do not depend neither on a_2 nor on a_3. □

Theorem 4.3 *Let $k_0 = \min\{k\in\mathbb{N}^* : t_k \geq \tau_1\}$. Assume that (3.17) holds and*

$$|a_2| + |a_3| < \frac{c_1}{c_2}, \tag{4.26}$$

where c_1 and c_2 are given in (4.26), $a = c_1$, $b = c_2(|a_2|+|a_3|)$ and a_k and b_k are defined in (4.28) below. Then there exist c_3, $c_4 > 0$ such that

$$E(t) \leq c_3 e^{-c_4 t}, \quad t\in[t_{k_0}, +\infty). \tag{4.27}$$

Proof: Let $k \in \{k_0, k_0+1, \cdots\}$. By exploiting (4.1), (4.2) and $(4.3)_4$, we see that

$$\begin{aligned}
\|\nabla u(t_k)\|^2 &= \|\nabla(g_{1,k}(u(t_k^-))) + \nabla(g_{2,k}(u(t_k^- - \tau_1)))\|^2 \\
&\le 2\max\left\{\|g_{1,k}'\|_\infty^2, \|g_{2,k}'\|_\infty^2\right\}\left(\|\nabla u(t_k^-)\|^2 + \|\nabla u(t_k^- - \tau_1)\|^2\right) \\
&\le 2\max\left\{\xi_{1,k}^2, \xi_{2,k}^2\right\}\left(\|\nabla u(t_k^-)\|^2 + \|\nabla u(t_k^- - \tau_1)\|^2\right) \\
&\le 2\max\left\{\xi_{1,k}^2, \xi_{2,k}^2\right\}\left(\|\nabla u(t_k^-)\|^2 + \sup_{t_k^- - \tau_1 \le s \le t_k^-} \|\nabla u(s)\|^2\right),
\end{aligned}$$

$$\begin{aligned}
\|u_t(t_k)\|^2 &= \|f_{1,k}(u_t(t_k^-)) + f_{2,k}(u_t(t_k^- - \tau_2))\|^2 \\
&\le 2\max\left\{\tilde\xi_{1,k}^2, \tilde\xi_{2,k}^2\right\}\left(\|u_t(t_k^-)\|^2 + \|u_t(t_k^- - \tau_2)\|^2\right) \\
&\le 2\max\left\{\tilde\xi_{1,k}^2, \tilde\xi_{2,k}^2\right\}\left(\|u_t(t_k^-)\|^2 + \sup_{t_k^- - \tau_1 \le s \le t_k^-} \|u_t(s)\|^2\right),
\end{aligned}$$

$$\int_{t_k-\tau_1}^{t_k} \|\nabla u(s)\|^2 ds = \int_{t_k^- - \tau_1}^{t_k^-} \|\nabla u(s)\|^2 ds$$

and

$$\int_{t_k-\tau_2}^{t_k} \|u_t(s)\|^2 ds = \int_{t_k^- - \tau_2}^{t_k^-} \|u_t(s)\|^2 ds,$$

then, for $\tilde a_k = 2\max\left\{\frac{1}{2}, \xi_{1,k}^2, \xi_{2,k}^2, \tilde\xi_{1,k}^2, \tilde\xi_{2,k}^2\right\}$ and $\tilde b_k = 2\max\left\{\xi_{1,k}^2, \xi_{2,k}^2, \tilde\xi_{1,k}^2, \tilde\xi_{2,k}^2\right\}$,

$$E_0(t_k) \le \tilde a_k E_0(t_k^-) + \tilde b_k \sup_{t_k^- - \tau_1 \le s \le t_k^-} E_0(s),$$

then, using (3.2) and (3.8) and puting

$$a_k = \frac{\tilde a_k \alpha_1 M_2}{\alpha_2 M_1} \quad \text{and} \quad b_k = \frac{\tilde b_k \alpha_1 M_2}{\alpha_2 M_1}, \tag{4.28}$$

we deduce that

$$L(t_k) \le a_k L(t_k^-) + b_k \sup_{t_k^- - \tau_1 \le s \le t_k^-} L(s), \quad k \in \{k_0, k_0+1, \cdots\}. \tag{4.29}$$

Consequently, according to Lemma 3.3 (with $h = L$), (3.8), (4.18) and (4.29) lead to (4.27). □

5. Comments and issues

1. Notice here that the present situation, where the time delay is considered at least in u, is more delicate in terms of restrictions on coefficients as $\|u(\cdot, t-\tau)\|^2$ cannot be dealt with $\|u_t(\cdot, t)\|^2$. When no time delay is considered (i.e. $a_2 = a_3 = 0$), we see from (3.4) and (4.14) that $E' \le 0$, so (1.4) and (4.3) are dissipative.

2. Our results are valid under the smallness conditions (3.18) and (4.26) on the sizes a_2 and a_3 of the time delays. In the literature, similar smallness conditions were assumed (even when no impulses are considered and the time delay is only in u_t) in order to guarantee the dissipativity of the system (i.e. the energy functional is decreasing). In this work, we proved the exponential stability even if our system is not dissipative, and the smallness conditions (3.18) and (4.26) on a_2 and a_3 were used only to manage the impulses.

3. Our approach can be applied to other systems with impulses (like Euler-Bernoulli and Timoshenko beams, laminated beams with slip, Bresse systems, Porous-media systems, thermoelastic problems, ...).

4. Other types of time delays can be considered also for which our approach can be adapted like, for example, variable discrete time delays (i.e. the constants time delays τ_1 and τ_2 are replaced by functions $\tau_1(t)$ and $\tau_2(t)$), multiple variable delays $\tau_j(t)$ and a distributed time delay in u_t; that is

$$\int_0^t f(s)u_t(x, t-s)ds,$$

for a given function $f : \mathbb{R} \to \mathbb{R}$.

5. In the present work, we have imposed the condition

$$\inf_{k\in\mathbb{N}^*} \{t_{k+1} - t_k\} > \tau \tag{5.1}$$

to establish our results. As a matter of fact, in [19], Wang dropped the condition (5.1) and proved an exponential decay result assuming only

$$\inf_{k\in\mathbb{N}^*} \{t_{k+1} - t_k\} > 0.$$

Indeed, he showed that the condition (5.1) is unnecessary. His proof relied on an already existing impulsive Halanay type inequality (proved with an extra distributed delay term).

6. Our results relied essentially on some impulsive type Halanay inequalities. The relation (3.18) was dictated by a crucial assumption in these inequalities. Clearly, we cannot withdraw any conclusion without it. Actually, it is well known that Halanay inequality may fail if this assumption is violated. One may start from this point (see also examples in [7]) to come up with an example showing possible instability. Actually, this should not be difficult for the second problem as it was already done in [16] for a similar problem (without the delay in the state and the impulses which will normally act in favor of this phenomena). For the variable coefficients case (in the Halanay inequalities), however, it is already known that the said condition may be relaxed considerably by replacing it rather by an average condition.

7. The case when the discrete delay occurs in the boundary may be tackled similarly. The main point is to pass, via some embedding properties, from boundary norms to the space norms of terms involved or may be controlled by corresponding ones in the energy or its modification. For the second problem, the condition that the coefficient of the delayed time derivative of the state be dominated by the coefficient of the damping (see (4.26)) has been relaxed recently (see [1]). This was established utilizing a different approach than the usually used one employing change of function and augmenting the treated system by a transport equation.

8. In this work, it is shown that a simple frictional damping is enough to stabilize the system exponentially even in the presence of delays and impulses. An interesting question is whether this is still the case if we replace the frictional damping by an even weaker one. Namely, by considering an intermittent (on/off) frictional damping. This means that the damping is not active all the time but rather on some time subintervals. This question has been tackled by numerous researchers (for problems without delays) and especially by engineers due to its applications in many fields. Several results already exist in the market imposing some conditions on the values as well as the size of the subintervals where the damping is effective in order to obtain exponential stability. This would be one of our future directions of research. We think of discussing the case of a general class of functions that may encompass the on/off case as a special case.

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Aissa Guesmia,

Institut Elie Cartan de Lorraine, UMR 7502, Université de Lorraine,

3 Rue Augustin Fresnel, BP 45112, 57073 Metz Cedex 03, France.

E-mail address: `aissa.guesmia@univ-lorraine.fr`

and

Nasser-eddine Tatar, Department of Mathematics,

IRC for Intelligent Manufacturing and Robotics,

King Fahd University of Petroleum and Minerals,

Dhahran 31261, Saudi Arabia.

E-mail address: `tatarn@kfupm.edu.sa`