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On Riesz-Caputo Fractional Differential Problems via Fixed Point Theory and Measure of Noncompactness

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ABSTRACT: This article deals with the existence and stability results for a class of initial value problems involving the Riesz-Caputo fractional derivative. In this paper, we will study two types of fractional differential problem, the result of the first one is based on Schaefer's fixed point theorem. The same problem but in Banach spaces is investigated using Sadovskii's fixed point theorem. The third problem is an implicit problem with retarded and advanced arguments, the results are based on Sadovskii's fixed point theorem combined with the technique of measure of noncompactness. Some examples are given to validate our main results.

Key Words: Riesz-Caputo fractional derivative, implicit problem, existence, fixed point, measure of noncompactness, retarded argument, advanced argument, Ulam stability.

Contents

1. Introduction

Over the last two decades, fractional calculus has shown to be a critical tool for dealing with the complexity structures encountered in a variety of fields. Its theory and applications are broad, and it is concerned with the extension of integer order differentiation and integration to arbitrary order. For more details on the applications of fractional calculus, the reader is directed to the books of Herrmann [14], Kilbas *et al.* [16] and Samko *et al.* [29] and the papers [4–6,21,22]. In [1–3], Abbas *et al.* studied several problems with advanced fractional differential and integral equations and presented various applications. Agrawal [7] introduced some generalizations of fractional integrals and derivatives and present some of their properties. In [23–27], the authors demonstrated the existence results for some generalized Hilfer fractional differential equations.

While solving differential equations precisely is difficult or impossible in several situations, along with nonlinear analysis and optimization, we investigate approximate solutions. It is important to stress that only stable estimates are acceptable. As a result, numerous methodologies for stability analysis are employed. Ulam, a mathematician, first raised the stability issue in functional equations in a 1940 lecture at Wisconsin University. S.M. Ulam posed the question, "Under what conditions does an additive mapping

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exist near an approximately additive mapping?" [31]. The succeeding year, Hyers addressed Ulam's issue for additive functions defined on Banach spaces in [15]. Rassias [19] showed the presence of unique linear mappings close to approximation additive mappings in 1978, generalizing Hyers' results. In [18], Luo *et al.* presented some existence, uniqueness and Hyers–Ulam stability results of Caputo fractional difference equations employing some new criteria, the Brouwer theorem and the contraction mapping principle. The authors of [28] investigated the Ulam stabilities of a k-generalized ψ-Hilfer fractional differential problems. Significant attention has been paid to the research of the Ulam-Hyers and Ulam-Hyers-Rassias stability of all types of functional equations, as evidenced by the book of Abbas *et al.* [3], and the work of Luo *et al.* [18] and Rus [20], which explored the Ulam-Hyers stability for operatorial equations.

The main feature of the Riesz-Caputo fractional operator is that it uses both left and right fractional derivatives which is a very important property in the fractional modeling, see [9–11] for more details.

The authors of [9] studied the existence of solution for the following boundary value problem:

$$\begin{cases} {}^{RC}_{0}D^{\nu}_{\varkappa}\varphi(\theta) = g(\theta,\varphi(\theta)), \quad \theta \in \Theta := [0,\varkappa], \\ \varphi(0) = \varphi_0, \quad \varphi(\varkappa) = \varphi_{\varkappa}, \end{cases}$$

where ${}^{RC}_{0}D^{\nu}_{\varkappa}$ is a Riesz-Caputo derivative of order $0 < \nu \leq 1$, $g : \Theta \times \mathbb{R} \to \mathbb{R}$ a continuous function and $\varphi_0, \varphi_{\varkappa} \in \mathbb{R}$. Their arguments are based on Leray-Schauder fixed point theorem, and Schauder's fixed point theorem.

In [17], Li and Wang discussed the following fractional problem:

$$\begin{gathered} {}^{RC}_{0}D^{\gamma}_{1}\varphi(\theta) = \Psi(\theta,\varphi(\theta)), \quad \theta \in [0,1], \quad 0 < \gamma \leq 1, \\ \varphi(0) = a, \quad \varphi(1) = b\varphi(\eta), \end{gathered}$$

where ${}^{RC}_{0}D^{\gamma}_{1}$ is the Riesz-Caputo derivative, $\Psi \in C([0,1]\times [0,+\infty),[0,+\infty)), 0 < \eta < 1, a > 0, 0 < b < 2$. They found the positive solutions by applying the technique of monotone iterative.

In [25], the authors considered the boundary valued problem for the nonlinear implicit k-generalized ψ-Hilfer type fractional differential equation involving both retarded and advanced arguments:

$$\begin{cases} \left({}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{\kappa_1+}\varphi\right)(\theta) = \Psi\left(\theta, x_\theta(\cdot), \left({}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{\kappa_1+}\varphi\right)(\theta)\right), \quad \theta \in (\kappa_1,\kappa_2], \\ \nu_1\left(\mathcal{J}^{k(1-\xi),k;\psi}_{\kappa_1+}\varphi\right)({\kappa_1}^{+}) + \nu_2\left(\mathcal{J}^{k(1-\xi),k;\psi}_{\kappa_1+}\varphi\right)(\kappa_2) = \nu_3, \\ \varphi(\theta) = \varpi(\theta), \quad \theta \in [\kappa_1 - \wp, \kappa_1], \ \wp > 0, \\ \varphi(\theta) = \tilde{\varpi}(\theta), \quad \theta \in [\kappa_2, \kappa_2 + \tilde{\wp}], \ \tilde{\wp} > 0, \end{cases}$$

where ${}^{H}_{k}\mathcal{D}^{\vartheta,r;\psi}_{\kappa_1+}$ and $\mathcal{J}^{k(1-\xi_1),k;\psi}_{\kappa_1+}$ are, respectively, the k-generalized ψ-Hilfer fractional derivative of order $\vartheta \in (0,k)$ and type $r \in [0,1]$, and k-generalized ψ-fractional integral of order $k(1-\xi_1)$, $\xi_1 = \frac{1}{k}(r(k-\vartheta)+\vartheta)$, $k > 0$, $\Psi : [\kappa_1,\kappa_2] \times C([-\wp,\tilde{\wp}],\mathbb{R}) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a given function, and $\nu_1,\nu_2,\nu_3 \in \mathbb{R}$ such that $\nu_1+\nu_2 \neq 0$, and $\varpi(\theta)$ and $\tilde{\varpi}(\theta)$ are, respectively, continuous functions on $[\kappa_1 - \wp, \kappa_1]$ and $[\kappa_2, \kappa_2 + \tilde{\wp}]$. They base their arguments on the Banach contraction principle and Schauder's fixed point theorem.

Motivated by the mentioned works, firstly, we investigate the existence results of the following problem:

$${}^{RC}_{0}D^{\nu}_{\varkappa}\varphi(\theta) = \Psi(\theta,\varphi(\theta)), \quad \theta \in [0,\varkappa], \tag{1.1}$$

$$\varphi(0) = \varphi_0, \tag{1.2}$$

where ${}_{0}^{RC}D_{\varkappa}^{\nu}$ represents the Riesz-Caputo fractional derivative of order $\nu \in (0,1]$, $\Psi : [0,\varkappa] \times \mathbb{R} \to \mathbb{R}$ is a given function and $\varphi_0 \in \mathbb{R}$. Next, using Sadovskii's fixed point theorem, we establish the existence result for the following problem in Banach space:

$$ {}_{0}^{RC}D_{\varkappa}^{\nu}\varphi(\theta) = \Psi(\theta, \varphi(\theta)), \quad \theta \in [0,\varkappa], \tag{1.3} $$

$$ \varphi(0) = \varphi_0, \tag{1.4} $$

where $\Psi : [0,\varkappa] \times \Xi \to \Xi$ is a given function, $(\Xi, \|.\|)$ is a Banach space, and $\varphi_0 \in \Xi$.

Finally, we study the existence and stability of the following implicit problem with retarded and advanced arguments in a Banach space:

$$ {}_{0}^{RC}D_{\varkappa}^{\nu}\varphi(\theta) = \Psi(\theta, \varphi^{\theta}, {}_{0}^{RC}D_{\varkappa}^{\nu}\varphi(\theta)), \quad \theta \in [0,\varkappa], \tag{1.5} $$

$$ \varphi(\theta) = \phi(\theta), \quad \theta \in [-\widehat{\delta}, 0], \ \widehat{\delta} > 0, \tag{1.6} $$

$$ \varphi(\theta) = \psi(\theta), \quad \theta \in [\varkappa, \varkappa+\delta], \ \delta > 0, \tag{1.7} $$

where ${}_{0}^{RC}D_{\varkappa}^{\nu}$ is the Riesz-Caputo fractional derivative of order $\nu \in (0,1]$, $\Psi : [0,\varkappa] \times C([-\widehat{\delta}, \delta], \Xi) \times \Xi \to \Xi$ is a given function, $\phi \in C([-\widehat{\delta}, 0], \Xi)$, and $\psi \in C([\varkappa, \varkappa+\delta], \Xi)$. We denote by φ^{θ} the element of $C([-\widehat{\delta}, \delta], \Xi)$ defined by

$$ \varphi^{\theta} = \varphi(\theta + s) : s \in [-\widehat{\delta}, \delta]. $$

The present paper is organized as follows. Section 2 presents various notations as well as some preliminary facts on the Riesz-Caputo fractional derivative and supplementary findings. Section 3 presents, in the first part, an existence result to the system (1.1)-(1.2) based on Schaefer's fixed point theorem and in the second part, an existence result to the problem (1.3)-(1.4) based on Sadovskii's fixed point theorem combined with the technique of measure of noncompactness. In Section 4, the existence and Ulam-Hyers-Rassias stability results for problem (1.5)-(1.7) is discussed that are based on Sadovskii's fixed point theorem. For each problem, examples are presented to explain how our study results could be applied.

2. Preliminaries

In this section, we recall some notations, definitions and previous results which are used throughout this paper.
We denote by $C(\Theta, \Xi)$, where $\Theta = [0,\varkappa]$, the Banach space of all continuous functions from Θ to Ξ with the norm

$$ \|\varphi\|_{\infty} = \sup\{\|\varphi(\theta)\| : \theta \in \Theta\}. $$

Consider the following sets:

The Banach space $C([-\widehat{\delta}, 0], \Xi)$ with the norm

$$ \|\varphi\|_{[-\widehat{\delta},0]} = \sup\{\|\varphi(\theta)\| : \theta \in [-\widehat{\delta}, 0]\}, $$

the Banach space $C([\varkappa, \varkappa+\delta], \Xi)$ with the norm

$$ \|\varphi\|_{[\varkappa,\varkappa+\delta]} = \sup\{\|\varphi(\theta)\| : \theta \in [\varkappa, \varkappa+\delta]\}, $$

and the Banach space $C([-\widehat{\delta}, \delta], \Xi)$ with the norm

$$ \|\varphi\|_{[-\widehat{\delta},\delta]} = \sup\{\|\varphi(\theta)\| : \theta \in [-\widehat{\delta}, \delta]\}. $$

Let

$$ \Upsilon = \Big\{ \varphi : [-\widehat{\delta}, \varkappa+\delta] \to \Xi : \varphi|_{[0,\varkappa]} \in C(\Theta, \Xi), \varphi|_{[-\widehat{\delta},0]} \in C([-\widehat{\delta}, 0] $$

$$\text{and } \varphi|_{[\varkappa,\varkappa+\delta]} \in C([\varkappa,\varkappa+\delta],\Xi)\Big\}.$$

We note that Υ is a Banach space with the norm

$$\|\varphi\|_{\Upsilon} = \sup_{\theta\in[-\widehat{\delta},\varkappa+\delta]} \|\varphi(\theta)\|.$$

Definition 2.1 ([16]) *Let $\nu > 0$. The left and right Riemann-Liouville fractional integrals of a function $\varphi \in C(\Theta,\Xi)$ of order ν are given respectively by*

$${}_0I_\theta^\nu\varphi(\theta) = \frac{1}{\Gamma(\nu)}\int_0^\theta (\theta-\varrho)^{\nu-1}\varphi(\varrho)d\varrho,$$

and

$${}_\theta I_\varkappa^\nu\varphi(\theta) = \frac{1}{\Gamma(\nu)}\int_\theta^\varkappa (\varrho-\theta)^{\nu-1}\varphi(\varrho)d\varrho.$$

Definition 2.2 ([16]) *Let $\nu > 0$. The Riesz fractional integral of a function $\varphi \in C(\Theta,\Xi)$ of order ν is defined by*

$$\begin{aligned} {}_0I_\varkappa^\nu\varphi(\theta) &= \frac{1}{\Gamma(\nu)}\int_0^\varkappa |\theta-\varrho|^{\nu-1}\varphi(\varrho)d\varrho \\ &= {}_0I_\theta^\nu\varphi(\theta) + {}_\theta I_\varkappa^\nu\varphi(\theta), \end{aligned}$$

where ${}_0I_\theta^\nu$ and ${}_\theta I_\varkappa^\nu$ are the left and right fractional integrals of Riemann-Liouville.

Definition 2.3 ([16]) *Let $\nu \in (\gamma,\gamma+1]$, $\gamma \in \mathbb{N}_0$. The Caputo fractional derivatives of a function $\varphi \in C^{\gamma+1}(\Theta,\Xi)$ of order ν is given by*

$${}_0^CD_\theta^\nu\varphi(\theta) = \frac{1}{\Gamma(\gamma+1-\nu)}\int_0^\theta (\theta-\varrho)^{\gamma-\nu}\varphi^{(\gamma+1)}(\varrho)d\varrho,$$

and

$${}_\theta^CD_\varkappa^\nu\varphi(\theta) = \frac{(-1)^{\gamma+1}}{\Gamma(\gamma+1-\nu)}\int_\theta^\varkappa (\varrho-\theta)^{\gamma-\nu}\varphi^{(\gamma+1)}(\varrho)d\varrho.$$

Definition 2.4 ([16]) *Let $\nu \in (\gamma,\gamma+1]$, $\gamma \in \mathbb{N}_0$. The Riesz-Caputo fractional derivative of a function $\varphi \in C^{\gamma+1}(\Theta,\Xi)$ of order ν is given by*

$$\begin{aligned} {}_0^{RC}D_\varkappa^\nu\varphi(\theta) &= \frac{1}{\Gamma(\gamma+1-\nu)}\int_0^\varkappa |\theta-\varrho|^{\gamma-\nu}\varphi^{(\gamma+1)}(\varrho)d\varrho \\ &= \frac{1}{2}({}_0^CD_\theta^\nu\varphi(\theta) + (-1)^{\gamma+1}{}_\theta^CD_\varkappa^\nu\varphi(\theta)), \end{aligned}$$

where ${}_0^CD_\theta^\nu$ is the left Caputo derivative and ${}_\theta^CD_\varkappa^\nu$ is the right one.

If we take $0 < \nu \leq 1$ and $\varphi \in C(\Theta,\Xi)$, we obtain

$${}_0^{RC}D_\varkappa^\nu\varphi(\theta) = \frac{1}{2}({}_0^CD_\theta^\nu\varphi(\theta) - {}_\theta^CD_\varkappa^\nu\varphi(\theta)).$$

Lemma 2.1 ([16]) *If $\varphi \in C^{\gamma+1}(\Theta,\Xi)$ and $\nu \in (\gamma,\gamma+1]$, then we have*

$${}_0I_\theta^\nu\,{}_0^CD_\theta^\nu\varphi(\theta) = \varphi(\theta) - \sum_{k=0}^{\gamma}\frac{\varphi^{(k)}(0)}{k!}\theta^k,$$

and

$$ {}_{\theta}I^{\nu}_{\varkappa}\, {}^{C}_{\theta}D^{\nu}_{\varkappa}\varphi(\theta) = (-1)^{\gamma+1}\left[\varphi(\theta) - \sum_{k=0}^{\gamma}\frac{(-1)^{k}\varphi^{(k)}(\varkappa)}{k!}(\varkappa-\theta)^{k}\right]. $$

Consequently, we may have

$$ {}_{0}I^{\nu}_{\varkappa}\, {}^{RC}_{0}D^{\nu}_{\varkappa}\varphi(\theta) = \frac{1}{2}({}_{0}I^{\nu}_{\theta}\, {}^{C}_{0}D^{\nu}_{\theta}\varphi(\theta) + (-1)^{\gamma+1}\, {}_{\theta}I^{\nu}_{\varkappa}\, {}^{C}_{\theta}D^{\nu}_{\varkappa}\varphi(\theta)). $$

In particular, if $0 < \nu \leq 1$, *then we obtain*

$$ {}_{0}I^{\nu}_{\varkappa}\, {}^{RC}_{0}D^{\nu}_{\varkappa}\varphi(\theta) = \varphi(\theta) - \frac{1}{2}(\varphi(0) + \varphi(\varkappa)). $$

2.1. Measure of Noncompactness

Definition 2.5 ([8]) *Let* $\mho$ *be a Banach space and let* $\digamma_{\mho}$ *be the family of bounded subsets of* $\mho$. *The Kuratowski measure of noncompactness is the map* $\zeta : \digamma_{\mho} \longrightarrow [0, \infty)$ *defined by*

$$ \zeta(\chi) = inf\left\{\varsigma > 0 : \chi \subset \bigcup_{\jmath=1}^{m}\chi_{\jmath}, diam(\chi_{\jmath}) \leq \varsigma\right\}, $$

where $\chi \in \digamma_{\mho}$.

The map ζ satisfies the following properties:

- $\zeta(\chi) = 0 \Leftrightarrow \overline{\chi}$ is compact (χ is relatively compact).
- $\zeta(\chi) = \zeta(\overline{\chi})$.
- $\chi_1 \subset \chi_2 \Rightarrow \zeta(\chi_1) \leq \zeta(\chi_2)$.
- $\zeta(\chi_1 + \chi_2) \leq \zeta(\digamma_1) + \zeta(\digamma_2)$.
- $\zeta(c\chi) = |c|\zeta(\chi)$, $c \in \mathbb{R}$.
- $\zeta(conv\chi) = \zeta(\chi)$.

Lemma 2.2 ([12]) *Let* $\digamma \subset \Upsilon$ *be a bounded and equicontinuous set. Then,*

a) *The function* $\theta \to \zeta(\digamma(\theta))$ *is continuous on* Θ, *and*

$$ \zeta_{\Upsilon}(\digamma) = \sup_{\theta\in[-\widehat{\delta},\varkappa+\delta]}\zeta(\digamma(\theta)), \quad \zeta_{C}(\digamma) = \sup_{\theta\in[0,\varkappa]}\zeta(\digamma(\theta)), $$

b) $\zeta\left(\int_0^{\varkappa}\varphi(\varrho)d\varrho : \varphi \in \digamma\right) \leq \int_0^{\varkappa}\zeta(\digamma(\varrho))d\varrho$, *where*

$$ \digamma(\theta) = \{\varphi(\theta) : \varphi \in \digamma, \theta \in \Theta\}. $$

Definition 2.6 ([8]) *Let* $\mho$ *be a Banach space and* $\mathcal{H} : \mho \longrightarrow \mho$ *a continuous mapping.* $\mathcal{H}$ *is said to be a condensing mapping if for each bounded set* $\Bbbk$ *with* $\zeta(\Bbbk) \neq 0$, *we have*

$$ \zeta(\mathcal{H}(\Bbbk)) < \zeta(\Bbbk). $$

2.2. Some Fixed Point Theorems

Theorem 2.1 (Schaefer's fixed point theorem [30]) *Let $\mho$ be a Banach space, and $\mathcal{H} : \mho \longrightarrow \mho$ a continuous and completely continuous operator. If the set*

$$\{\varphi \in \mho : \varphi = \varpi \mathcal{H}\varphi \text{ for some } \varpi \in (0,1)\}$$

is bounded, then $\mathcal{H}$ has at least one fixed point in $\mho$.

Theorem 2.2 (Sadovskii's fixed point Theorem [13]) *Let D be a non-empty, closed, bounded and convex subset of a Banach space $\mho$, and let $\mathcal{H}$ be a condensing mapping of D. Then $\mathcal{H}$ has a fixed point in D.*

3. Existence Results

3.1. Existence Results of the problem (1.1)-(1.2)

Consider the following fractional differential problem:

$$ {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta) = \mu(\theta), \quad \theta \in [0,\varkappa], \tag{3.1}$$

$$\varphi(0) = \varphi_0, \tag{3.2}$$

where $\mu : \Theta \to \mathbb{R}$ is a continuous function.

Lemma 3.1 *Let $\nu \in (0,1]$, and $\mu : \Theta \to \mathbb{R}$ be continuous function. The problem (3.1)-(3.2) has a unique solution given by*

$$\varphi(\theta) = \varphi_0 - \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} \varrho^{\nu-1}\mu(\varrho)d\varrho + \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} |\theta-\varrho|^{\nu-1}\mu(\varrho)d\varrho. \tag{3.3}$$

Proof. Suppose that φ satisfies (3.1)-(3.2). Then from Lemma 2.1, we have

$${}_0I_{\varkappa}^{\nu}\, {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta) = \varphi(\theta) - \frac{1}{2}(\varphi(0)+\varphi(\varkappa)),$$

this implies that

$$\begin{aligned}\varphi(\theta) &= \frac{1}{2}(\varphi(0)+\varphi(\varkappa)) + \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} |\theta-\varrho|^{\nu-1}\mu(\varrho)d\varrho \\ &= \frac{1}{2}(\varphi(0)+\varphi(\varkappa)) + \frac{1}{\Gamma(\nu)}\int_0^{\theta} (\theta-\varrho)^{\nu-1}\mu(\varrho)d\varrho + \frac{1}{\Gamma(\nu)}\int_{\theta}^{\varkappa} (\varrho-\theta)^{\nu-1}\mu(\varrho)d\varrho.\end{aligned}$$

For $\theta = 0$, we have

$$\varphi(\varkappa) = \varphi_0 - \frac{2}{\Gamma(\nu)}\int_0^{\varkappa} \varrho^{\nu-1}\mu(\varrho)d\varrho.$$

Thus,

$$\varphi(\theta) = \varphi_0 - \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} \varrho^{\nu-1}\mu(\varrho)d\varrho + \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} |\theta-\varrho|^{\nu-1}\mu(\varrho)d\varrho.$$

Conversely, we can easily show by Lemma 2.1, that if φ satisfies (3.3), then it satisfies the equation (3.1) and the condition (3.2).

Definition 3.1 *By a solution of problem (1.1)-(1.2) we mean a function $\varphi \in C(\Theta,\mathbb{R})$ that satisfies the equation (1.1) and the condition (1.2).*

Lemma 3.2 *Let $\Psi : \Theta \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Then the problem (1.1)-(1.2) is equivalent to the following integral equation:*

$$\varphi(\theta) = \varphi_0 - \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho + \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho.$$

We are now in a position to prove the existence result of the problem (1.1)-(1.2) based on Schaefer's fixed point theorem.

Let us put the following conditions:

(*D1*) The function $\theta \longmapsto \Psi(\theta, \varpi_1)$ is measurable on Θ for each $\varpi_1 \in \mathbb{R}$ and the function $\varpi_1 \longmapsto \Psi(\theta, \varpi_1)$ is continuous on $\mathbb{R}$ for almost each $\theta \in \Theta$.

(*D2*) There exist continuous functions $\xi_1, \xi_2 : \Theta \to \mathbb{R}_+$ such that

$$|\Psi(\theta, \varpi_1)| \leq \xi_1(\theta) + \xi_2(\theta)|\varpi_1|, \text{ for any } \varpi_1 \in \mathbb{R} \text{ and } \theta \in \Theta.$$

Set ${\xi_1}^* = \sup_{\theta \in \Theta} \xi_1(\theta)$ and ${\xi_2}^* = \sup_{\theta \in \Theta} \xi_2(\theta)$.

Theorem 3.1 *Assume that the assumptions* ($D1$)-($D2$) *hold. If*

$$\frac{2{\xi_2}^* \varkappa^{\nu}}{\Gamma(\nu+1)} < 1, \tag{3.4}$$

then the problem (1.1)-(1.2) has at least one solution on Θ.

Proof. Consider the operator $\aleph : C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ defined by:

$$\aleph\varphi(\theta) = \varphi_0 - \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho + \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho.$$

Clearly the fixed points of the operator $\aleph$ are solutions of the problem (1.1)-(1.2). The proof will be given in several steps.

Step 1: The operator $\aleph : C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ is continuous.
Let $\{\varphi_\gamma\}_{\gamma \in \mathbb{N}}$ be a sequence such that $\varphi_\gamma \longrightarrow \varphi$ in $C(\Theta, \mathbb{R})$. Then, for each $\theta \in [0, \varkappa]$, we have

$$\begin{aligned} |\aleph\varphi_\gamma(\theta) - \aleph\varphi(\theta)| &\leq \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} |\Psi(\varrho, \varphi_\gamma(\varrho)) - \Psi(\varrho, \varphi(\varrho))| d\varrho \\ &+ \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} |\Psi(\varrho, \varphi_\gamma(\varrho)) - \Psi(\varrho, \varphi(\varrho))| d\varrho. \end{aligned}$$

Since $\varphi_\gamma \longrightarrow \varphi$ as $\gamma \longrightarrow \infty$ and Ψ is continuous, then by the Lebesgue dominated convergence theorem, we get

$$|\aleph\varphi_\gamma(\theta) - \aleph\varphi(\theta)| \longrightarrow 0 \quad as \ \gamma \longrightarrow \infty,$$

hence,

$$\|\aleph\varphi_\gamma - \aleph\varphi\|_\infty \longrightarrow 0 \quad as \ \gamma \longrightarrow \infty.$$

Which implies that $\aleph$ is continuous.

Let $R > 0$ and define the ball

$$\widehat{\Bbbk}_R = \{\varphi \in C(\Theta, \mathbb{R}) : \|\varphi\|_\infty \leq R\}.$$

Step 2: $\aleph$ maps bounded sets to bounded sets in $C(\Theta,\mathbb{R})$.
Let $\varphi \in \widehat{\Bbbk}_R$ and $\theta \in [0,\varkappa]$. Then, we have

$$\begin{aligned}|\aleph\varphi(\theta)| &\leq |\varphi_0| + \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} \varrho^{\nu-1}|\Psi(\varrho,\varphi(\varrho))|d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} |\theta-\varrho|^{\nu-1}|\Psi(\varrho,\varphi(\varrho))|d\varrho.\end{aligned}$$

By hypothesis $(D2)$, we have

$$\begin{aligned}|\aleph\varphi(\theta)| &\leq |\varphi_0| + \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} \varrho^{\nu-1}(\xi_1(\varrho)+\xi_2(\varrho)|\varphi(\varrho)|)d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} |\theta-\varrho|^{\nu-1}(\xi_1(\varrho)+\xi_2(\varrho)|\varphi(\varrho)|)d\varrho \\ &\leq |\varphi_0| + \frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}d\varrho + \frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}d\varrho \\ &\leq |\varphi_0| + \frac{2({\xi_1}^*+{\xi_2}^*R)\varkappa^{\nu}}{\Gamma(\nu+1)}.\end{aligned}$$

Thus, $\aleph$ maps bounded sets to bounded sets.

Step 3: $\aleph$ is equicontinuous.
Let $\theta_1,\theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $\varphi \in \widehat{\Bbbk}_R$. Then, we have

$$\begin{aligned}|\aleph\varphi(\theta_2)-\aleph\varphi(\theta_1)| &= \Bigg| -\frac{1}{\Gamma(\nu)}\int_0^{\theta_2}\varrho^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho - \frac{1}{\Gamma(\nu)}\int_{\theta_2}^{\varkappa}\varrho^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_0^{\theta_2}(\theta_2-\varrho)^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho + \frac{1}{\Gamma(\nu)}\int_{\theta_2}^{\varkappa}(\varrho-\theta_2)^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_0^{\theta_1}\varrho^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho + \frac{1}{\Gamma(\nu)}\int_{\theta_1}^{\varkappa}\varrho^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho \\ &- \frac{1}{\Gamma(\nu)}\int_0^{\theta_1}(\theta_1-\varrho)^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho - \frac{1}{\Gamma(\nu)}\int_{\theta_1}^{\varkappa}(\varrho-\theta_1)^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho\Bigg| \\ &\leq \frac{2}{\Gamma(\nu)}\int_{\theta_1}^{\theta_2}\varrho^{\nu-1}|\Psi(\varrho,\varphi(\varrho))|d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_0^{\theta_1}[(\theta_2-\varrho)^{\nu-1}-(\theta_1-\varrho)^{\nu-1}]|\Psi(\varrho,\varphi(\varrho))|d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_{\theta_1}^{\theta_2}(\theta_2-\varrho)^{\nu-1}|\Psi(\varrho,\varphi(\varrho))|d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_{\theta_2}^{\varkappa}[(\varrho-\theta_2)^{\nu-1}-(\varrho-\theta_1)^{\nu-1}]|\Psi(\varrho,\varphi(\varrho))|d\varrho \\ &+ \frac{1}{\Gamma(\nu)}\int_{\theta_1}^{\theta_2}(\varrho-\theta_1)^{\nu-1}|\Psi(\varrho,\varphi(\varrho))|d\varrho \\ &\leq \frac{2({\xi_1}^*+{\xi_2}^*R)}{\Gamma(\nu+1)}(\theta_2^{\nu}-\theta_1^{\nu}) + \frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu+1)}(\theta_2^{\nu}-\theta_1^{\nu}) \\ &+ \frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu)}\int_{\theta_1}^{\theta_2}(\theta_2-\varrho)^{\nu-1}d\varrho + \frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu+1)}[(\varkappa-\theta_2)^{\nu}-(\varkappa-\theta_1)^{\nu}] \\ &+ \frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu)}(\theta_2-\theta_1)^{\nu},\end{aligned}$$

as $\theta_1 \longrightarrow \theta_2$, the right-hand side of the above inequality tend to zero. According to the three steps and the Arzela-Ascoli theorem, we can conclude that the operator $\aleph$ is completely continuous.

Step 4: A priori bounds. Now we show that the set

$$\digamma = \{\varphi \in C(\Theta, \mathbb{R}) : \varphi = \kappa \aleph \varphi \ for\ some\ 0 < \kappa < 1\}$$

is bounded. Let $\varphi \in \digamma$, then for each $\theta \in \Theta$, we have

$$\begin{aligned} \varphi(\theta) &= \kappa \aleph \varphi(\theta) \\ &= \kappa \varphi_0 - \frac{\kappa}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho \\ &+ \frac{\kappa}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho. \end{aligned}$$

Then,

$$\begin{aligned} |\varphi(\theta)| &\leq |\varphi_0| + \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} |\Psi(\varrho, \varphi(\varrho))| d\varrho \\ &+ \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} |\Psi(\varrho, \varphi(\varrho))| d\varrho \\ &\leq |\varphi_0| + \frac{{\xi_1}^*}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} d\varrho + \frac{{\xi_1}^*}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} d\varrho \\ &\leq |\varphi_0| + \frac{2({\xi_1}^* + {\xi_2}^* \|\varphi\|_\infty) \varkappa^\nu}{\Gamma(\nu+1)}. \end{aligned}$$

Then,

$$\|\varphi\|_\infty \leq \frac{|\varphi_0| + \frac{2{\xi_1}^* \varkappa^\nu}{\Gamma(\nu+1)}}{1 - \frac{2{\xi_2}^* \varkappa^\nu}{\Gamma(\nu+1)}} := K.$$

Thus,

$$\|\varphi\|_\infty \leq K.$$

This shows that $\digamma$ is a bounded set. As a consequence of Schaefer's fixed point theorem, the operator $\aleph$ has at least a fixed point which is solution of the problem (1.1)-(1.2).

3.2. Existence Results of the problem (1.3)-(1.4)

Definition 3.2 *By a solution of problem* (1.3)-(1.4) *we mean a function* $\varphi \in C(\Theta, \Xi)$ *that satisfies the equation* (1.3) *and the condition* (1.4).

Lemma 3.3 *Let* $\Psi : \Theta \times \Xi \to \Xi$ *be a continuous function. Then the problem* (1.3)-(1.4) *is equivalent to the following integral equation:*

$$\varphi(\theta) = \varphi_0 - \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} \varrho^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho + \frac{1}{\Gamma(\nu)} \int_0^{\varkappa} |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho.$$

We are now in a position to prove the existence result of the problem (1.3)-(1.4) based on the concept of measure of noncompactness and Sadovskii's fixed point theorem.

Let us put the following conditions:

(*H1*) The function $\theta \longmapsto \Psi(\theta, \varpi_1)$ is measurable on Θ for each $\varpi_1 \in \Xi$ and the function $\varpi_1 \longmapsto \Psi(\theta, \varpi_1)$ is continuous on Ξ for almost each $\theta \in \Theta$.

(*H2*) There exist continuous functions $\xi_1, \xi_2 : \Theta \to \mathbb{R}_+$ such that

$$\|\Psi(\theta, \varpi_1)\| \leq \xi_1(\theta) + \xi_2(\theta)\|\varpi_1\|, \text{ for any } \varpi_1 \in \Xi \text{ and } \theta \in \Theta.$$

(*H3*) For each $\theta \in \Theta$ and bounded sets $\Bbbk_1 \subset \Xi$, we have

$$\zeta(\Psi(\theta, \Bbbk_1)) \leq \xi_2(\theta)\zeta(\Bbbk_1),$$

where ζ is the Kuratowski measure of noncompactnes of the Banach space Ξ. Set ${\xi_1}^* = \sup\limits_{\theta\in\Theta} \xi_1(\theta)$ and ${\xi_2}^* = \sup\limits_{\theta\in\Theta} \xi_2(\theta)$.

Theorem 3.2 *Assume that the assumptions* $(H1)$-$(H3)$ *hold. If*

$$\frac{2{\xi_2}^* \varkappa^\nu}{\Gamma(\nu+1)} < 1, \tag{3.5}$$

then the problem (1.3)-(1.4) *has at least one solution on* Θ*.*

Proof. Consider the operator $\aleph : C(\Theta, \Xi) \longrightarrow C(\Theta, \Xi)$ defined by:

$$\aleph\varphi(\theta) = \varphi_0 - \frac{1}{\Gamma(\nu)} \int_0^\varkappa \varrho^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho + \frac{1}{\Gamma(\nu)} \int_0^\varkappa |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi(\varrho)) d\varrho.$$

Clearly the fixed points of the operator $\aleph$ are solutions of the problem (1.3)-(1.4). The proof will be given in several steps.

Step 1: The operator $\aleph : C(\Theta, \Xi) \longrightarrow C(\Theta, \Xi)$ is continuous.
Let $\{\varphi_\gamma\}_{\gamma\in\mathbb{N}}$ be a sequence such that $\varphi_\gamma \longrightarrow \varphi$ in $C(\Theta, \Xi)$. Then, for each $\theta \in [0, \varkappa]$, we have

$$\begin{aligned} \|\aleph\varphi_\gamma(\theta) - \aleph\varphi(\theta)\| &\leq \frac{1}{\Gamma(\nu)} \int_0^\varkappa \varrho^{\nu-1} \|\Psi(\varrho, \varphi_\gamma(\varrho)) - \Psi(\varrho, \varphi(\varrho))\| d\varrho \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa |\theta - \varrho|^{\nu-1} \|\Psi(\varrho, \varphi_\gamma(\varrho)) - \Psi(\varrho, \varphi(\varrho))\| d\varrho. \end{aligned}$$

Since $\varphi_\gamma \longrightarrow \varphi$ as $\gamma \longrightarrow \infty$ and Ψ is continuous, then by the Lebesgue dominated convergence theorem, we get

$$\|\aleph\varphi_\gamma(\theta) - \aleph\varphi(\theta)\| \longrightarrow 0 \quad as\ \gamma \longrightarrow \infty.$$

Hence,

$$\|\aleph\varphi_\gamma - \aleph\varphi\|_\infty \longrightarrow 0 \quad as\ \gamma \longrightarrow \infty.$$

Thus, $\aleph$ is continuous.

Let $R > 0$ such that

$$R \geq \frac{\|\varphi_0\| + \frac{2{\xi_1}^* \varkappa^\nu}{\Gamma(\nu+1)}}{1 - \frac{2{\xi_2}^* \varkappa^\nu}{\Gamma(\nu+1)}}.$$

Define the ball

$$\Bbbk_R = \{\varphi \in C(\Theta, \Xi) : \|\varphi\|_\infty \leq R\}.$$

Step 2: $\aleph(\Bbbk_R) \subset \Bbbk_R$.
Let $\varphi \in \Bbbk_R$ and $\theta \in [0, \varkappa]$. Then, we have

$$|\aleph\varphi(\theta)| \quad \leq \quad \|\varphi_0\| + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \varrho^{\nu-1} \|\Psi(\varrho, \varphi(\varrho))\| d\varrho$$

$$+\quad \frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\|\Psi(\varrho,\varphi(\varrho))\|d\varrho.$$

By hypothesis $(H2)$, we have

$$\begin{aligned}
\|\aleph\varphi(\theta)\| &\leq \|\varphi_0\|+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}(\xi_1(\varrho)+\xi_2(\varrho)\|\varphi(\varrho)\|)d\varrho \\
&+ \frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}(\xi_1(\varrho)+\xi_2(\varrho)\|\varphi(\varrho)\|)d\varrho \\
&\leq \|\varphi_0\|+\frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}d\varrho+\frac{{\xi_1}^*+{\xi_2}^*R}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}d\varrho \\
&\leq \|\varphi_0\|+\frac{2({\xi_1}^*+{\xi_2}^*R)\varkappa^{\nu}}{\Gamma(\nu+1)}.
\end{aligned}$$

Thus,

$$\|\aleph\varphi\|_{\infty}\leq R.$$

Consequently, $\aleph$ maps $\Bbbk_R$ to $\Bbbk_R$.

Step 3: We may apply the same steps as in the previous problem to demonstrate that $\aleph(\Bbbk_R)$ is equicontinuous. Now, we show that $\aleph(\Bbbk_R)$ is condensing.
Let G be a subset of $\Bbbk_R$. For each $\theta\in[0,\varkappa]$, we have

$$\begin{aligned}
\zeta(\aleph G(\theta)) &= \zeta\{\aleph\varphi(\theta),\quad \varphi\in G\} \\
&= \left\{\phi(0)-\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\Psi(\varrho,\varphi(\varrho))d\varrho,\quad \varphi\in G\right\} \\
&\leq \frac{1}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\{\zeta(\Psi(\varrho,\varphi(\varrho)))d\varrho,\quad \varphi\in G\} \\
&+ \frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\{\zeta(\Psi(\varrho,\varphi(\varrho)))d\varrho,\quad \varphi\in G\}.
\end{aligned}$$

From $(H3)$, we have

$$\begin{aligned}
\zeta(\aleph G(\theta)) &\leq \frac{{\xi_2}^*}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\{\zeta(\varphi(\varrho))d\varrho,\quad \varphi\in G\} \\
&+ \frac{{\xi_2}^*}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\{\zeta(\varphi(\varrho))d\varrho,\quad \varphi\in G\} \\
&\leq \left[\frac{2\varkappa^{\nu}{\xi_2}^*}{\Gamma(\nu+1)}\right]\zeta_C(G).
\end{aligned}$$

Therefore,

$$\zeta_C(\aleph G)\leq\left[\frac{2\varkappa^{\nu}{\xi_2}^*}{\Gamma(\nu+1)}\right]\zeta_C(G).$$

By condition (3.5), we have that $\aleph$ is a condensing operator. As a consequence of Sadovslii's fixed point theorem, the operator $\aleph$ has a fixed point which is a solution of the problem (1.5)-(1.7).

3.3. An Example

Consider the following fractional problem:

$$ {}^{RC}_{0}D^{\frac{1}{2}}_{1}\varphi(\theta)=\frac{e^{-\theta}\cos(\theta)}{100(1+|\varphi(\theta)|)},\qquad \theta\in[0,1], \tag{3.6}$$

$$\varphi(0) = 1. \tag{3.7}$$

Set

$$\Psi(\theta, \varpi_1) = \frac{e^{-\theta}\cos(\theta)}{100(1+|\varpi_1|)}, \quad \theta \in [0,1], \ \varpi_1 \in \mathbb{R}.$$

Clearly, the function Ψ satisfies the hypothesis $(D1)$. Also, we have

$$\begin{aligned} |\Psi(\theta, \varpi_1)| &= \left| \frac{e^{-\theta}\cos(\theta)}{100(1+|\varpi_1|)} \right| \\ &\leq \frac{e^{-\theta}|\cos(\theta)|}{100}(1+|\varpi_1|). \end{aligned}$$

Then, the condition $(D2)$ is satisfied with

$$\xi_1(\theta) = \xi_2(\theta) = \frac{e^{-\theta}|\cos(\theta)|}{100}.$$

Moreover, we have

$$\frac{2{\xi_2}^* \varkappa^{\nu}}{\Gamma(\nu+1)} = \frac{1}{25\sqrt{\pi}} \approx 0.0225675833419103.$$

Thus, the condition (3.4) is verified. Consequently, Theorem 3.1 implies that the problem (3.6)-(3.7) has at least one solution on [0,1].

4. Implicit Problem with Retarded and Advanced Arguments in Banach Space

4.1. Existence results

In this part, we will investigate the problem (1.5)-(1.7).

Definition 4.1 *By a solution of problem* (1.5)-(1.7) *we mean a function* $\varphi \in \Upsilon$ *that satisfies the equation* (1.5) *and the conditions* (1.6)-(1.7).

Lemma 4.1 *Let* $\Psi : \Theta \times C([-\widehat{\delta}, \delta], \Xi) \times \Xi \to \Xi$ *be a continuous function. Then the problem* (1.5)-(1.7) *is equivalent to the following integral equation:*

$$\varphi(\theta) = \begin{cases} \phi(0) - \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^{\varkappa} \varrho^{\nu-1} \Psi(\varrho, \varphi^{\varrho}, \widehat{\Psi}(\varrho)) d\varrho + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^{\varkappa} |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi^{\varrho}, \widehat{\Psi}(\varrho)) d\varrho, \\ \phi(\theta), \qquad if \ \theta \in [-\widehat{\delta}, 0], \\ \psi(\theta), \qquad if \ \theta \in [\varkappa, \varkappa+\delta], \end{cases}$$

where $\widehat{\Psi} \in C(\Theta, \Xi)$ *satisfies the following functional equation*

$$\widehat{\Psi}(\theta) = \Psi(\theta, \varphi^{\theta}, \widehat{\Psi}(\theta)).$$

Let us set the following assumptions:

(*T1*) The function $\theta \longmapsto \Psi(\theta, \varpi_1, \varpi_2)$ is measurable on Θ for each $\varpi_1 \in C([-\widehat{\delta}, \delta], \Xi)$, $\varpi_2 \in \Xi$ and the functions $\varpi_1 \longmapsto \Psi(\theta, \varpi_1, \varpi_2)$ and $\varpi_2 \longmapsto \Psi(\theta, \varpi_1, \varpi_2)$ are continuous on Ξ for almost each $\theta \in \Theta$.

(*T2*) There exist continuous functions $\xi_1, \xi_2 : \Theta \to \mathbb{R}_+$ such that

$$\|\Psi(\theta, \varpi_1, \varpi_2)\| \leq \xi_1(\theta)\|\varpi_1\|_{[-\widehat{\delta},\delta]} + \xi_2(\theta)\|\varpi_2\|,$$

for each $\theta \in \Theta$, $\varpi_1 \in C([-\widehat{\delta}, \delta], \Xi)$ and $\varpi_2 \in \Xi$.

(*T3*) For each $\theta \in \Theta$ and bounded sets $\mathbb{k}_1 \subset C([-\widehat{\delta}, \delta], \Xi)$, $\mathbb{k}_2 \subset \Xi$, we have

$$\zeta(\Psi(\theta, \mathbb{k}_1, \mathbb{k}_2)) \leq \xi_1(\theta) \sup_{\varrho \in [-\widehat{\delta}, \delta]} \zeta(\mathbb{k}_1(\varrho)) + \xi_2(\theta)\zeta(\mathbb{k}_2),$$

where ζ is the Kuratowski measure of noncompactnes of the Banach space Ξ. Let ${\xi_1}^* = \sup_{\theta \in \Theta} \xi_1(\theta)$ and ${\xi_2}^* = \sup_{\theta \in \Theta} \xi_2(\theta) < 1$.

Theorem 4.1 *Assume that the assumptions* $(T1)$-$(T3)$ *are verified. If*

$$\frac{2{\xi_1}^* \varkappa^\nu}{(1 - {\xi_2}^*)\Gamma(\nu + 1)} < 1,$$

then the problem (1.5)-(1.7) *has at least one solution.*

To prove the existence of solution of the problem (1.5)-(1.7), we will use Sadovskii's fixed point theorem.

Proof. Transform the problem (1.5)-(1.7) into a fixed point problem. Consider the operator $\aleph : C(\Theta, \mathbb{R}) \longrightarrow C(\Theta, \mathbb{R})$ defined by:

$$\aleph\varphi(\theta) = \begin{cases} \phi(0) - \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^\varkappa \varrho^{\nu-1} \Psi(\varrho, \varphi^\varrho, \widehat{\Psi}(\varrho)) d\varrho + \dfrac{1}{\Gamma(\nu)} \displaystyle\int_0^\varkappa |\theta - \varrho|^{\nu-1} \Psi(\varrho, \varphi^\varrho, \widehat{\Psi}(\varrho)) d\varrho, \\ \phi(\theta), \qquad if\ \theta \in [-\widehat{\delta}, 0], \\ \psi(\theta), \qquad if\ \theta \in [\varkappa, \varkappa + \delta], \end{cases}$$

where $\widehat{\Psi} \in C(\Theta, \Xi)$ satisfies the following functional equation

$$\widehat{\Psi}(\theta) = \Psi(\theta, \varphi^\theta, \widehat{\Psi}(\theta)).$$

The proof will be given in several steps.

Step 1: The operator $\aleph$ is continuous.
Let $\{\varphi_\gamma\}_{\gamma \in \mathbb{N}}$ be a sequence such that $\varphi_\gamma \longrightarrow \varphi$ in Υ. If $\theta \in [-\widehat{\delta}, 0]$ or $\theta \in [\varkappa, \varkappa + \delta]$, then

$$\|\aleph\varphi_\gamma(\theta) - \aleph\varphi(\theta)\| = 0.$$

If $\theta \in [0, \varkappa]$, we have

$$\begin{aligned} \|\aleph\varphi_\gamma(\theta) - \aleph\varphi(\theta)\| &\leq \frac{1}{\Gamma(\nu)} \int_0^\varkappa \varrho^{\nu-1} \|g_\gamma(\varrho) - \widehat{\Psi}(\varrho)\| d\varrho \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa |\theta - \varrho|^{\nu-1} \|g_\gamma(\varrho) - \widehat{\Psi}(\varrho)\| d\varrho. \end{aligned}$$

Since $\varphi_\gamma \longrightarrow \varphi$ as $\gamma \longrightarrow \infty$ and Ψ is continuous, then by the Lebesgue dominated convergence theorem, we get

$$\|g_\gamma(\theta) - \widehat{\Psi}(\theta)\| \longrightarrow 0 \quad as\ \gamma \longrightarrow \infty,$$

which implies that

$$\|\aleph\varphi_\gamma - \aleph\varphi\|_\Upsilon \longrightarrow 0 \quad as\ \gamma \longrightarrow \infty.$$

Hence, $\aleph$ is continuous.

Let $R > 0$ such that

$$R \geq \max\left\{ \frac{\|\phi(0)\|}{1 - \frac{2{\xi_1}^* \varkappa^\nu}{(1-{\xi_2}^*)\Gamma(\nu+1)}}, \|\phi\|_{[-\widehat{\delta},0]}, \|\psi\|_{[\varkappa,\varkappa+\delta]} \right\}.$$

Define the ball

$$\widehat{\mathbb{k}}_R = \{\varphi \in \Upsilon : \|\varphi\|_\Upsilon \leq R\}.$$

Step 2: $\aleph(\widehat{\mathbb{k}}_R) \subset \widehat{\mathbb{k}}_R$.
Let $\varphi \in \widehat{\mathbb{k}}_R$. If $\theta \in [-\widehat{\delta}, 0]$, then

$$\|\aleph\varphi(\theta)\| \leq \|\phi\|_{[-\widehat{\delta},0]} \leq R,$$

and if $\theta \in [\varkappa, \varkappa + \delta]$, then

$$\|\aleph\varphi(\theta)\| \leq \|\psi\|_{[\varkappa,\varkappa+\delta]} \leq R.$$

For each $\theta \in [0, \varkappa]$, we have

$$\begin{aligned} \|\aleph\varphi(\theta)\| &\leq \|\phi(0)\| + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \varrho^{\nu-1} \|\widehat{\Psi}(\varrho)\| d\varrho \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa |\theta - \varrho|^{\nu-1} \|\widehat{\Psi}(\varrho)\| d\varrho. \end{aligned}$$

By $(T2)$, we have

$$\begin{aligned} \|\widehat{\Psi}(\theta)\| &= \|\Psi(\theta, \varphi^\theta, \widehat{\Psi}(\theta))\| \\ &\leq \xi_1(\theta)\|\varphi^\theta\|_{[-\widehat{\delta},\delta]} + \xi_2(\theta)\|\widehat{\Psi}(\theta)\| \\ &\leq {\xi_1}^*\|\varphi^\theta\|_{[-\widehat{\delta},\delta]} + {\xi_2}^*\|\widehat{\Psi}(\theta)\|. \end{aligned}$$

Thus,

$$\|\widehat{\Psi}(\theta)\| \leq \frac{{\xi_1}^*}{1 - {\xi_2}^*}\|\varphi^\theta\|_{[-\widehat{\delta},\delta]}.$$

Then,

$$\begin{aligned} \|\aleph\varphi(\theta)\| &\leq \|\phi(0)\| + \frac{1}{\Gamma(\nu)} \int_0^\varkappa \varrho^{\nu-1} \frac{{\xi_1}^*}{1 - {\xi_2}^*}\|\varphi^\theta\|_{[-\widehat{\delta},\delta]} d\varrho \\ &+ \frac{1}{\Gamma(\nu)} \int_0^\varkappa |\theta - \varrho|^{\nu-1} \frac{{\xi_1}^*}{1 - {\xi_2}^*}\|\varphi^\theta\|_{[-\widehat{\delta},\delta]} d\varrho \\ &\leq \|\phi(0)\| + \frac{2{\xi_1}^* R \varkappa^\nu}{(1 - {\xi_2}^*)\Gamma(\nu + 1)} \\ &\leq R. \end{aligned}$$

Thus, for each $\theta \in [-\widehat{\delta}, \varkappa + \delta]$, $\|\aleph\varphi(\theta)\| \leq R$, which mean that

$$\|\aleph\varphi\|_\Upsilon \leq R.$$

So $\aleph(\widehat{\mathbb{k}}_R) \subset \widehat{\mathbb{k}}_R$

Step 3: $\aleph(\widehat{\mathbb{k}}_R)$ is equicontinuous.
Let $\theta_1, \theta_2 \in \Theta$, where $\theta_1 < \theta_2$ and $\varphi \in \widehat{\mathbb{k}}_R$. Then, we have

$$\|\aleph\varphi(\theta_2) - \aleph\varphi(\theta_1)\| \leq \frac{2}{\Gamma(\nu)} \int_{\theta_1}^{\theta_2} \varrho^{\nu-1} \|\widehat{\Psi}(\varrho)\| d\varrho$$

$$\begin{aligned}
&+ \quad \frac{1}{\Gamma(\nu)}\int_0^{\theta_1}[(\theta_2-\varrho)^{\nu-1}-(\theta_1-\varrho)^{\nu-1}]\|\widehat{\Psi}(\varrho)\|d\varrho \\
&+ \quad \frac{1}{\Gamma(\nu)}\int_{\theta_1}^{\theta_2}(\theta_2-\varrho)^{\nu-1}\|\widehat{\Psi}(\varrho)\|d\varrho \\
&+ \quad \frac{1}{\Gamma(\nu)}\int_{\theta_2}^{\varkappa}[(\varrho-\theta_2)^{\nu-1}-(\varrho-\theta_1)^{\nu-1}]\|\widehat{\Psi}(\varrho)\|d\varrho \\
&+ \quad \frac{1}{\Gamma(\nu)}\int_{\theta_1}^{\theta_2}(\varrho-\theta_1)^{\nu-1}\|\widehat{\Psi}(\varrho)\|d\varrho \\
&\leq \quad \frac{2{\xi_1}^*R}{(1-{\xi_2}^*)\Gamma(\nu+1)}(\theta_2^\nu-\theta_1^\nu)+\frac{{\xi_1}^*R}{(1-{\xi_2}^*)\Gamma(\nu+1)}(\theta_2^\nu-\theta_1^\nu) \\
&+ \quad \frac{{\xi_1}^*R}{(1-{\xi_2}^*)\Gamma(\nu+1)}[(\varkappa-\theta_2)^\nu-(\varkappa-\theta_1)^\nu] \\
&+ \quad \frac{2{\xi_1}^*R(\theta_2-\theta_1)^\nu}{(1-{\xi_2}^*)\Gamma(\nu+1)},
\end{aligned}$$

then, when $\theta_1 \longrightarrow \theta_2$, the right-hand side of the preceding inequality tend to zero. Therefore the operator $\aleph$ is equicontinuous.

Step 4: $\aleph(\widehat{\Bbbk}_R)$ is condensing.
Let $\Bbbk$ be a subset of $\widehat{\Bbbk}_R$. If $\theta \in [-\widehat{\delta}, 0]$, then

$$\begin{aligned}
\zeta(\aleph\Bbbk(\theta)) &= \zeta\big\{\aleph\varphi(\theta), \quad \varphi\in\Bbbk\big\} \\
&= \zeta\big\{\phi(\theta), \quad \varphi\in\Bbbk\big\} \\
&= 0,
\end{aligned}$$

and if $\theta \in [\varkappa, \varkappa+\delta]$, then

$$\begin{aligned}
\zeta(\aleph\Bbbk(\theta)) &= \zeta\big\{\aleph\varphi(\theta), \quad \varphi\in\Bbbk\big\} \\
&= \zeta\big\{\psi(\theta), \quad \varphi\in\Bbbk\big\} \\
&= 0.
\end{aligned}$$

For each $\theta \in [0, \varkappa]$, we have

$$\begin{aligned}
\zeta(\aleph\Bbbk(\theta)) &= \zeta\big\{\aleph\varphi(\theta), \quad \varphi\in\Bbbk\big\} \\
&= \left\{\phi(0)-\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\widehat{\Psi}(\varrho)d\varrho+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\widehat{\Psi}(\varrho)d\varrho, \quad \varphi\in\Bbbk\right\} \\
&\leq \frac{1}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\{\zeta(\widehat{\Psi}(\varrho))d\varrho, \quad \varphi\in\Bbbk\} \\
&+ \frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\{\zeta(\widehat{\Psi}(\varrho))d\varrho, \quad \varphi\in\Bbbk\}.
\end{aligned}$$

From $(T3)$, we have

$$\begin{aligned}
\zeta(\widehat{\Psi}(\theta)) &= \zeta(\Psi(\theta,\varphi^\theta,\widehat{\Psi}(\theta)) \\
&\leq \xi_1(\theta)\sup_{\varrho\in[-\widehat{\delta},\delta]}\zeta(\varphi^\theta)(\varrho)+\xi_2(\theta)\zeta(\widehat{\Psi}(\theta)) \\
&\leq {\xi_1}^*\sup_{\varrho\in[-\widehat{\delta},\varkappa+\delta]}\zeta(\varphi(\theta))+{\xi_2}^*\zeta(\widehat{\Psi}(\theta)).
\end{aligned}$$

Thus,

$$\zeta(\widehat{\Psi}(\theta)) \leq \frac{{\xi_1}^*}{1-{\xi_2}^*}\sup_{\varrho\in[-\widehat{\delta},\varkappa+\delta]}\zeta(\varphi(\theta)).$$

Then,

$$\begin{aligned}\zeta(\aleph\Bbbk(\theta)) &\leq \frac{{\xi_1}^*}{(1-{\xi_2}^*)\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\{\sup_{\varrho\in[-\widehat{\delta},\varkappa+\delta]}\zeta(\varphi(\varrho))d\varrho,\quad \varphi\in\Bbbk\}\\ &+ \frac{{\xi_1}^*}{(1-{\xi_2}^*)\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\{\sup_{\varrho\in[-\widehat{\delta},\varkappa+\delta]}\zeta(\varphi(\varrho))d\varrho,\quad \varphi\in\Bbbk\}\\ &\leq \left[\frac{2\varkappa^{\nu}{\xi_1}^*}{(1-{\xi_2}^*)\Gamma(\nu+1)}\right]\zeta_{\Upsilon}(\Bbbk).\end{aligned}$$

Therefore,

$$\zeta_{\Upsilon}(\aleph\Bbbk)\leq\left[\frac{2\varkappa^{\nu}}{(1-{\xi_1}^*)\Gamma(\nu+1)}\right]\zeta_{\Upsilon}(\Bbbk)<\zeta_{\Upsilon}(\Bbbk).$$

Then, we have $\zeta_{\Upsilon}(\aleph\Bbbk)<\zeta_{\Upsilon}(\Bbbk)$, which implies that $\aleph$ is a condensing operator. As a consequence of Sadovslii's fixed point theorem, the operator $\aleph$ has a fixed point which is a solution of the problem (1.5)-(1.7).

4.2. Ulam-Hyers-Rassias Stability

Now we are concerned with the generalized Ulam-Hyers-Rassias stability of our problem (1.5)-(1.7).

Definition 4.2 **([1, 28])** *The problem (1.5)-(1.7) is generalized Ulam-Hyers-Rassias stable with respect to $\beta\in C(\Theta,\mathbb{R}_+)$, if there exists a real number $C_{\Psi,\beta}>0$ such that for each solution $\varphi\in\Upsilon$ of inequality*

$$\left\|{}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta)-\Psi\left(\theta,\varphi^{\theta},\ {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta)\right)\right\|<\beta(\theta),\quad \theta\in\Theta, \tag{4.1}$$

there exists a solution $\bar{\varphi}\in\Upsilon$ of the problem (1.5)-(1.7) with

$$\|\varphi(\theta)-\bar{\varphi}(\theta)\|\leq C_{\Psi,\beta}\beta(\theta),\qquad \theta\in\Theta.$$

Remark 4.1 *A function $\varphi\in\Upsilon$ is a solution of the inequality (4.1) if and only if there exists a function $\ell\in C^1(\Theta,\Xi)$ (which depend on φ) such that*

1. $\|\ell(\theta)\|\leq\beta(\theta),\quad for\ each\ \theta\in\Theta.$
2. ${}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta)=\Psi\left(\theta,\varphi^{\theta},\ {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta)\right)+\ell(\theta),\quad for\ each\ \theta\in\Theta.$

Lemma 4.2 *The solution of the following perturbed problem*

$${}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta)=\Psi\left(\theta,\varphi^{\theta},\ {}_0^{RC}D_{\varkappa}^{\nu}\varphi(\theta)\right)+\ell(\theta),\qquad \theta\in\Theta:=[0,\varkappa],$$

$$\varphi(\theta)=\phi(\theta),\qquad \theta\in[-\widehat{\delta},0],$$

$$\varphi(\theta)=\psi(\theta),\qquad \theta\in[\varkappa,\varkappa+\delta],$$

is given by

$$\varphi(\theta)=\begin{cases}\phi(0)-\dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^{\varkappa}\varrho^{\nu-1}\widehat{\Psi}(\varrho)d\varrho+\dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\widehat{\Psi}(\varrho)d\varrho\\ -\dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^{\varkappa}\varrho^{\nu-1}\ell(\varrho)d\varrho+\dfrac{1}{\Gamma(\nu)}\displaystyle\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\ell(\varrho)d\varrho\\ \phi(\theta), \qquad if\ \theta\in[-\widehat{\delta},0],\\ \psi(\theta), \qquad if\ \theta\in[\varkappa,\varkappa+\delta].\end{cases}$$

Moreover, the solution satisfies the following inequality

$$\left\|\varphi(\theta)-\left[\phi(0)-\frac{1}{\Gamma(\nu}\int_0^{\varkappa}\varrho^{\nu-1}\widehat{\Psi}(\varrho)d\varrho+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\widehat{\Psi}(\varrho)d\varrho\right]\right\|$$
$$\leq 2\beta(\theta), \quad \text{for each } \theta\in\Theta.$$

Theorem 4.2 *Assume that* $(T1)$*-*$(T3)$ *and the following hold:*

(T4) *There exists a nondecreasing function* $\beta\in C(\Theta,\mathbb{R}_+)$*, and* $\wp_\beta>0$ *such that, for any* $\theta\in\Theta$

$$_0I^{\nu}_{\varkappa}\beta(\theta)\leq\wp_\beta\beta(\theta).$$

(T5) *There exist continuous functions* $\widehat{p}:\Theta\to\mathbb{R}_+$ *such that*

$$(1+\|\varpi_1\|_{[-\widehat{\delta},\delta]}+\|\varpi_2\|)\|\Psi(\theta,\varpi_1,\varpi_2)\|\leq\widehat{p}(\theta)(\|\varpi_1\|_{[-\widehat{\delta},\delta]}+\|\varpi_2\|),$$

for each $\theta\in\Theta$*,* $\varpi_1\in C([-\widehat{\delta},\delta],\Xi)$ *and* $\varpi_2\in\Xi$*.*

(T6) *There exists a continuous function* $q:\Theta\longrightarrow\mathbb{R}_+$ *such that*

$$\widehat{p}(\theta)\leq q(\theta)\beta(\theta),$$

where $q^*=\sup_{\theta\in[0,\varkappa]}q(\theta)$*.*
Then the problem (1.5)*-*(1.7) *is Ulam-Hyers-Rassias stable.*

Proof. Let $\varphi\in\Upsilon$ be a solution of the inequality (4.1) and $\bar{\varphi}\in\Upsilon$ the solution of the problem (1.5)-(1.7), then,

$$\begin{aligned}
\|\varphi(\theta)-\bar{\varphi}(\theta)\| &\leq 2\wp_\beta\varsigma\beta(\theta)+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\|\widehat{\Psi}(\varrho)-h(\varrho)\|d\varrho\\
&+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\|\widehat{\Psi}(\varrho)-h(\varrho)\|d\varrho\\
&\leq 2\wp_\beta\varsigma\beta(\theta)+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}2\varrho^{\nu-1}\widehat{p}(\varrho)d\varrho\\
&+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}2|\theta-\varrho|^{\nu-1}\widehat{p}(\varrho)d\varrho\\
&\leq 2\wp_\beta\varsigma\beta(\theta)+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}2\varrho^{\nu-1}q(\varrho)\beta(\varrho)d\varrho\\
&+\frac{1}{\Gamma(\nu)}\int_0^{\varkappa}2|\theta-\varrho|^{\nu-1}q(\varrho)\beta(\varrho)d\varrho\\
&\leq 2\wp_\beta\varsigma\beta(\theta)+\frac{2q^*}{\Gamma(\nu)}\int_0^{\varkappa}\varrho^{\nu-1}\beta(\varrho)d\varrho\\
&+\frac{2q^*}{\Gamma(\nu)}\int_0^{\varkappa}|\theta-\varrho|^{\nu-1}\beta(\varrho)d\varrho\\
&\leq 2\wp_\beta\beta(\theta)[\varsigma+2q^*]:=C_{\Psi,\beta}\beta(\theta).
\end{aligned}$$

Thus, the problem (1.5)-(1.7) is Ulam-Hyers-Rassias stable.

4.3. An Example

Set

$$\Xi=l^1=\left\{\varphi=(\varphi_1,\varphi_2,\ldots,\varphi_\gamma,\ldots),\sum_{\gamma=1}^{\infty}|\varphi_\gamma|<\infty\right\},$$

where Ξ is a Banach space with the norm $\|\varphi\| = \sum_{\gamma=1}^{\infty} |\varphi_\gamma|$.

Consider the following implicit problem:

$$ {}_0^{RC}D^{\frac{1}{2}}_{\varkappa}\varphi_\gamma(\theta) = \frac{9\sqrt{\pi}\sin(\sqrt{\theta+1})\left(\|\varphi_\gamma{}^{\theta}\|_{[-\widehat{\delta},\delta]} + \left\| {}_0^{RC}D^{\frac{1}{2}}_{\varkappa}\varphi(\theta)\right\|\right)}{7e^{11}\left(1+\|\varphi_\gamma{}^{\theta}\|_{[-\widehat{\delta},\delta]} + \left\| {}_0^{RC}D^{\frac{1}{2}}_{\varkappa}\varphi(\theta)\right\|\right)}, \ \theta \in \left[0, \tfrac{1}{10}\right], \tag{4.2}$$

$$\varphi_\gamma(\theta) = \phi(\theta), \quad \theta \in \left[-\tfrac{1}{10}, 0\right], \tag{4.3}$$

$$\varphi_\gamma(\theta) = \psi(\theta), \quad \theta \in \left[\tfrac{1}{10}, 1\right], \tag{4.4}$$

where $\varkappa = \frac{1}{10}$, $\phi \in C\left(\left[-\frac{1}{10}, 0\right], \Xi\right)$ and $\psi \in C\left(\left[\frac{1}{10}, 1\right], \Xi\right)$.

Set

$$\Psi(\theta, \varpi_1, \varpi_2) = \frac{9\sqrt{\pi}\sin(\sqrt{\theta+1})(\|\varpi_1\|_{[-\widehat{\delta},\delta]} + \|\varpi_2\|)}{7e^{11}(1+\|\varpi_1\|_{[-\widehat{\delta},\delta]} + \|\varpi_2\|)}, \quad \theta \in \left[0, \tfrac{1}{10}\right],$$

where $\varpi_1 \in C\left(\left[-\frac{1}{10}, \frac{9}{10}\right], \Xi\right), \varpi_2 \in \Xi$. Clearly, the function Ψ satisfies the condition $(T1)$. Also we have

$$\|\Psi(\theta, \varpi_1, \varpi_2)\| \leq \frac{9\sqrt{\pi}|\sin(\sqrt{\theta+1})|}{7e^{11}}(\|\varpi_1\|_{[-\widehat{\delta},\delta]} + \|\varpi_2\|).$$

And as

$$\begin{aligned} \frac{2\xi_1{}^*\varkappa^\nu}{(1-\xi_2{}^*)\Gamma(\nu+1)} &= \frac{36}{\sqrt{10}(7e^{11} - 9\sqrt{\pi})} \\ &\approx 2.71632474408497 \cdot 10^{-5} \\ &< 1. \end{aligned}$$

Thus, by Theorem 4.1, the problem (4.2)-(4.4) has at least one solution.

For any $\theta \in \left[0, \frac{1}{10}\right]$, we take $\beta(\theta) = e^2$. Then,

$${}_0I^\nu_\varkappa\beta(\theta) = \frac{1}{\Gamma(\nu)}\int_0^{\varkappa} |\theta - \varrho|^{\nu-1}\beta(\varrho)d\varrho.$$

Therefore,

$${}_0I^\nu_{\frac{1}{10}}(e^2) \leq \frac{\left(\frac{1}{10}\right)^{\frac{3}{2}}}{\Gamma\left(\frac{3}{2}\right)}e^2 = \frac{2}{10^{\frac{3}{2}}\sqrt{\pi}}e^2 := \frac{2}{10^{\frac{3}{2}}\sqrt{\pi}}\beta(\theta).$$

The conditions $(T5)$ and $(T6)$ are verified with

$$\widehat{p}(\theta) = \frac{9\sqrt{\pi}|\sin(\sqrt{\theta+1})|}{7e^{11}}, q(\theta) = \frac{9\sqrt{\pi}|\sin(\sqrt{\theta+1})|}{7e^{13}}.$$

Consequently, Theorem 4.2 assures that the problem (4.2)-(4.4) is Ulam-Hyers-Rassias stable.

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