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Stochastic Intervention Control of Mean-Field Jump System With Noisy Observation via L-Derivatives With Application to Finance

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ABSTRACT: In this paper, we investigate stochastic optimal intervention control of mean-field jump system with noisy observation via L-derivatives on Wasserstein space of probability measures We derive the necessary conditions of optimality for partially observed optimal intervention control problems of mean-field type. The coefficients depend on the state of the solution process as well as of its probability distribution and the control variable. The proof of our main results are obtained by applying L-derivatives in the sense of Lions. In our control problem, there are two models of jumps for the state process, the inaccessible ones which come from the Poission process and the predictable ones which come from the intervention control Finally, we apply our result to study conditional mean-variance portfolio selection problem with interventions, where the foreign exchange interventions are intended to contain excessive fluctuations in foreign exchange rates and to stabilize them.

Key Words: Stochastic optimal control, Probabilistic methods, stochastic differential equations; Intervention control, Mean-field stochastic system with jumps, L-derivatives, Conditional mean-variance portfolio selection problem.

Contents

1. Introduction

Since the development of nonlinear filtering theory, stochastic control problems under partial observation have received much attention and became a powerful tool in many fields with important applications, such as finance and economics, etc. In many situations, the states of the systems cannot be completely observed; however, some other processes related to the unobservable states can be observed. Such subjects have been discussed by many authors, such as Wang, Wu and Xiong [1], Wang, Zhang, and Zhang [2], Wang, Wu and Xiong [3], Wang and Wu [4], Bensoussan and Yam [5], Wang, Shi and Meng [6], Djehiche and Tembine [7], Lakhdari, Miloudi and Hafayed [8], Miloudi et al [9], Abada, Hafayed and Meherrem [10].

General mean-field type stochastic differential equations (SDEs) are Itô's stochastic differential equations, where the coefficients of the state equation depend on the time variable, the state of the solution process as well as of its probability law. In his course at Collége de France [11], (refer to Cardaliaguet [12] for the written version) P.L. Lions introduced and studied the innovative notion of new derivatives with respect to measure over Wasserstein spaces. Strongly motivated by these works, Buckdahn, Li and

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Ma, [13] proved the necessary conditions for general mean-field systems. Stochastic maximum principles for general mean-field models were later studied in [9,14,15].

Stochastic irregular (singular or impulse) control problems have received considerable attention in the literature. There are numerous papers by different authors investigating the stochastic optimal singular or impulse control problems, e.g., Cadenillas and Haussmann [16], Dufour and Miller [17], Hafayed and Abbas [18], Zhang [19], Jeanblanc-Piqué [20], Korn [21], Wu and Zhang [22]. An extensive list of recent references to singular control problem, with some applications in finance and economics can be found in [18,23,24,25]. Optimal control problems for SDEs with jump processes have been investigated by many authors, see for instance, [26,27,28,29,30]. A good account and an extensive list of references on jump processes can be founded in [31] for a comprehensive theoretical study of the topic.

In the present paper, we study a new mean-field type intervention control problem. We establish a new set of necessary conditions of optimal intervention control for general mean-field jump systems. Our mean-field dynamic is governed by SDEs with a random measures and an independent Brownian motion, with noisy observation. The coefficients of our mean-field dynamic depend nonlinearly on both the state process as well as of its probability law. The control domain is assumed to be convex. The L-derivatives with respect to probability measure and the associate Itô-formula are applied to prove our main results. Noting that our general mean-field partially observed control problem occur naturally in the probabilistic analysis of financial optimization problems. Our model of partially observed intervention control problem play an important role in different fields of economics and finance, as conditional mean variance portfolio selection problem with discrete movement in incomplete market. Also, optimal consumption and portfolio problem under proportional transaction costs. Moreover, the exchange rate under uncertainty, where government has two means of influencing the foreign exchange rate of its own currency:

1. At all times t the government can choose the domestic interest rate.
2. At selected times τ_i the government, or bank can intervene in the foreign exchange market by selling or buying large amounts of foreign currency.

In our model of mean-field control problem, there are two types of jumps for the state processes, the inaccessible ones which come from the Poission process and the predictable ones which come from the intervention control.

As an illustration, by applying our result, conditional mean-variance portfolio selection problem with interventions with incomplete market is discussed. In financial markets three important objectives of interventions: to influence the level of the exchange rate, to dampen exchange rate volatility or supply liquidity to foreign exchange markets; and to influence the amount of foreign reserves. Banks intervene in foreign exchange markets in order to achieve a variety of overall economic objectives, such as controlling inflation, maintaining competitiveness or maintaining financial stability.

The rest of the paper is organized as follows. Sect. 2 begins with a formulation of the partially observed control problem of general mean-field differential equations with Poisson jump processes. We give the notations and definitions of the L-derivatives on the Wasserstein space via P.L. Lions sense and assumptions used throughout the paper. In Sect. 3, we prove the necessary conditions of optimality which are our main results. Conditional mean-variance portfolio selection problem with interventions is also given in Sect. 4. At the end of this paper, some discussions with concluding remarks and future developments are presented in the last Section.

2. Formulation of the problem and preliminaries

Spaces and notations. Let T is a fixed terminal time and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$ be a complete filtered probability space on which are defined two independent standard one-dimensional Brownian motions $W(\cdot)$ and $Y(\cdot)$. Let $\mathbb{R}^n$ is a n-dimensional Euclidean space, $\mathbb{R}^{n\times d}$ the collection of $n \times d$ matrices. Let $k(\cdot)$ be a stationary $\mathcal{F}_t$-Poisson point process with the characteristic measure $m\,(\mathrm{d}\theta)$. We denote by $\eta\,(d\theta, dt)$ the counting measure or Poisson measure defined on $\Theta \times \mathbb{R}_+$, where Θ is a fixed nonempty subset of $\mathbb{R}$ with its Borel σ-field $\mathcal{B}\,(\Theta)$ and set $\widetilde{\eta}\,(\mathrm{d}\theta, \mathrm{d}t) = \eta\,(\mathrm{d}\theta, \mathrm{d}t) - m\,(\mathrm{d}\theta)\,\mathrm{d}t$ satisfying $\int_\Theta (1 \wedge |\theta|^2) m\,(\mathrm{d}\theta) < \infty$ and $m\,(\Theta) < +\infty$. Let $\mathcal{F}_t^W$, $\mathcal{F}_t^Y$ and $\mathcal{F}_t^\eta$ be the natural filtration generated by $W(\cdot)$, $Y(\cdot)$ and $\eta(\cdot,\cdot)$ respectively. We assume that $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{F}_t^\eta \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of $\mathcal{P}$-null sets. We denote by $\langle \cdot, \cdot \rangle$ (resp. $|\cdot|$) the scalar product (resp., norm), $\mathbb{E}\,(\cdot)$ denotes the expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. Throughout this work, we denote by $\mathbb{L}^2(\mathcal{F}_t;\mathbb{R}^n)$ the space of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variable X, such that $\mathbb{E}(|X|^2) < +\infty$ and by $\mathcal{M}^2([0,T];\mathbb{R})$: the space of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted measurable process $g(\cdot)$, such that $\mathbb{E}\int_0^T\int_\Theta |g(t,\theta)|^2 m(\mathrm{d}\theta)\,\mathrm{d}t < +\infty$. Let $\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ be the Hilbert space with inner product $(X,Y)_2 = \mathbb{E}[X.Y]$,where $X,Y \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ and the norm $\|X\|_2^2 = (X,X)_2$. Let $\mathbb{X}_2(\mathbb{R}^d)$ be the space of all probability measures μ on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$ with finite second moment, i.e, $\int_{\mathbb{R}^d}|x|^2\mu(\mathrm{d}x) < +\infty$, endowed with the following Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$; for $\mu,\nu \in \mathbb{X}_2(\mathbb{R}^d)$,

$$\mathbb{D}_2(\mu,\nu) = \inf_{\delta(\cdot,\cdot)\in\mathbb{X}_2(\mathbb{R}^{2d})}\left\{\left[\int_{\mathbb{R}^{2d}}|x-y|^2\,\delta(\mathrm{d}x,\mathrm{d}y)\right]^{\frac{1}{2}}\right\},$$

where $\delta(\cdot,\cdot) \in \mathbb{X}_2(\mathbb{R}^{2d})$, $\delta(A,\mathbb{R}^d) = \mu(A)$, $\delta(\mathbb{R}^d,B) = v(B)$. This distance is just the *Monge-Kankorovich distance* when $p = 2$.

2.1. L-derivatives on the Wasserstein space

Now, we recall briefly the innovative notion of L-derivatives with respect to probability distribution over Wasserstein spaces, which was studied by Lions [11], and Cardaliaguet [12] and the pioneering work by Cardaliaguet et. al. [32] in their study of the so-called master equation in mean field game systems. The main idea is to identify a distribution $\mu \in \mathbb{X}_2(\mathbb{R}^d)$ with a random variables $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ so that $\mu = \mathcal{P}_\vartheta$. To be more precise, we assume that probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathcal{P})$ is rich enough in the sense that for every $\mu \in \mathbb{X}_2(\mathbb{R}^d)$, there is a random variable $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ such that $\mu = \mathcal{P}_\vartheta$.
Definition 2.1 (Lift function) Let Φ be a given function such that $\Phi : \mathbb{X}_2(\mathbb{R}^d) \to \mathbb{R}$. We define the lift function $\widetilde{\Phi} : \mathbb{L}^2(\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$ such that $\widetilde{\Phi}(Z) = \Phi(\mathcal{P}_Z)$, $Z \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$.

Clearly, the lift function $\widetilde{\Phi}$ of Φ, depends only on the law of random variable $Z \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ and is independent of the choice of the representative Z.

Definition 2.2 A function $f : \mathbb{X}_2(\mathbb{R}^d) \to \mathbb{R}$ is said to be differentiable at $\mu_0 \in \mathbb{X}_2(\mathbb{R}^d)$ if there exists $Z_0 \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ with $\mu_0 = \mathcal{P}_{Z_0} \in \mathbb{X}_2(\mathbb{R}^d)$ such that its lift function $\widetilde{f}$ is *Fréchet differentiable* at Z_0. More precisely, there exists a continuous linear functional $D\widetilde{f}(\cdot) : \mathbb{L}^2(\mathcal{F};\mathbb{R}^d) \to \mathbb{R}$ such that

$$\begin{aligned}\widetilde{f}(Z_0+\tau) - \widetilde{f}(Z_0) &= \left\langle D\widetilde{f}(Z_0),\tau\right\rangle + o(\|\tau\|_2) \\ &= D_\tau f(\mu_0) + o(\|\tau\|_2),\end{aligned} \tag{2.1}$$

where $\langle\cdot,\cdot\rangle$ is the dual product on $\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, and we will refer to $D_\tau f(\mu_0)$ as the Fréchet derivative of f at μ_0 in the direction τ. In this case, we have

$$D_\tau f(\mu_0) = \left\langle D\widetilde{f}(Z_0),\tau\right\rangle = \frac{d}{dt}\widetilde{f}(Z_0+t\tau)\bigg|_{t=0},\ \text{with } \mu_0 = \mathcal{P}_{Z_0}.$$

So,

$$D_\tau f(\mathcal{P}_{Z_0}) = \frac{d}{dt}\left[\widetilde{f}(Z_0+t\tau)\right]\bigg|_{t=0}. \tag{2.2}$$

From (2.2), then we obtain the following form of the Taylor expansion

$$f(\mathcal{P}_Z) - f(\mathcal{P}_{Z_0}) = D_\alpha f(\mathcal{P}_Z) + \mathcal{E}(\tau), \tag{2.3}$$

where $\mathcal{E}(\tau)$ is of order $o(\|\tau\|_2)$ with $o(\|\tau\|_2) \to 0$ for $\tau(\cdot) \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$.
By using the *Riesz' representation theorem*, there is a unique random variable Z_0 in the Hilbert space $\mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ such that $\left\langle D\widetilde{f}(Z),\tau\right\rangle = (Z_0,\tau)_2 = \mathbb{E}[(Z_0,\tau)_2]$, where $\tau(\cdot) \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$. It was shown, see the works of Lions [11], see also Cardaliaguet [12], Buckdahn, Li, and Ma [13], that there exists a Boral function $\psi[\mu_0] : \mathbb{R}^d \to \mathbb{R}^d$, depending only on the law $\mu_0 = \mathcal{P}_Z$ but not on the particular choice of the representative Z such that $Z_0 = \psi[\mu_0](Z)$.

Thus, we can write (2.1) as $\forall\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$.

$$f(\mathcal{P}_\vartheta) - f(\mathcal{P}_Z) = (\psi[\mu_0](Z), \vartheta - Z)_2 + o(\|\vartheta - Z\|_2).$$

We denote $\partial_\mu f(\mathcal{P}_Z, y) = \psi[\mu_0](y)$, $y \in \mathbb{R}^d$. We note that for each $\mu \in \mathbb{X}_2(\mathbb{R}^d)$, $\partial_\mu f(\mathcal{P}_Z, \cdot) = \psi[\mathcal{P}_Z](\cdot)$ is only defined in a $\mathcal{P}_Z(dx) - a.e$ sense, where $\mu = \mathcal{P}_Z$.

Definition 2.3 (Space of differentiable functions in $\mathbb{X}_2(\mathbb{R}^d)$). We say that the function $f \in \mathbb{C}_b^{1,1}(\mathbb{X}_2(\mathbb{R}^d))$ if for all $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, there exists a $\mathcal{P}_\vartheta$-modification of $\partial_\mu f(\mathcal{P}_\vartheta, \cdot)$ such that $\partial_\mu f : \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C > 0$, it holds that

(i) $|\partial_\mu f(\mu, x)| \leq C$, $\forall \mu \in \mathbb{X}_2(\mathbb{R}^d)$, $\forall x \in \mathbb{R}^d$;

(ii) $|\partial_\mu f(\mu_1, x) - \partial_\mu f(\mu_2, y)| \leq C(\mathbb{D}_2(\mu_1, \mu_2) + |x - y|)$, $\forall\ \mu_1, \mu_2 \in \mathbb{X}_2(\mathbb{R}^d)$, $\forall\ x, y \in \mathbb{R}^d$.

We would like to point out that the version of $\partial_\mu f(\mathcal{P}_\vartheta, \cdot)$, $\vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ indicated in the above definition is unique (see Buckdahn et al. [13] for more information).

Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, \widehat{\mathcal{P}})$ be a copy of the probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$. For any pair of random variable $(\vartheta_1, \vartheta_2) \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d) \times \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$, we let $(\widehat{\vartheta_1}, \widehat{\vartheta_2})$ be an independent copy of $(\vartheta_1, \vartheta_2)$ defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, \widehat{\mathcal{P}})$. We consider the product probability space $(\Omega \times \widehat{\Omega}, \mathcal{F} \otimes \widehat{\mathcal{F}}, \mathcal{F}_t \otimes \widehat{\mathcal{F}}_t, \mathcal{P} \otimes \widehat{\mathcal{P}})$ and setting $(\widehat{\vartheta_1}, \widehat{\vartheta_2})(w, \widehat{w}) = (\vartheta_1(\widehat{w}), \vartheta_2(\widehat{w}))$ for any $(w, \widehat{w}) \in \Omega \times \widehat{\Omega}$. Let $(\widehat{u}(t), \widehat{x}(t))$ be an independent copy of $(u(t), x(t))$ so that $\mathcal{P}_{x(t)} = \widehat{\mathcal{P}}_{\widehat{x(t)}}$. We denote by $\widehat{\mathbb{E}}(\cdot) = \widehat{\mathbb{E}}_{\widehat{\mathcal{P}}}(\cdot)$ the expectation under probability measure $\widehat{\mathcal{P}}$ and $\mathcal{P}_X = \mathcal{P} \circ X^{-1}$ denotes the law of the random variable X.

Let $\mathbb{A}_1$ be a closed convex subset of $\mathbb{R}^k$ and $\mathbb{A}_2 := [0, +\infty)^m$.

Definition 2.4. An admissible continuous control $u(\cdot)$ is an $\mathcal{F}_t^Y$-adapted process with values in $\mathbb{A}_1$ satisfies $\sup_{t\in[0,T]}(\mathbb{E}|u(t)|^n) < \infty$, $n = 2, 3, \ldots$. We denote by $\mathcal{U}_1^Y$ the set of the admissible regular control variables.

Definition 2.5. An intervention control is a stochastic irregular process $\xi(\cdot)$ of measurable $\mathbb{A}_2-$valued, F^Y-adapted processes, such that the process $\xi(\cdot) : [0, T] \times \Omega \to \mathbb{A}_2$ is non-decreasing continuous on the right with left-limits, with bounded variation and $\xi(0) = 0$. Moreover, $\mathbb{E}(|\xi(T)|^p) < \infty$ for any $p \geq 2$. We denote by $\mathcal{U}_2^Y$ the set of the admissible intervention control variables.

Definition 2.6. An admissible combined control is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, F^Y-adapted processes, such that the process $u(\cdot) : [0, T] \times \Omega \to \mathbb{A}_1$ is regular process satisfies *Definition 2.4* and $\xi(\cdot) : [0, T] \times \Omega \to \mathbb{A}_2$ is an intervention control given by *Definition 2.5.* We denote by $\mathcal{U}_1^Y \times \mathcal{U}_2^Y$ the set of the admissible combined control variables.

2.2. Partially observed optimal intervention control Model

In this paper, we formulate this problem mathematically as a combined stochastic continuous control and irregular control problem. We study partially observed optimal stochastic intervention control problem for systems governed by mean-field SDEs with correlated noisy between the system and the observation, allowing both classical and intervention control of the form: $t \in [0, T]$

$$\begin{cases} \mathrm{d}x^{u,\xi}(t) = f(t, x^{u,\xi}(t), \mathcal{P}[x^{u,\xi}(t)], u(t))\mathrm{d}t + \sigma(t, x^{u,\xi}(t), \mathcal{P}[x^{u,\xi}(t)], u(t))\mathrm{d}W(t) \\ \qquad\qquad + \int_\Theta g(t, x^{u,\xi}(t_-), \mathcal{P}[x^{u,\xi}(t_-)], u(t), \theta)\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}t) \\ \qquad\qquad + c(t, x^{u,\xi}(t), \mathcal{P}[x^{u,\xi}(t)], u(t))\mathrm{d}\widetilde{W}(t) + G(t)\mathrm{d}\xi(t), \\ x^{u,\xi}(0) = x_0, \end{cases} \tag{2.4}$$

where $\mathcal{P}[x^{u,\xi}(t)] = \mathcal{P} \circ (x^{u,\xi})^{-1}$ denotes the law of the random variable $x^{u,\xi}$. The mappings

$$\begin{aligned} f &: [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \to \mathbb{R}^n \\ \sigma &: [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \to \mathcal{M}(\mathbb{R}^{n\times d}) \\ c &: [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \to \mathcal{M}(\mathbb{R}^{n\times d}) \\ g &: [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}^d) \times \mathbb{A}_1 \times \Theta \to \mathcal{M}(\mathbb{R}^{n\times d}) \\ G &: [0,T] \to \mathbb{R}^n \end{aligned}$$

are given deterministic functions.
Suppose that the state processes $x^{u,\xi}(\cdot)$ cannot be observed directly, but the controllers can observe a related noisy process $Y(\cdot)$, which is governed by the following equation

$$\begin{cases} dY(t) &= h(t, x^{u,\xi}(t), u(t))dt + d\widetilde{W}(t) \\ Y(0) &= 0, \end{cases} \tag{2.5}$$

where $h : [0,T] \times \mathbb{R}^n \times \mathbb{A}_1 \to \mathbb{R}^r$, and $\widetilde{W}(\cdot)$ is a stochastic process depending on the control $u(\cdot)$.
Consider the cost functional

$$\begin{aligned} J(u(\cdot), \xi(\cdot)) &= \mathbb{E}^u \Bigg[\int_0^T l(t, x^{u,\xi}(t), \mathcal{P}[x^{u,\xi}(t)], u(t))\mathrm{d}t \\ &\quad + \psi(x^{u,\xi}(T), \mathcal{P}[x^{u,\xi}(t)]) + \int_{[0,T]} M(t)\mathrm{d}\xi(t) \Bigg]. \end{aligned} \tag{2.6}$$

Where $l : [0,T] \times \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}) \times \mathbb{A}_1 \to \mathbb{R}$, $\psi : \mathbb{R}^n \times \mathbb{X}_2(\mathbb{R}) \to \mathbb{R}$ and $\mathbb{E}^u$ stands for the mathematical expectation on $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^u)$ defined by

$$\mathbb{E}^u(X) = \mathbb{E}_{\mathcal{P}^u}(X) = \int_\Omega X(w)\mathrm{d}\mathcal{P}^u(w).$$

In this paper, we shall make use of the following standing assumptions.
Assumption (H1) The maps $f, \sigma, c, l : [0,T] \times \mathbb{R} \times \mathbb{X}_2(\mathbb{R}) \times \mathbb{A}_1 \to \mathbb{R}$ and $\psi : \mathbb{R} \times \mathbb{X}_2(\mathbb{R}) \to \mathbb{R}$ are measurable in all variables. Moreover, $f(t,\cdot,\cdot,u)$, $\sigma(t,\cdot,\cdot,u)$, $c(t,\cdot,\cdot,u)$, $l(t,\cdot,\cdot,u)$, $g(t,\cdot,\cdot,u,\theta) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times \mathbb{X}_2(\mathbb{R}), \mathbb{R})$ and $\psi(\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times \mathbb{X}_2(\mathbb{R}), \mathbb{R})$ for all $u \in \mathbb{A}_1$.

Assumption (H2) Let $\varphi(x,\mu) = f(t,x,\mu,u), \sigma(t,x,\mu,u), c(t,x,\mu,u), l(t,x,\mu,u), g(t,x,\mu,u,\theta), \psi(x,\mu)$, the function $\varphi(\cdot,\cdot)$ satisfies the following properties:
(**1**) For fixed $x \in \mathbb{R}$ and $\mu \in \mathbb{X}_2(\mathbb{R})$, the function $\varphi(\cdot,\mu) \in \mathbb{C}_b^1(\mathbb{R})$ and $\varphi(x,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{X}_2(\mathbb{R}^d), \mathbb{R})$. All the derivatives φ_x and $\partial_\mu \varphi$, for $\varphi = f, \sigma, c, l, \psi$ are bounded and Lipschitz continuous, with Lipschitz constants independent of $u \in \mathbb{A}_1$. Moreover, there exists a constants $C(T, m(\Theta)) > 0$ such that

$$\sup_{\theta\in\Theta} |g_x(t,x,\mu,u,\theta)| + \sup_{\theta\in\Theta} |\partial_\mu g(t,x,\mu,u,\theta)| \leq C.$$

$$\begin{aligned} &\sup_{\theta\in\Theta} |g_x(t,x,\mu,u,\theta) - g_x(t,x',\mu',u,\theta)| + \sup_{\theta\in\Theta} |\partial_\mu g(t,x,\mu,u,\theta) - \partial_\mu g(t,x',\mu',u,\theta)| \\ \leq\ & C\left[|x - x'| + \mathbb{D}_2(\mu,\mu')\right]. \end{aligned}$$

(**2**) The functions f, σ, c, g and l are continuously differentiable with respect to control variable $u(\cdot)$, and all their derivatives are continuous and bounded. Moreover, there exists a constants $C = C(T, m(\Theta)) > 0$ such that

$$\sup_{\theta\in\Theta} |g_u(t,x,\mu,u,\theta)| \leq C.$$

The function h is continuously differentiable in x and continuous in v, its derivatives and h are all uniformly bounded which satisfies the following *Novikov's condition*:

$$\mathbb{E}\left(\exp\left[\frac{1}{2}\int_0^t \left|h(s, x^{u,\xi}(s), u(s))\right|^2 \mathrm{d}s\right]\right) < \infty. \tag{2.7}$$

Assumption (H3) The functions $G(\cdot) : [0,T] \times \Omega \to \mathbb{R}$, and $M(\cdot) : [0,T] \times \Omega \to \mathbb{R}^+$ are continuous and bounded.

Clearly, assumption (H3) allows us to define integrals of the form $\int_{[0,T]} G(t)\mathrm{d}\xi(t)$ and $\int_{[0,T]} M(t)\mathrm{d}\xi(t)$. Moreover, under assumptions (H1), (H2) and (H3), for any $(u(\cdot), \xi(\cdot)) \in \mathcal{U}_1^Y \times \mathcal{U}_2^Y$ the mean-field equation (2.4) admits a unique strong solution $x^{u,\xi}(t)$ given by

$$\begin{aligned} x^{u,\xi}(t) &= x_0 + \int_0^t f(s, x^{u,\xi}(s), \mathcal{P}[x^{u,\xi}(s)], u(s))\mathrm{d}s + \sigma(s, x^{u,\xi}(s), \mathcal{P}[x^{u,\xi}(s)], u(s))\mathrm{d}W(s) \\ &\quad + c(s, x^{u,\xi}(s), \mathcal{P}[x^{u,\xi}(s)], u(s))\mathrm{d}\widetilde{W}(s) \\ &\quad + \int_0^t \int_\Theta g(s, x^{u,\xi}(s_-), \mathcal{P}[x^{u,\xi}(s_-)], u(s), \theta)\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}s) \\ &\quad + \int_{[0,T]} G(s)\mathrm{d}\xi(s). \end{aligned}$$

We define the $\mathcal{F}_t^Y-$martingale $\alpha^u(t)$ which is the solution of the equation

$$\begin{cases} \mathrm{d}\alpha^u(t) = \alpha^u(t)h\left(t, x^{u,\xi}(t), u(t)\right)\mathrm{d}Y(t), \\ \alpha^u(0) = 1. \end{cases} \tag{2.8}$$

This martingale allowed to define a new probability $\mathcal{P}^u$ on the space $(\Omega, \mathcal{F})$, to emphasize the fact that it depend on the control $u(\cdot)$. It is given by the *Radon-Nikodym derivative*:

$$\left.\frac{\mathrm{d}\mathcal{P}^u}{\mathrm{d}\mathcal{P}}\right|_{\mathcal{F}_t^Y} = \alpha^u(t). \tag{2.9}$$

From the linear equation (2.8), and by a simple computation, we can get

$$\alpha^u(t) = \exp\left[\int_0^t h(s, x^{u,\xi}(s), u(s))\mathrm{d}Y(s) - \frac{1}{2}\int_0^t \left|h(s, x^{u,\xi}(s), u(s))\right|^2 \mathrm{d}s\right]. \tag{2.10}$$

This type of equations are called *Doléan-Dade's exponential.* We note that $\mathbb{E}^u(\varphi(X))$ refers to the expected value of $\Psi(X)$ with respect to the probability law $\mathcal{P}^u$. Moreover, since $\mathrm{d}\mathcal{P}^u = \alpha^u(t)\mathrm{d}\mathcal{P}$, we have

$$\begin{aligned} \mathbb{E}^u(\varphi(X)) &= \mathbb{E}_{\mathcal{P}^u}(\varphi(X)) = \int_\Omega \varphi(X(w))\mathrm{d}\mathcal{P}^u(w), \\ &= \int_\Omega \varphi(X(w))\alpha^u(t)\mathrm{d}\mathcal{P}(w), \\ &= \mathbb{E}_{\mathcal{P}}(\alpha^u(t)\varphi(X)) = \mathbb{E}\left[\alpha^u(t)\varphi(X)\right]. \end{aligned}$$

Applying Itô's formula, we can prove that $\sup_{t\in[0,T]} \mathbb{E}\left(|\alpha^u(t)|^n\right) < +\infty$, $n > 1$. By Girsanov's theorem and assumptions (H1), (H2) and (H3), $\mathcal{P}^u$ is a new probability measure of density $\alpha^u(t)$. The process

$$\widetilde{W}(t) = Y(t) - \int_0^t h(s, x^{u,\xi}(s), u(s))\mathrm{d}s,$$

is a standard Brownian motion independent of $B(\cdot)$ and x_0 on the new probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P}^u)$. By Radon-Nikodym derivative (2.9), with the martingale property of $\alpha^u(t)$, the cost functional (2.6) can be written as

$$\begin{aligned} J(u(\cdot), \xi(\cdot)) &= \mathbb{E}\left[\int_0^T \alpha^u(t)l(t, x^{u,\xi}(t), \mathcal{P}[x^{u,\xi}(t)], u(t))\mathrm{d}t + \alpha^u(T)\psi(x^{u,\xi}(T), \mathcal{P}[x^{u,\xi}(T)])\right. \\ &\quad \left. + \int_{[0,T]} \alpha^u(t)M(t)\mathrm{d}\xi(t)\right]. \end{aligned} \tag{2.11}$$

The main purpose of this paper is to prove stochastic maximum principle, also called necessary optimality conditions for the partially observed optimal control of mean-field Poisson jumps.

Notice that the jumps of a singular control $\xi(\cdot)$ at any time t_j denote by $\Delta\xi(t_j) = \xi(t_j) - \xi(t_{j-})$ and we define the continuous part of the intervention control by

$$|\xi|(t) = \xi(t) - \sum_{0\le t_j \le t} \Delta\xi(t_j).$$

Here $|\xi|(t)$ the process obtained by removing the jumps of $\xi(t)$.

Throughout this paper, we distinguish between the jumps caused by the intervention control $\xi(\cdot)$ and the jumps caused by the random Poisson measure at any jumping time t.The jumps of $x^{u,\xi}(t)$ caused by the intervention control $\xi(\cdot)$ by

$$\triangle_{\xi} x^{u,\xi}(t) = G(t)\triangle\xi(t) = G(t)(\xi(t) - \xi(t_-)), \tag{2.12}$$

and the jumps of $x^{u,\xi}(t)$ caused by the Poisson measure of $\widetilde{\eta}(\theta, t)$ by

$$\begin{aligned}\triangle_{\eta} x^{u,\xi}(t) &= \int_{\Theta} g\left(t, x^{u,\xi}(t_-), \mathcal{P}\left[x^{u,\xi}(t_-)\right], u(t_-), \theta\right) \widetilde{\eta}\left(\mathrm{d}\theta, \{t\}\right) \\ &= \begin{cases} g\left(t, x^{u,\xi}(t_-), \mathcal{P}[x^{u,\xi}(t_-)], u(t_-), \theta\right) : \text{if } \xi \text{ has a jump of size } \theta \text{ at time } t. \\ 0 : \text{otherwise,} \end{cases}\end{aligned} \tag{2.13}$$

where $\widetilde{\eta}\left(\mathrm{d}\theta, \{t\}\right)$ means the jump in the Poisson random measure, occurring at time t.

Finally, the general jump of the state processes $x^{u,\xi}(\cdot)$ at any jumping time t is given by

$$\triangle x^{u,\xi}(t) = x^{u,\xi}(t) - x^{u,\xi}(t_-) = \triangle_{\xi} x^{u,\xi}(t) + \triangle_{\eta} x^{u,\xi}(t). \tag{2.14}$$

3. Necessary conditions for optimal intervention control in Wasserstein space

In this section, we prove the necessary conditions of optimality for our partially observed optimal intervention control problem of general mean-field stochastic differential equations with jumps. The proof is based on Girsanov's theorem, the derivatives with respect to probability measure in Wasserstein space and by introducing the variational equations with some estimates of their solutions.

3.1. Main results

Hamiltonian. We define the Hamiltonian

$$H : [0, T] \times \mathbb{R} \times \mathbb{X}_2(\mathbb{R}) \times \mathbb{A}_1 \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

associated with our control problem by

$$\begin{aligned} & H(t, x, \mu, u, \Phi(t), Q(t), \overline{Q}(t), K(t), R(t,\theta)) \\ = \quad & l(t, x, \mu, u) + f(t, x, \mu, u)\Phi(t) + \sigma(t, x, \mu, u)Q(t) \\ & + c(t, x, \mu, u)\overline{Q}(t) + h(t, x, u) K(t) + \int_{\Theta} g(t, x, \mu, u, \theta) R(t, \theta) m(\mathrm{d}\theta). \end{aligned} \tag{3.1}$$

Adjoint equations. We are now ready to introduce two new adjoint equations that will be the building blocks of the stochastic maximum principle and

$$\begin{cases} -\mathrm{d}\Phi(t) &= \Big[f_x(t)\Phi(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(t)\widehat{\Phi}(t)\right] + \sigma_x(t) Q(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(t)\widehat{Q}(t)\right] \\ & \quad + c_x(t)\overline{Q}(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{c}(t)\widehat{\overline{Q}}(t)\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{l}(t)\right] \\ & \quad + \int_\Theta \left[g_x(t,\theta) R(t,\theta) + \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(t,\theta)\widehat{R}(t,\theta)\right]\right] m(\mathrm{d}\theta) + h_x(t) K(t) \Big] \mathrm{d}t \\ & \quad - Q(t)\mathrm{d}W(t) - \overline{Q}(t)\mathrm{d}\widetilde{W}(t) - \int_\Theta R(t,\theta)\widetilde{\eta}(\mathrm{d}\theta, \mathrm{d}t), \\ \Phi(T) &= \psi_x(x(T), \mathcal{P}[x(t)]) + \widehat{\mathbb{E}}\left[\partial_\mu \psi(\widehat{x}(T), \mathcal{P}[x(T)]; x(T))\right]. \end{cases} \tag{3.2}$$

and

$$\begin{cases} -\mathrm{d}y(t) & = l(t)dt - z\left(t\right)\mathrm{d}W\left(t\right) - K\left(t\right)\mathrm{d}\widetilde{W}\left(t\right) - \int_{\Theta} R\left(t,\theta\right)\widetilde{\eta}\left(\mathrm{d}\theta,\mathrm{d}t\right), \\ y(T) & = \psi(x(T),\mathcal{P}\left[x(t)\right]), \end{cases} \tag{3.3}$$

Clearly, under assumptions (H1) and (H2), it is easy to prove that BSDEs (3.3) and (3.2) admits a unique strong solutions, given by

$$\begin{aligned} y(t) &= \psi(x(T),\mathcal{P}\left[x(T)\right]) + \int_t^T l(s)\mathrm{d}s - \int_t^T z\left(s\right)\mathrm{d}W(s) - \int_t^T K\left(s\right)\mathrm{d}\widetilde{W}\left(s\right) \\ &\quad - \int_t^T\int_{\Theta} R\left(s,\theta\right)\widetilde{\eta}\left(\mathrm{d}\theta,\mathrm{d}s\right). \end{aligned}$$

and

$$\begin{aligned} \Phi\left(t\right) &= \psi_x(x\left(T\right),\mathcal{P}\left[x(t)\right]) + \widehat{\mathbb{E}}\left[\partial_\mu\psi(\widehat{x}\left(T\right),\mathcal{P}\left[x(t)\right];x(T))\right]. \\ &\quad + \int_t^T \Big[f_x\left(s\right)\Phi\left(s\right) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{f}\left(s\right)\widehat{\Phi}\left(s\right)\right] + \sigma_x\left(s\right)Q\left(s\right) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}\left(s\right)\widehat{Q}\left(s\right)\right] \\ &\quad + c_x\left(s\right)\overline{Q}\left(s\right) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{c}\left(s\right)\widehat{\overline{Q}}\left(s\right)\right] + l_x\left(s\right) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{l}\left(s\right)\right] \\ &\quad + \int_{\Theta}\left[g_x\left(s,\theta\right)R\left(s,\theta\right) + \widehat{\mathbb{E}}\left[\partial_\mu\widehat{g}\left(s,\theta\right)\widehat{R}\left(s,\theta\right)\right]\right]m\left(d\theta\right) + h_x\left(s\right)K(s)\Big]\mathrm{d}s \\ &\quad - \int_t^T Q(s)\mathrm{d}W(s) - \int_t^T \overline{Q}(s)\mathrm{d}\widetilde{W}(s) - \int_t^T\int_{\Theta} R\left(s,\theta\right)\widetilde{\eta}\left(\mathrm{d}\theta,\mathrm{d}s\right). \end{aligned}$$

The main result of this paper is stated in the following theorem.

Theorem 3.1 Let assumptions (H1) and (H2) hold. Let $(u^*(\cdot),\xi^*(t),x^*(\cdot))$ be the optimal solution of the control problem (2.4)-(2.6). *Then there exists* $(\Phi\left(\cdot\right),Q\left(\cdot\right),\overline{Q}(\cdot),K\left(\cdot\right),R\left(\cdot,\theta\right))$ *solution of* (3.2)-(3.3) *such that for any* $(u,\xi)\in\mathbb{A}_1\times\mathbb{A}_2$, we have $\mathbb{P}-a.s.,\ a.e.t\in[0,T]$,

$$\begin{aligned} 0 \leq\ & \mathbb{E}^u\left[H_u(t,x^*(t),\mathcal{P}\left[x^*(t)\right],u^*\left(t\right),\Phi\left(t\right),Q\left(t\right),\overline{Q}\left(t\right),K\left(t\right),R\left(t,\theta\right))\left(u\left(t\right)-u^*\left(t\right)\right)\mid\mathcal{F}_t^Y\right] \\ & +\mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\mathrm{d}\left(\xi-\xi^*\right)(t)\mid\mathcal{F}_t^Y\right], \end{aligned} \tag{3.4}$$

where the Hamiltonian function H *is defined by* (3.1).

3.2. Proof of main results

Double convex perturbation. To prove our main result, the approach that we use is based on a double perturbation of the optimal control. This perturbation is described as follows:
Let $(u(\cdot),\xi(\cdot))\in\mathcal{U}_1^Y\times\mathcal{U}_2^Y$, be any given admissible control. Let $\varepsilon\in(0,1)$, and write

$$u^\varepsilon(\cdot) = u^*(\cdot)+\varepsilon v(\cdot)\quad \text{ where } v(t)=u(t)-u^*(t), \tag{3.5}$$

and

$$\xi^\varepsilon(t)=\xi^*(t)+\varepsilon\zeta(t)\text{ where }\zeta(t)=\xi(t)-\xi^*(t), \tag{3.6}$$

where ε a sufficiently small $\varepsilon>0$. Here $(u^\varepsilon(\cdot),\xi^\varepsilon(\cdot))$ is the so called convex perturbation of $(u^*(\cdot),\xi^*(\cdot))$ defined as follows: for any $t\in[0,T]$

$$(u^\varepsilon(t),\xi^\varepsilon(t)) = (u^*(t),\xi^*(t))+\varepsilon\left[(u(t),\xi(t))-(u^*(t),\xi^*(t))\right],$$

Denote by $x^\varepsilon(\cdot)=x^{u^\varepsilon,\xi^\varepsilon}(\cdot)$ the solution of (2.4) associated with $(u^\varepsilon(\cdot),\xi^\varepsilon(\cdot))$ and by $\alpha^\varepsilon(\cdot)$ the solution of (2.8) corresponding to $u^\varepsilon(\cdot)$.

We denote by $x^\varepsilon(\cdot),x(\cdot),\alpha^\varepsilon(\cdot),\alpha(\cdot)$ the state trajectories of (2.4) and (2.8) corresponding respectively to $u^\varepsilon(\cdot)$ and $u(\cdot)$.

Short-hand notation. For simplification, we introduce the short-hand notation

$$\begin{aligned}\varphi(t) &= \varphi\left(t, x^{u,\xi}(t), \mathcal{P}[x^{u,\xi}(t)], u(t)\right), \\ \varphi^{\varepsilon}(t) &= \varphi(t, x^{\varepsilon}(t), \mathcal{P}[x^{\varepsilon}(t)], u^{\varepsilon}(t)),\end{aligned}$$

and

$$g(t,\theta) = g(t, x^{u,\xi}(t_-), \mathcal{P}\left[x^{u,\xi}(t_-)\right], u(t), \theta), \quad h(t) = h\left(t, x^{u,\xi}(t), u(t)\right),$$

$$g^{\varepsilon}(t,\theta) = g(t, x^{\varepsilon}(t_-), \mathcal{P}[x^{\varepsilon}(t_-)], u^{\varepsilon}(t), \theta), \qquad h^{\varepsilon}(t) = h(t, x^{\varepsilon}(t), u^{\varepsilon}(t)),$$

where g, h and $\varphi = f, \sigma, c, l$ as well as their partial derivatives with respect to x and u. Also, we will denote for $\varphi = f, \sigma, c, l$ and g :

$$\begin{aligned}&\partial_{\mu}\varphi(t) = \partial_{\mu}\varphi(t, x(t), \mathcal{P}[x(t)], u(t); \widehat{x}(t)), \\ &\partial_{\mu}\widehat{\varphi}(t) = \partial_{\mu}\varphi(t, \widehat{x}(t), \mathcal{P}[\widehat{x}(t)], \widehat{u}(t); x(t)),\end{aligned}$$

and

$$\begin{aligned}&\partial_{\mu}g(t,\theta) = \partial_{\mu}g(t, x(t_-), \mathcal{P}[x(t_-)], u(t), \theta; \widehat{x}(t)), \\ &\partial_{\mu}\widehat{g}(t,\theta) = \partial_{\mu}g(t, \widehat{x}(t), \mathcal{P}[\widehat{x}(t_-)], \widehat{u}(t), \theta; x(t)).\end{aligned}$$

In order to prove our main result in Theorem 3.1, we present some auxiliary results

Lemma 3.2 Suppose that assumptions (H1), (H2) and (H3) hold. Then, we have

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}|x^{\varepsilon}(t) - x^{*}(t)|^{2}\right] = 0. \tag{3.7}$$

Proof Applying standard estimates, the *Burkholder-Davis-Gundy inequality*, and Proposition A1 (Appendix), we have

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}|x^{\varepsilon}(t) - x^{*}(t)|^{2}\right] \\ &\le \mathbb{E}\int_{0}^{t}|f^{\varepsilon}(s) - f^{*}(s)|^{2}ds + \mathbb{E}^{u}\int_{0}^{t}|\sigma^{\varepsilon}(s) - \sigma^{*}(s)|^{2}\mathrm{d}s \\ &+ \mathbb{E}\int_{0}^{t}|c^{\varepsilon}(s) - c^{*}(s)|^{2}\mathrm{d}s + \mathbb{E}\int_{0}^{t}\int_{\Theta}|g^{\varepsilon}(s,\theta) - g^{*}(s,\theta)|^{2}m(\mathrm{d}\theta)\,ds \\ &+ \mathbb{E}\left|\int_{[0,t]}G(s)\mathrm{d}\left(\xi^{\varepsilon} - \xi^{*}\right)(s)\right|^{2},\end{aligned}$$

According to the Lipschitz conditions on the coefficients f, σ, c and g with respect to x, μ and u, (assumptions (H2)-(H3)), we obtain the following estimation:

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|x^{\varepsilon}(t) - x^{*}(t)|^{2}\right] &\le C_{T}\mathbb{E}\int_{0}^{t}\left[|x^{\varepsilon}(s) - x^{*}(s)|^{2} + |\mathbb{D}_{2}(\mathcal{P}[x^{\varepsilon}(s)], \mathcal{P}[x^{*}(s)])|^{2}\right]\mathrm{d}s \\ &+ C_{T}\varepsilon^{2}\mathbb{E}\int_{0}^{t}|u^{\varepsilon}(s) - u^{*}(s)|^{2}\mathrm{d}s \\ &+ C_{T}\varepsilon^{2}\mathbb{E}\,|\xi^{\varepsilon}(T) - \xi^{*}(T)|^{2}.\end{aligned} \tag{3.8}$$

Applying the definition of *Wasserstein metric* $\mathbb{D}_2(\cdot,\cdot)$, we have

$$\begin{aligned}\mathbb{D}_{2}(\mathcal{P}[x^{\varepsilon}(s)], \mathcal{P}[x^{*}(s)]) &= \inf\Big\{\left[\mathbb{E}\,|\widetilde{x}^{\varepsilon}(s) - \widetilde{x}^{*}(s)|^{2}\right]^{\frac{1}{2}}, \text{ for } \widetilde{x}^{\varepsilon}(\cdot), \widetilde{x}^{*}(\cdot) \in \mathbb{L}^{2}\left(\mathcal{F};\mathbb{R}^{d}\right), \\ &\qquad \mathcal{P}[x^{\varepsilon}(s)] = \mathcal{P}[\widetilde{x}^{\varepsilon}(s)] \text{ and } \mathcal{P}[x^{*}(s)] = \mathcal{P}[\widetilde{x}^{*}(s)]\Big\} \\ &\le \left[\mathbb{E}\,|x^{\varepsilon}(s) - x^{*}(s)|^{2}\right]^{\frac{1}{2}}.\end{aligned} \tag{3.9}$$

By Definition 2.2 and from (3.8) and (3.9), we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x^{\varepsilon}(t)-x^{*}(t)\right|^{2}\right]\leq C_{T}\mathbb{E}\int_{0}^{t}\sup_{r\in[0,s]}\left|x^{\varepsilon}(r)-x^{*}(r)\right|^{2}\mathrm{d}s+M_{T}\varepsilon^{2}.$$

Finally, applying Gronwall's inequality, the desired result (3.7) follows immediately by letting ε go to 0. This achieve the proof of Lemma 3.2. □

Variational equations. Now, we introduce the following variational equations involved in the stochastic maximum principle for our control problem

$$\begin{cases}\mathrm{d}\mathcal{Z}(t)=\left[f_{x}\left(t\right)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_{\mu}f\left(t\right)\widehat{\mathcal{Z}}\left(t\right)\right]+f_{u}(t)(u(t)-u^{*}(t))\right]\mathrm{d}t\\ \qquad+\left[\sigma_{x}(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_{\mu}\sigma\left(t\right)\widehat{\mathcal{Z}}\left(t\right)\right]+\sigma_{u}(t)(u(t)-u^{*}(t))\right]\mathrm{d}W(t)\\ \qquad+\left[c_{x}(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_{\mu}c\left(t\right)\widehat{\mathcal{Z}}\left(t\right)\right]+c_{u}\left(t\right)\left(u(t)-u^{*}(t)\right)\right]\mathrm{d}\widetilde{W}\left(t\right)\\ \quad+\int_{\Theta}\left[g_{x}(t,\theta)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_{\mu}g\left(t,\theta\right)\widehat{\mathcal{Z}}\left(t\right)\right]+g_{u}(t,\theta)(u(t)-u^{*}(t))\right]\widetilde{\eta}\left(\mathrm{d}\theta,\mathrm{d}t\right),\\ \quad+G(t)\mathrm{d}\left(\xi-\xi^{*}\right)(t),\\ \mathcal{Z}(0)=0,\end{cases}\tag{3.10}$$

and

$$\begin{cases}\mathrm{d}\alpha_{1}(t)= & \left[\alpha_{1}(t)h(t)+\alpha(t)h_{x}(t)\mathcal{Z}(t)+\alpha(t)h_{u}(t)(u(t)-u^{*}(t))\right]\mathrm{d}Y(t),\\ \alpha_{1}(0)= & 0.\end{cases}\tag{3.11}$$

Under assumptions (H1) and (H2), equations (3.10) and (3.11) admits a unique adapted solutions $\mathcal{Z}\left(\cdot\right)$ and $\alpha_{1}\left(\cdot\right)$, respectively.

Lemma 3.3 Suppose that assumptions (H1), (H2) and (H3) hold. Then, we have

$$\lim_{\varepsilon\rightarrow0}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{x^{\varepsilon}(t)-x(t)}{\varepsilon}-\mathcal{Z}(t)\right|^{2}=0.\tag{3.12}$$

Proof Let $\gamma^{\varepsilon}(t)=\frac{x^{\varepsilon}(t)-x^{*}(t)}{\varepsilon}-\mathcal{Z}(t),\ t\in[0,T]$. To simplify, we will use the following notations, for $\varphi=f,\sigma,c,l$ and g :

$$\begin{aligned}\varphi_{x}^{\lambda,\varepsilon}\left(t\right) &= \varphi_{x}\left(t,x^{\lambda,\varepsilon}\left(t\right),\mathcal{P}\left[x^{\varepsilon}(t)\right],u^{\varepsilon}(t)\right),\\ g_{x}^{\lambda,\varepsilon}\left(t,\theta\right) &= g_{x}\left(t,x^{\lambda,\varepsilon}\left(t\right),\mathcal{P}\left[x^{\varepsilon}(t)\right],u^{\varepsilon}(t),\theta\right),\\ \partial_{\mu}^{\lambda,\varepsilon}\varphi\left(t\right) &= \partial_{\mu}\varphi(s,x^{\varepsilon}(t),\mathcal{P}\left[\widehat{x}^{\lambda,\varepsilon}\left(t\right)\right],u^{\varepsilon}(t);\widehat{x}(t)),\\ \partial_{\mu}^{\lambda,\varepsilon}g\left(t,\theta\right) &= \partial_{\mu}g(t,x^{\varepsilon}(t),\mathcal{P}\left[\widehat{x}^{\lambda,\varepsilon}\left(t\right)\right],u^{\varepsilon}(t),\theta;\widehat{x}(t)),\end{aligned}$$

and

$$\begin{aligned}&x^{\lambda,\varepsilon}\left(s\right)=x^{*}\left(s\right)+\lambda\varepsilon\left(\gamma^{\varepsilon}\left(s\right)+\mathcal{Z}\left(s\right)\right),\\ &\widehat{x}^{\lambda,\varepsilon}\left(s\right)=x^{*}(s)+\lambda\varepsilon(\widehat{\gamma}^{\varepsilon}(s)+\widehat{\mathcal{Z}}\left(s\right)),\\ &u^{\lambda,\varepsilon}\left(s\right)=u^{*}\left(s\right)+\lambda\varepsilon v\left(s\right).\end{aligned}$$

By simple computations, we get

$$
\begin{aligned}
\gamma^{\varepsilon}(t) = & \frac{1}{\varepsilon}\int_0^t \left[f^{\varepsilon}(s)-f^{*}(s)\right]\mathrm{d}s+\frac{1}{\varepsilon}\int_0^t \left[\sigma^{\varepsilon}(s)-\sigma^{*}(s)\right]\mathrm{d}W(s) \\
& +\frac{1}{\varepsilon}\int_0^t \left[c^{\varepsilon}(s)-c^{*}(s)\right]\mathrm{d}\widetilde{W}(s)+\frac{1}{\varepsilon}\int_0^t\int_{\Theta}\left[g^{\varepsilon}(s,\theta)-g^{*}(s,\theta)\right]\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s) \\
& +\frac{1}{\varepsilon}\int_{[0,t]} G(s)\mathrm{d}\left(\xi^{\varepsilon}-\xi^{*}\right)(s) \\
& -\int_0^t\left[f_x(s)\mathcal{Z}(s)+\widehat{\mathbb{E}}\left[\partial_{\mu}f(s)\widehat{\mathcal{Z}}(s)\right]+f_u(s)\left(u(s)-u^{*}(s)\right)\right]\mathrm{d}s \\
& -\int_0^t\left[\sigma_x(s)\mathcal{Z}(s)+\widehat{\mathbb{E}}\left[\partial_{\mu}\sigma(s)\widehat{\mathcal{Z}}(s)\right]+\sigma_u(s)\left(u(s)-u^{*}(s)\right)\right]\mathrm{d}W(s) \\
& -\int_0^t\left[c_x(s)\mathcal{Z}(s)+\widehat{\mathbb{E}}\left[\partial_{\mu}c(s)\widehat{\mathcal{Z}}(s)\right]+c_u(s)\left(u(s)-u^{*}(s)\right)\right]\mathrm{d}\widetilde{W}(s) \\
& -\int_0^t\int_{\Theta}\left[g_x(s,\theta)\mathcal{Z}(s)+\widehat{\mathbb{E}}\left[\partial_{\mu}g(s,\theta)\widehat{\mathcal{Z}}(s)\right]+g_u(s,\theta)\left(u(s)-u^{*}(s)\right)\right]\widetilde{\eta}(\mathrm{d}\theta,ds) \\
& -\int_{[0,t]} G(s)\mathrm{d}\left(\xi-\xi^{*}\right)(s).
\end{aligned}
$$

Now, we decompose $\frac{1}{\varepsilon}\int_0^t\left[f^{\varepsilon}(s)-f^{*}(s)\right]\mathrm{d}s$ into the following parts

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t\left[f^{\varepsilon}(s)-f^{*}(s)\right]\mathrm{d}s \\
& =\frac{1}{\varepsilon}\int_0^t\left[f(s,x^{\varepsilon}(s),\mathcal{P}\left[x^{\varepsilon}(s)\right],u^{\varepsilon}(s))-f(s,x^{*}(s),\mathcal{P}\left[x^{*}(s)\right],u^{*}(s))\right]\mathrm{d}s \\
& =\frac{1}{\varepsilon}\int_0^t\left[f(s,x^{\varepsilon}(s),\mathcal{P}\left[x^{\varepsilon}(s)\right],u^{\varepsilon}(s))-f(s,x^{*}(s),\mathcal{P}\left[x^{\varepsilon}(s)\right],u^{\varepsilon}(s))\right]\mathrm{d}s \\
& \quad+\frac{1}{\varepsilon}\int_0^t\left[f(s,x^{*}(s),\mathcal{P}\left[x^{\varepsilon}(s)\right],u^{\varepsilon}(s))-f(s,x^{*}(s),\mathcal{P}\left[x^{*}(s)\right],u^{\varepsilon}(s))\right]\mathrm{d}s \\
& \quad+\frac{1}{\varepsilon}\int_0^t\left[f(s,x^{*}(s),\mathcal{P}\left[x^{*}(s)\right],u^{\varepsilon}(s))-f(s,x^{*}(s),\mathcal{P}\left[x^{*}(s)\right],u^{*}(s))\right]\mathrm{d}s.
\end{aligned}
$$

We notice that

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t\left[f^{\varepsilon}(s)-f(s,x^{*}(s),\mathcal{P}\left[x^{\varepsilon}(s)\right],u^{\varepsilon}(s))\right]ds=\int_0^t\int_0^1\left[f_x^{\lambda,\varepsilon}(s)\left(\gamma^{\varepsilon}(s)+\mathcal{Z}(s)\right)\right]\mathrm{d}\lambda\mathrm{d}s, \\
& \frac{1}{\varepsilon}\int_0^t\left[f^{\varepsilon}(s)-f(s,x^{\varepsilon}(s),\mathcal{P}\left[x^{*}(s)\right],u^{\varepsilon}(s)\right]ds=\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\partial_{\mu}^{\lambda,\varepsilon}f(s)\left(\widehat{\gamma}^{\varepsilon}(s)+\widehat{\mathcal{Z}}(s)\right)\right]\mathrm{d}\lambda\mathrm{d}s,
\end{aligned}
$$

and

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_0^t\left[f\left(s,x(s),\mathcal{P}\left[x(s)\right],u^{\varepsilon}(s)\right)-f(s,x^{*}(s),\mathcal{P}\left[x^{*}(s)\right],u^{*}(s))\right]\mathrm{d}s \\
= & \int_0^t\int_0^1\left[f_u\left(s,x(s),\mathcal{P}\left[x(s)\right],u^{\lambda,\varepsilon}(s)\right)\left(u(s)-u^{*}(s)\right)\right]\mathrm{d}\lambda\mathrm{d}s.
\end{aligned}
$$

By applying similar method developed above, the analogue approachs hold for the coefficients σ, c and g. Moreover, from (3.6), we obtain

$$
\frac{1}{\varepsilon}\int_{[0,t]}G(s)\mathrm{d}\left(\xi^{\varepsilon}-\xi^{*}\right)(s)-\int_{[0,t]}G(s)\mathrm{d}\left(\xi-\xi^{*}\right)(s)=0.
$$

Now, we turn our attention to estimate $\gamma^{\varepsilon}(s)$, then we get

$$\begin{aligned}
\mathbb{E}\left[\sup_{s\in[0,t]}\left|\gamma^{\varepsilon}(s)\right|^{2}\right] &= C(t)\,\mathbb{E}\left[\int_{0}^{t}\int_{0}^{1}\left|f_{x}^{\lambda,\varepsilon}(s)\,\gamma^{\varepsilon}(s)\right|^{2}\mathrm{d}\lambda\mathrm{d}s\right.\\
&\quad+\int_{0}^{t}\int_{0}^{1}\widehat{\mathbb{E}}\left|\partial_{\mu}^{\lambda,\varepsilon}f(s)\,\widehat{\gamma}^{\varepsilon}(s)\right|^{2}\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\left|\sigma_{x}^{\lambda,\varepsilon}(s)\,\gamma^{\varepsilon}(s)\right|^{2}\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\widehat{\mathbb{E}}\left|\partial_{\mu}^{\lambda,\varepsilon}\sigma(s)\widehat{\gamma}^{\varepsilon}(s)\right|^{2}\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\left|c_{x}^{\lambda,\varepsilon}(s)\,\gamma^{\varepsilon}(s)\right|^{2}\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\widehat{\mathbb{E}}\left|\partial_{\mu}^{\lambda,\varepsilon}c(s)\widehat{\gamma}^{\varepsilon}(s))\right|^{2}\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{\Theta}\int_{0}^{1}\left|g_{x}^{\lambda,\varepsilon}(s,\theta)\,\gamma^{\varepsilon}(s)\right|^{2}\mathrm{d}\lambda m(\mathrm{d}\theta)\,\mathrm{d}s\\
&\quad\left.+\int_{0}^{t}\int_{\Theta}\int_{0}^{1}\widehat{\mathbb{E}}\left|\partial_{\mu}^{\lambda,\varepsilon}g(s,\theta)\,\widehat{\gamma}^{\varepsilon}(s))\right|^{2}\mathrm{d}\lambda m(\mathrm{d}\theta)\,\mathrm{d}s\right]\\
&\quad+C(t)\,\mathbb{E}\left[\sup_{s\in[0,t]}\left|\pi^{\varepsilon}(s)\right|^{2}\right],
\end{aligned}$$

where

$$\begin{aligned}
\pi^{\varepsilon}(t) &= \int_{0}^{t}\int_{0}^{1}\left[f_{x}^{\lambda,\varepsilon}(s)-f_{x}(s)\right]\mathcal{Z}(s)\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\widehat{\mathbb{E}}\left[\left(\partial_{\mu}^{\lambda,\varepsilon}f(s)-\partial_{\mu}f(s)\right)\widehat{\mathcal{Z}}(s)\right]\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\left[f_{u}\left(s,x(s),\mathcal{P}[x(s)],v^{\lambda,\varepsilon}(s)\right)-f_{u}(s)\right](u(s)-u^{*}(s))\,\mathrm{d}\lambda\mathrm{d}s\\
&\quad+\int_{0}^{t}\int_{0}^{1}\left[\sigma_{x}^{\lambda,\varepsilon}(s)-\sigma_{x}(s)\right]\mathcal{Z}(s)\mathrm{d}\lambda\mathrm{d}W(s)\\
&\quad+\int_{0}^{t}\int_{0}^{1}\widehat{\mathbb{E}}\left[\left(\partial_{\mu}^{\lambda,\varepsilon}\sigma(s)-\partial_{\mu}\sigma(s)\right)\widehat{\mathcal{Z}}(s)\right]\mathrm{d}\lambda\mathrm{d}W(s)\\
&\quad+\int_{0}^{t}\int_{0}^{1}\left[\sigma_{u}\left(s,x(s),\mathcal{P}[x(s)],v^{\lambda,\varepsilon}(s)\right)-\sigma_{u}(s)\right](u(s)-u^{*}(s))\,\mathrm{d}\lambda\mathrm{d}W(s)
\end{aligned}$$

$$+\int_0^t\int_0^1\left[c_x^{\lambda,\varepsilon}(s)-c_x(s)\right]\mathcal{Z}(s)\mathrm{d}\lambda\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}c(s)-\partial_\mu c(s)\right)\widehat{\mathcal{Z}}(s)\right]\mathrm{d}\lambda\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_0^1\left[c_u\left(s,x(s),\mathcal{P}[x(s)],u^{\lambda,\varepsilon}(s)\right)-c_u(s)\right](u(s)-u^*(s))\mathrm{d}\lambda\mathrm{d}\widetilde{W}(s)$$
$$+\int_0^t\int_\Theta\int_0^1\left[g_x^{\lambda,\varepsilon}(s,\theta)-g_x(s,\theta)\right]\mathcal{Z}(s_-)\mathrm{d}\lambda\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s)$$
$$+\int_0^t\int_\Theta\int_0^1\widehat{\mathbb{E}}\left[\left(\partial_\mu^{\lambda,\varepsilon}g(s,\theta)-\partial_\mu g(s,\theta)\right)\widehat{\mathcal{Z}}(s_-)\right]\mathrm{d}\lambda\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s)$$
$$+\int_0^t\int_\Theta\int_0^1\left[g_u\left(s,x(s),\mathcal{P}[x(s)],u^{\lambda,\varepsilon}(s),\theta\right)-g_u(s,\theta)\right](u(s)-u^*(s))\mathrm{d}\lambda\widetilde{\eta}(\mathrm{d}\theta,\mathrm{d}s).$$

Now, the derivatives of f,σ,c and g with respect to (x,μ,u) are Lipschitz continuous in (x,μ,u), we get

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{s\in[0,T]}|\pi^\varepsilon(s)|^2\right]=0.$$

Note that since the derivatives of the coefficients f,σ,c and γ are bounded with respect to (x,μ,u), we obtain

$$\mathbb{E}\left[\sup_{s\in[0,t]}|\gamma^\varepsilon(s)|^2\right]\leq C(t)\left\{\mathbb{E}\int_0^t|\gamma^\varepsilon(s)|^2ds+\mathbb{E}\left[\sup_{s\in[0,t]}|\pi^\varepsilon(s)|^2\right]\right\}.$$

By applying Gronwall's lemma, we obtain $\forall t\in[0,T]$

$$\mathbb{E}\left[\sup_{s\in[0,t]}|\gamma^\varepsilon(s)|^2\right]\leq C(t)\left\{\mathbb{E}\left[\sup_{s\in[0,t]}|\pi^\varepsilon(s)|^2\right]\exp\left\{\int_0^t C(s)\,ds\right\}\right\}.$$

Finally, the proof of Lemma 3.3 is fulfilled by putting $t=T$ and letting ε go to zero. $\square$

Now, we introduce the following lemma which play an important role in computing the variational inequality.

Lemma 3.4. Let assumption (H1) hold. Then, we have

$$\lim_{\varepsilon\to 0}\sup_{0\leq t\leq T}\mathbb{E}\left|\frac{\alpha^\varepsilon(t)-\alpha^*(t)}{\varepsilon}-\alpha_1(t)\right|^2=0. \tag{3.13}$$

Proof. From the definition of $\alpha^*(\cdot)$ and $\alpha_1(\cdot)$, we obtain

$$\begin{aligned}\alpha^*(t)+\varepsilon\alpha_1(t)&=\alpha^*(0)+\int_0^t\alpha^*(s)h^*(s)\mathrm{d}Y(s)\\&+\varepsilon\int_0^t\left[\alpha_1(s)h^*(s)+\alpha^*(s)h_x(s)\mathcal{Z}(s)+\alpha^*(s)h_u(s)(u(s)-u^*(s))\right]\mathrm{d}Y(s)\\&=\alpha^*(0)+\varepsilon\int_0^t\alpha_1(s)h^*(s)\mathrm{d}Y(s)\\&+\int_0^t\alpha^*(s)h(s,x^*(s)+\varepsilon\mathcal{Z}(s),u^*(s)+\varepsilon v(s))\mathrm{d}Y(s)\\&-\varepsilon\int_0^t\alpha^*(s)\left[\ell_0^\varepsilon(s)\right]\mathrm{d}Y(s),\end{aligned}$$

where

$$\ell_0^\varepsilon(s)=\int_0^1\left[h_x(s,x^*(s)+\lambda\varepsilon\mathcal{Z}(s),u^*(s)+\lambda\varepsilon v(s))-h_x(s)\right]\mathcal{Z}(s)d\lambda$$
$$+\int_0^1\left[h_u(s,x^*(s)+\lambda\varepsilon\mathcal{Z}(s),u^*(s)+\lambda\varepsilon v(s))-h_u(s)\right](u(s)-u^*(s))d\lambda.$$

Then, we have

$$\begin{aligned}
&\alpha^\varepsilon(t)-\alpha^*(t)-\varepsilon\alpha_1(t)\\
&=\int_0^t\alpha^\varepsilon(s)\,h^\varepsilon(t)\,dY(s)-\varepsilon\int_0^t\alpha_1(s)h^*(s)\mathrm{d}Y(s)\\
&\quad-\int_0^t\alpha^*(s)h\left(s,x^*(s)+\varepsilon\mathcal{Z}(s),u^*(s)+\varepsilon v(s)\right)dY(s)+\varepsilon\int_0^t\alpha^*(s)\ell_0^\varepsilon(s)\,\mathrm{d}Y(s)\\
&=\int_0^t\left(\alpha^\varepsilon(s)-\alpha^*(s)-\varepsilon\alpha_1(s)\right)h^\varepsilon(s)\,\mathrm{d}Y(s)\\
&\quad+\int_0^t\left(\alpha^*(s)+\varepsilon\alpha_1(s)\right)\left[h^\varepsilon(s)-h\left(s,x^*(s)+\varepsilon\mathcal{Z}(s),u^*(s)+\varepsilon v(s)\right)\right]\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_0^t\alpha_1(s)\,h(s,x^*(s)+\varepsilon\mathcal{Z}(s),u^*(s)+\varepsilon v(s))\mathrm{d}Y(s)\\
&\quad-\varepsilon\int_0^t\alpha_1(s)\,h^*(s)\mathrm{d}Y(s)+\varepsilon\int_0^t\alpha^*(s)\ell_0^\varepsilon(s)\,\mathrm{d}Y(s)\\
&=\int_0^t\left(\alpha^\varepsilon(s)-\alpha^*(s)-\varepsilon\alpha_1(s)\right)h^\varepsilon(s)\,\mathrm{d}Y(s)\\
&\quad+\int_0^t(\alpha^*(s)+\varepsilon\alpha_1(s))\ell_1^\varepsilon(s)\mathrm{d}Y(s)+\varepsilon\int_0^t\alpha_1(s)\ell_2^\varepsilon(s)\mathrm{d}Y(s)\\
&\quad+\varepsilon\int_0^t\alpha^*(s)\ell_0^\varepsilon(s)\mathrm{d}Y(s),
\end{aligned}$$

where

$$\begin{aligned}
\ell_1^\varepsilon(s)&=h^\varepsilon(s)-h\left(s,x^*(s)+\varepsilon\mathcal{Z}(s),u^*(s)+\varepsilon v(s)\right),\\
\ell_2^\varepsilon(s)&=h(s,x^*(s)+\varepsilon\mathcal{Z}(s),u^*(s)+\varepsilon v(s))-h^*(s).
\end{aligned}\tag{3.14}$$

From (3.14), we have

$$\begin{aligned}
\ell_1^\varepsilon(s)\quad=\quad&\int_0^1\left[h_x(s,x^*(s)+\varepsilon\mathcal{Z}(s)+\lambda(x^\varepsilon(s)-x^*(s)-\varepsilon\mathcal{Z}(s)),v^\varepsilon(s))\right]\\
&\times(x^\varepsilon(s)-x^*(s)-\varepsilon\mathcal{Z}(s))\mathrm{d}\lambda.
\end{aligned}$$

By Lemma 3.3, we have

$$\mathbb{E}\int_0^t\left|(\alpha^*(s)+\varepsilon\alpha_1(s))\ell_1^\varepsilon(s)\right|^2\mathrm{d}s\leq\varepsilon^2C(\varepsilon),\tag{3.15}$$

here $C(\varepsilon)$ denotes some nonnegative constant such that $C(\varepsilon)\to 0$ as $\varepsilon\to 0$.
Moreover, it is easy to see that

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\varepsilon\int_0^t\alpha^*(s)\ell_0^\varepsilon(s)\mathrm{d}Y(s)\right]^2\leq\varepsilon^2C(\varepsilon),\tag{3.16}$$

and

$$\sup_{0\leq t\leq T}\mathbb{E}\left[\varepsilon\int_0^t\alpha_1(s)\ell_2^\varepsilon(s)\mathrm{d}Y(s)\right]^2\leq\varepsilon^2C(\varepsilon).\tag{3.17}$$

From (3.15), (3.16) and (3.17), we get

$$\begin{aligned}
&\mathbb{E}\left|(\alpha^{\varepsilon}(t)-\alpha^{*}(t))-\varepsilon\alpha_{1}(t)\right|^{2} \\
&\leq C\left[\int_{0}^{t}\mathbb{E}\left|(\alpha^{\varepsilon}(s)-\alpha^{*}(s))-\varepsilon\alpha_{1}(s)\right|^{2}+\mathbb{E}\int_{0}^{t}\left|(\alpha^{*}(s)+\varepsilon\alpha_{1}(s))\ell_{1}^{\varepsilon}(s)\right|^{2}\mathrm{d}s\right. \\
&\quad\left.+\sup_{0\leq s\leq t}\mathbb{E}\left(\varepsilon\int_{0}^{t}\alpha^{*}(s)\ell_{0}^{\varepsilon}(s)\mathrm{d}Y(s)\right)^{2}+\sup_{0\leq s\leq t}\mathbb{E}\left(\varepsilon\int_{0}^{t}\alpha_{1}(s)\ell_{2}^{\varepsilon}(s)\mathrm{d}Y(s)\right)^{2}\right] \\
&\leq C\int_{0}^{t}\mathbb{E}\left|\alpha^{\varepsilon}(s)-\alpha^{*}(s)-\varepsilon\alpha_{1}(s)\right|^{2}\mathrm{d}s+C(\varepsilon)\varepsilon^{2}.
\end{aligned}$$

Finally, by using Gronwall's inequality, the proof of Lemma 3.4 is complete. □

Lemma 3.5. Let assumption (H1), (H2) and (H3) hold. Then, we have

$$\begin{aligned}
0\leq\mathbb{E}\int_{0}^{T}&\Big[\alpha_{1}(t)\,l(t)+\alpha^{*}(t)\,l_{x}(t)\mathcal{Z}(t)+\alpha^{*}(t)\widehat{\mathbb{E}}\left[\partial_{\mu}l(t)\right]\mathcal{Z}(t) \\
&+\alpha^{*}(t)l_{u}(t)(u(t)-u^{*}(t))\Big]\,dt \\
&+\mathbb{E}\left[\alpha_{1}(T)\,\psi(x(T),\mathcal{P}\left[x(T)\right])\right]+\mathbb{E}\left[\alpha^{*}(T)\,\psi_{x}(x(T),\mathcal{P}\left[x(T)\right])\mathcal{Z}(T)\right] \\
&+\mathbb{E}\left[\alpha^{*}(T)\,\widehat{\mathbb{E}}\left[\partial_{\mu}\psi(x(T),\mathcal{P}\left[x(T)\right];\widehat{x}(T))\right]\mathcal{Z}(T)\right] \\
&+\mathbb{E}\int_{[0,T]}\alpha^{*}(t)\,M(t)\mathrm{d}\left(\xi-\xi^{*}\right)(t).
\end{aligned}\tag{3.18}$$

Proof. From (2.6), we have

$$\begin{aligned}
0&\leq\frac{1}{\varepsilon}\left[J\left(u^{\varepsilon}(t),\xi^{\varepsilon}(t)\right)-J\left(u^{*}(t),\xi^{*}(t)\right)\right] \\
&=\frac{1}{\varepsilon}\left[J\left(u^{\varepsilon}(t),\xi^{\varepsilon}(t)\right)-J\left(u^{*}(t),\xi^{\varepsilon}(t)\right)\right] \\
&\quad+\frac{1}{\varepsilon}\left[J\left(u^{*},\xi^{\varepsilon}(t)\right)-J\left(u^{*}(t),\xi^{*}(t)\right)\right] \\
&=\mathcal{J}_{1}+\mathcal{J}_{2}.
\end{aligned}\tag{3.19}$$

From (2.11), we get

$$\begin{aligned}
\mathcal{J}_{1}&=\frac{1}{\varepsilon}\left[J\left(u^{\varepsilon}(t),\xi^{\varepsilon}(t)\right)-J\left(u^{*}(t),\xi^{\varepsilon}(t)\right)\right] \\
&=\frac{1}{\varepsilon}\mathbb{E}\int_{0}^{T}\left[\alpha^{\varepsilon}(t)l^{\varepsilon}(t)-\alpha^{*}(t)l(t)\right]\mathrm{d}t \\
&\quad+\frac{1}{\varepsilon}\mathbb{E}\left[\alpha^{\varepsilon}(T)\,\psi(x^{\varepsilon}(T),\mathcal{P}\left[x^{\varepsilon}(T)\right])-\alpha^{*}(T)\,\psi(x^{*}(T),\mathcal{P}\left[x^{*}(T)\right])\right],
\end{aligned}\tag{3.20}$$

and by simple computation, the second term $\mathcal{J}_2$ being

$$\begin{aligned}
\mathcal{J}_{2}&=\frac{1}{\varepsilon}\left[J\left(u^{*}(t),\xi^{\varepsilon}(t)\right)-J\left(u^{*}(t),\xi^{*}(t)\right)\right] \\
&=\frac{1}{\varepsilon}\left[\mathbb{E}\int_{[0,T]}\alpha^{*}(t)M(t)\mathrm{d}\xi^{\varepsilon}(t)-\int_{[0,T]}\alpha^{*}(t)M(t)\mathrm{d}\xi^{*}(t)\right].
\end{aligned}\tag{3.21}$$

Using the Taylor expansion, Lemmas 3.3 and Lemma 3.4, we get

$$\begin{aligned}
&\lim_{\varepsilon\to0}\varepsilon^{-1}\mathbb{E}\left[\alpha^{\varepsilon}(T)\,\psi(x^{\varepsilon}(T),\mathcal{P}\left[x^{\varepsilon}(T)\right])-\alpha^{*}(T)\,\psi(x^{*}(T),\mathcal{P}\left[x^{*}(T)\right])\right] \\
&=\mathbb{E}\left[\alpha_{1}(T)\psi(x(T),\mathcal{P}\left[x(t)\right])+\alpha^{*}(T)\psi_{x}(x(T),\mathcal{P}\left[x(T)\right])\mathcal{Z}(T)\right] \\
&+\mathbb{E}\left[\alpha^{*}(T)\,\widehat{\mathbb{E}}\left[\partial_{\mu}\psi(x(T),\mathcal{P}\left[x(t)\right];\widehat{x}(T))\right]\mathcal{Z}(T)\right],
\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon^{-1}\mathbb{E}\int_0^T\left[\alpha^{\varepsilon}(t)l^{\varepsilon}(t)-\alpha^*(t)l(t)\right]dt\\
&=\mathbb{E}\int_0^T\Big[\alpha_1(t)l(t)+\alpha^*(t)l_x(t)\mathcal{Z}(t)+\alpha^*(t)\widehat{\mathbb{E}}\left[\partial_\mu l\left(t\right)\right]\widehat{\mathcal{Z}}(t) \\
&+\alpha^*(t)l_u(t)(u\left(t\right)-u^*(t))\Big]\,dt
\end{aligned}\tag{3.23}$$

From (3.6), and since $\xi^{\varepsilon}(t)-\xi^*(t)=\varepsilon(\xi(t)-\xi^*(t))$, we get

$$\begin{aligned}
\mathcal{J}_2&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\mathbb{E}\int_{[0,T]}\alpha^*(t)M(t)\mathrm{d}\xi^{\varepsilon}(t)-\int_{[0,T]}\alpha^*(t)M(t)\mathrm{d}\xi^*(t)\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\mathbb{E}\int_{[0,T]}\alpha^*(t)M(t)\mathrm{d}(\xi^{\varepsilon}-\xi^*)(t)\right]\\
&=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\mathbb{E}\int_{[0,T]}\varepsilon\alpha^*(t)M(t)\mathrm{d}(\xi-\xi^*)(t)\right]\\
&=\mathbb{E}\int_{[0,T]}\alpha^*(t)M(t)\mathrm{d}(\xi-\xi^*)(t).
\end{aligned}\tag{3.24}$$

Substituting (3.22), (3.23) and (3.24) into (3.19), the desired result (3.18) fulfilled immediately. This achieve the proof of Lemma 3.5 . □

Let $\widetilde{\alpha}(t)=\frac{\alpha_1(t)}{\alpha^*(t)}$ then we have

$$\begin{cases}
d\widetilde{\alpha}(t)=\left\{h_x(t)\mathcal{Z}(t)+h_u(t)(u\left(t\right)-u^*(t))\right\}d\widetilde{W}(t),\\
\widetilde{\alpha}(0)=0,
\end{cases}\tag{3.25}$$

Lemma 3.6 Let $\Phi(\cdot)$ and $\mathcal{Z}\left(\cdot\right)$ be the solutions of (3.2) and (3.10) respectively. Then we have

$$\begin{aligned}
\mathbb{E}^u\left[\Phi\left(T\right)\mathcal{Z}\left(T\right)\right]&=\mathbb{E}^u\int_0^T\Phi\left(t\right)f_u(t)(u\left(t\right)-u^*(t))\mathrm{d}t+\mathbb{E}^u\int_0^T q(t)\sigma_u(t)(u\left(t\right)-u^*(t))\mathrm{d}t\\
&+\mathbb{E}^u\int_0^T\overline{q}(t)c_u(t)(u\left(t\right)-u^*(t))\mathrm{d}t-\mathbb{E}^u\int_0^T\mathcal{Z}\left(t\right)(l_x\left(t\right)+\widehat{\mathbb{E}}(\partial_\mu\widehat{l}\left(t\right)))\mathrm{d}t\\
&+\mathbb{E}^u\int_0^T\int_\Theta R\left(t,\theta\right)g_u(t,\theta)(u\left(t\right)-u^*(t))m\left(d\theta\right)dt\\
&+\mathbb{E}^u\int_0^T\Phi(t)G(t)\mathrm{d}(\xi-\xi^*)(t),
\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}
\mathbb{E}^u\left[y\left(T\right)\widetilde{\alpha}\left(T\right)\right]&=\mathbb{E}^u\int_0^T k\left(t\right)\left[h_x(t)\mathcal{Z}(t)+h_u(t)(u\left(t\right)-u^*(t))\right]\mathrm{d}t.\\
&-\mathbb{E}^u\int_0^T\widetilde{\alpha}\left(t\right)l(t)\mathrm{d}t.
\end{aligned}\tag{3.27}$$

Proof. By applying Itô's formula to $\Phi\left(t\right)\mathcal{Z}\left(t\right)$, $y\left(t\right)\widetilde{\alpha}\left(t\right)$ and taking expectation respectively, where

$\mathcal{Z}(0)=0$ and $\widetilde{\alpha}(0)=0$, we obtain

$$\begin{aligned}
&\mathbb{E}^u\left[\Phi\left(T\right)\mathcal{Z}\left(T\right)\right]\\
&=\mathbb{E}^u\int_0^T\Phi\left(t\right)\mathrm{d}\mathcal{Z}\left(t\right)+\mathbb{E}^u\int_0^T\mathcal{Z}\left(t\right)\mathrm{d}\Phi\left(t\right)\\
&+\mathbb{E}^u\int_0^T Q(t)\left[\sigma_x(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_\mu\sigma(t)\widehat{\mathcal{Z}}(t)\right]+\sigma_u(t)(u\left(t\right)-u^*(t))\right]\mathrm{d}t\\
&+\mathbb{E}^u\int_0^T\overline{Q}(t)\left[c_x(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_\mu c(t)\widehat{\mathcal{Z}}(t)\right]+c_u(t)(u\left(t\right)-u^*(t))\right]\mathrm{d}t\\
&+\mathbb{E}^u\int_0^T\int_\Theta R\left(t,\theta\right)\left[g_x(t,\theta)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_\mu g\left(t,\theta\right)\widehat{\mathcal{Z}}\left(t\right)\right]+g_u(t,\theta)(u\left(t\right)-u^*(t))\right]m\left(d\theta\right)\mathrm{d}t\\
&=I_1\left(T\right)+I_2\left(T\right)+I_3\left(T\right)+I_4\left(T\right).
\end{aligned}\tag{3.28}$$

First, note that

$$\begin{aligned}
I_1\left(T\right)&=\mathbb{E}^u\int_0^T\Phi\left(t\right)\mathrm{d}\mathcal{Z}\left(t\right)\\
&=\mathbb{E}^u\int_0^T\Phi\left(t\right)\left[f_x(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\mathcal{Z}}(t)\right]+f_u(t)(u\left(t\right)-u^*(t))\right]\mathrm{d}t\\
&+\mathbb{E}^u\int_0^T\Phi(t)G(t)\mathrm{d}(\xi-\xi^*)(t),\\
&=\mathbb{E}^u\int_0^T\Phi\left(t\right)f_x(t)\mathcal{Z}(t)dt+\mathbb{E}^u\int_0^T\Phi\left(t\right)\widehat{\mathbb{E}}\left[\partial_\mu f(t)\widehat{\mathcal{Z}}(t)\right]\mathrm{d}t\\
&+\mathbb{E}^u\int_0^T\Phi\left(t\right)f_u(t)(u\left(t\right)-u^*(t))dt+\mathbb{E}^u\int_0^T\Phi(t)G(t)\mathrm{d}(\xi-\xi^*)(t).
\end{aligned}\tag{3.29}$$

We proceed to estimate $I_2\left(T\right)$, From equation (3.2), we have

$$\begin{aligned}
I_2\left(T\right)&=\mathbb{E}^u\int_0^T\mathcal{Z}\left(t\right)d\Phi\left(t\right)\\
&=-\mathbb{E}^u\int_0^T\mathcal{Z}\left(t\right)\left[f_x\left(t\right)\Phi\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{f}\left(t\right)\widehat{\Phi}\left(t\right)\right]+\sigma_x\left(t\right)Q(t)\right.\\
&+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{\sigma}\left(t\right)\widehat{Q}(t)\right]+c_x\left(t\right)\overline{Q}(t)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{c}\left(t\right)\widehat{\overline{Q}}(t)\right]+l_x\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{l}\left(t\right)\right]\\
&\left.+\int_\Theta\left[g_x\left(t,\theta\right)R\left(t,\theta\right)+\widehat{\mathbb{E}}\left[\partial_\mu\widehat{g}\left(t,\theta\right)\widehat{R}\left(t,\theta\right)\right]\right]m\left(d\theta\right)+h_x\left(t\right)K(t)\right]dt.
\end{aligned}\tag{3.30}$$

By simple computation, we have

$$
\begin{aligned}
I_2(T) = &-\mathbb{E}^u \int_0^T \mathcal{Z}(t)\, f_x(t)\, \Phi(t)\, dt - \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(t)\, \widehat{\Phi}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \sigma_x(t)\, Q(t) dt - \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(t)\, \widehat{Q}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, c_x(t)\, \overline{Q}(t) dt - \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{c}(t)\, \widehat{\overline{Q}}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, l_x(t)\, dt - \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{l}(t)\right] dt \\
&- \mathbb{E}^u \int_0^T \int_\Theta \mathcal{Z}(t)\, g_x(t,\theta)\, R(t,\theta)\, m(d\theta)\, dt \\
&- \mathbb{E}^u \int_0^T \int_\Theta \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(t,\theta)\, \widehat{R}(t,\theta)\right] m(d\theta)\, dt \\
&- \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, h_x(t)\, K(t) dt.
\end{aligned} \tag{3.31}
$$

Similarly, we can obtain

$$
\begin{aligned}
I_3(T) = &\mathbb{E}^u \int_0^T Q(t) \left[\sigma_x(t)\mathcal{Z}(t) + \widehat{\mathbb{E}}\left[\partial_\mu \sigma(t) \widehat{\mathcal{Z}}(t)\right] + \sigma_u(t)(u(t) - u^*(t))\right] dt \\
&+ \mathbb{E}^u \int_0^T \overline{Q}(t) \left[c_x(t)\mathcal{Z}(t) + \widehat{\mathbb{E}}\left[\partial_\mu c(t) \widehat{\mathcal{Z}}(t)\right] + c_u(t)(u(t) - u^*(t))\right] dt,
\end{aligned} \tag{3.32}
$$

and

$$
I_4(T) = \mathbb{E}^u \int_0^T \int_\Theta R(t,\theta) \left[g_x(t,\theta)\mathcal{Z}(t) + \widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta)\, \widehat{\mathcal{Z}}(t)\right] + g_u(t,\theta)(u(t) - u^*(t))\right] m(d\theta)\, dt. \tag{3.33}
$$

Now, by applying Fubini's theorem, we obtain

$$
\mathbb{E}^u \int_0^T \Phi(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{f}(t)\, \widehat{\mathcal{Z}}(t)\right] dt = \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu f(t) \widehat{\Phi}(t)\right] dt, \tag{3.34}
$$

$$
\mathbb{E}^u \int_0^T Q(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\sigma}(t)\, \widehat{\mathcal{Z}}(t)\right] dt = \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \sigma(t) \widehat{Q}(t)\right] dt, \tag{3.35}
$$

$$
\mathbb{E}^u \int_0^T \overline{Q}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{c}(t)\, \widehat{\mathcal{Z}}(t)\right] dt = \mathbb{E}^u \int_0^T \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu c(t) \widehat{\overline{Q}}(t)\right] dt, \tag{3.36}
$$

and

$$
\mathbb{E}^u \int_0^T \int_\Theta R(t,\theta)\, \widehat{\mathbb{E}}\left[\partial_\mu \widehat{g}(t,\theta)\, \widehat{\mathcal{Z}}(t)\right] m(d\theta)\, dt = \mathbb{E}^u \int_0^T \int_\Theta \mathcal{Z}(t)\, \widehat{\mathbb{E}}\left[\partial_\mu g(t,\theta) \widehat{R}(t,\theta)\right] m(d\theta)\, dt. \tag{3.37}
$$

By substituting (3.29), (3.31), (3.32) and (3.33) into (3.28), with the helps of (3.34), (3.35), (3.36) and (3.37) the desired result (3.26) follows immediately.

By applying Itô's formula to $y(t)\,\widetilde{\alpha}(t)$ and taking expectation, we get

$$
\begin{aligned}
\mathbb{E}^u\left[y(T)\,\widetilde{\alpha}(T)\right] &= \mathbb{E}^u \int_0^T y(t)\, \mathrm{d}\widetilde{\alpha}(t) + \mathbb{E}^u \int_0^T \widetilde{\alpha}(t)\, \mathrm{d}y(t) \\
&\quad + \mathbb{E}^u \int_0^T K(t) \left\{h_x(t)\mathcal{Z}(t) + h_u(t)(u(t) - u^*(t))\right\} \mathrm{d}t \\
&= J_1(T) + J_2(T) + J_3(T),
\end{aligned} \tag{3.38}
$$

where,

$$\begin{aligned} J_1(T) &= \mathbb{E}^u \int_0^T y(t)\, \mathrm{d}\widetilde{\alpha}(t) \\ &= \mathbb{E}^u \int_0^T y(t)\left(h_x(t)\mathcal{Z}(t) + h_u(t)(u(t) - u^*(t))\right) \mathrm{d}\widetilde{W}(t), \end{aligned} \tag{3.39}$$

is a martingale with zero expectation. Moreover, by a simple computations, we get

$$J_2(T) = \mathbb{E}^u \int_0^T \widetilde{\alpha}(t)\, \mathrm{d}y(t) = -\mathbb{E}^u \int_0^T \widetilde{\alpha}(t)\, l(t)\mathrm{d}t, \tag{3.40}$$

and

$$J_3(T) = \mathbb{E}^u \int_0^T K(t)\left[h_x(t)\mathcal{Z}(t) + h_u(t)(u(t) - u^*(t))\right] \mathrm{d}t. \tag{3.41}$$

Substituting (3.39), (3.40), (3.41), into (3.38), the desired result (3.27) fulfilled.
Proof of Theorem 3.1. From Lemma 3.5 and based on the fact that $y(T) = \psi(x^{u,\xi}(T), \mathcal{P}[x^{u,\xi}(T)])$, and $\Phi(T) = \psi_x(x^{u,\xi}(T), \mathcal{P}[x^{u,\xi}(T)]) + \widehat{\mathbb{E}}\left[\partial_\mu \psi(\widehat{x}(T), \mathcal{P}[\widehat{x}(T)]; x^{u,\xi}(T))\right]$ we have

$$\begin{aligned} 0 \leq\; & \mathbb{E} \int_0^T \left[\alpha_1(t)\, l(t) + \alpha^*(t)\, l_x(t)\mathcal{Z}(t) + \alpha^*(t)\widehat{\mathbb{E}}\left[\partial_\mu l(t)\right]\mathcal{Z}(t) + \alpha^*(t) l_u(t)(u(t) - u^*(t))\right] \mathrm{d}t \\ & + \mathbb{E}\left[\alpha_1(T)\, y(T)\right] + \mathbb{E}\left[\alpha^*(T)\, \Phi(T)\, \mathcal{Z}(T)\right] \\ & + \mathbb{E} \int_{[0,T]} \alpha^*(t)\, M(t) \mathrm{d}(\xi - \xi^*)(t). \end{aligned} \tag{3.42}$$

Since

$$\begin{aligned} \mathbb{E}\left[\alpha_1(T)\, y(T)\right] &= \mathbb{E}\left[\alpha^*(T)\, \widetilde{\alpha}(T)\, y(T)\right] = \mathbb{E}^u\left[\widetilde{\alpha}(T)\, y(T)\right], \\ \mathbb{E}\left[\alpha^*(T)\, \Phi(T)\, \mathcal{Z}(T)\right] &= \mathbb{E}^u\left[\Phi(T)\, \mathcal{Z}(T)\right], \\ \mathbb{E} \int_{[0,T]} \alpha^*(t)\, M(t) \mathrm{d}(\xi - \xi^*)(t) &= \mathbb{E}^u \int_{[0,T]} M(t) \mathrm{d}(\xi - \xi^*)(t). \end{aligned}$$

Finally, by substituting (3.26) and (3.27) of Lemma 3.6 into (3.42), we get

$$\begin{aligned} 0 \leq\; & \mathbb{E} \int_0^T \alpha^*(t) \left[\Phi(t)\, f_u(t) + Q(t)\, \sigma_u(t) + \overline{Q}(t)\, c_u(t)\right. \\ & \left. + \int_\Theta R(t,\theta)\, g_u(t,\theta) m(d\theta) + K(t) h_u(t) + l_u(t)\right] (u(t) - u^*(t)) \mathrm{d}t \\ & + \mathbb{E} \int_{[0,T]} \alpha^*(t)\left(M(t) + \Phi(t) G(t)\right) \mathrm{d}(\xi - \xi^*)(t). \end{aligned} \tag{3.43}$$

This completes the proof of Theorem 3.1. □

4. Application: Conditional mean-variance portfolio selection problem associated with interventions

In this section, we study a conditional mean-variance portfolio selection problem in incomplete market, where the system is governed by Lévy measure associated with some Gamma process and an independent Brownian motion. The Gamma process is a Lévy process (of bounded variation) $(\Gamma(t))_{t \geq 0}$, with Lévy measure given by

$$\mu(\mathrm{d}x) = \frac{e^{-x}}{x} \chi_{\{x>0\}} \mathrm{d}x. \tag{4.1}$$

It is called *Gamma process* because the probability law of $\Gamma(\cdot)$ is a Gamma distribution with mean t and scale parameter equal to one. The Lévy measure $\mu(\mathrm{d}x)$ dictates how the jumps occur.

Let $(\Gamma(t))_{t\in[0,T]}$ be a $\mathbb{R}$-valued Gamma process, independent of the Brownian motion $W(\cdot)$. Assume that the Lévy measure $\mu(\mathrm{d}x)$ corresponding to the Gamma process $\Gamma(\cdot)$ has a moments of all orders. This implies that $\int_{(-\delta,\delta)^c} e^{|x|}\mu(\mathrm{d}x) < \infty$ for every $\delta > 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\mu(\mathrm{d}x) < \infty$. We assume that $\mathcal{F}_t$ is $\mathcal{P}-$augmentation of the natural filtration $\mathcal{F}_t^{(W,\Gamma)}$ defined as follows

$$\mathcal{F}_t^{(W,\Gamma)} = \mathcal{F}_t^W \vee \sigma\left\{\Gamma(r) : 0 \leq r \leq t\right\} \vee \mathcal{F}_0,$$

where $\mathcal{F}_t^W := \sigma\left\{W(s) : 0 \leq s \leq t\right\}$, $\mathcal{F}_0$ denotes the totality of $\mathcal{P}^u-$null sets and $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. We denote by $\Delta\Gamma(\tau_j) = \Gamma(\tau_j) - \Gamma(\tau_{j-})$ the jump size at time τ_j. We denote by $\Gamma^j(t) = \sum_{0\leq s\leq t}\left(\Delta\Gamma(s)\right)^j : j : 1, ..., n$ the power jump processes of $\Gamma(\cdot)$. By using *Exponential formula* proved in Bertoin [33], we obtain

$$\mathbb{E}^u\left(\exp(i\theta\Gamma^j(t))\right) = \exp\left[t\int_0^{+\infty}(\exp(i\theta x^j) - 1)\frac{e^{-x}}{x}\mathrm{d}x\right].$$

Let $\Gamma_0(n)$ Gamma function defined by $\Gamma_0(n) = \int_0^{+\infty} x^{n-1}e^{-x}\mathrm{d}x$, and $\varphi_{\Gamma^j(t)}(t)$: the moment generating function $\varphi_{\Gamma^j(t)}(t) = \mathbb{E}^u(\exp(t\Gamma^j(t)))$. Now, based on $\varphi^{(k)}_{\Gamma^j(t)}(0) = \mathbb{E}^u\left(\left(\Gamma^j(t)\right)^k\right)$, we deduce

$$\mathbb{E}^u\left(\Gamma^j(t)\right) = \varphi'_{\Gamma^j(t)}(0) = t\Gamma_0(j) = (j-1)!t : j : 1, ..., n$$

Now, we proceed to obtain $\mathbb{V}^u_{ar}\left(\Gamma^j(t)\right)$, then we have

$$\begin{aligned}
\mathbb{V}^u_{ar}\left(\Gamma^j(t)\right) &= \mathbb{E}^u\left[\left(\Gamma^j(t)\right)^2\right] - \left[\mathbb{E}\left(\Gamma^j(t)\right)\right]^2 \\
&= \varphi''_{\Gamma^j(t)}(0) - \left[\varphi'_{\Gamma^j(t)}(0)\right]^2 \\
&= t\int_0^{+\infty} x^{2j-1}e^{-x}\mathrm{d}x \\
&= t\Gamma_0(2j), \; j : 1, ..., n,
\end{aligned}$$

Let

$$\mathcal{L}^j(t) = \frac{\Gamma^j(t) - \mathbb{E}^u\left(\Gamma^j(t)\right)}{\mathbb{V}^u_{ar}\left(\Gamma^j(t)\right)} = \frac{\sum_{0\leq s\leq t}\left(\Delta\Gamma(s)\right)^j - (j-1)!t}{t\Gamma_0(2j)}, \; j : 1, ..., n \tag{4.2}$$

then we have $\mathbb{E}^u\left(\mathcal{L}^j(t)\right) = 0$ and $\mathbb{V}^u_{ar}\left(\mathcal{L}^j(t)\right) = 1$.

Derivatives with respect to measure in the sense of P.L. Lions. Let $(\Gamma(t))_{t\geq 0}$ be Gamma process with Lévy measure $\mu(\cdot)$ given by (4.1). We give some examples.

1. If $\Phi(\mu) = \int_{\mathbb{R}^n}\varphi(x)\mu(\mathrm{d}x)$ then the derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$\partial_\mu\Phi(\mu)(z) = \frac{\partial\varphi}{\partial x}(z).$$

2. If $\Phi(\mu) = \int_{\mathbb{R}^n}\varphi(x,\mu)\mu(\mathrm{d}x)$ then the derivatives of $\Phi(\mu)$ with respect to measure at z is given by

$$\partial_\mu\Phi(\mu)(z) = \frac{\partial\varphi}{\partial x}(z,\mu) + \int_{\mathbb{R}^n}\frac{\partial\varphi}{\partial\mu}(x,\mu)(z)\,\mu(\mathrm{d}x).$$

Conditional mean-variance portfolio selection problem with interventions. In this section, we study a conditional mean-variance portfolio selection problem in incomplete market with interventions. As example, foreign exchange interventions are conducted by monetary authorities (Bank or minister of finance) to influence foreign exchange rates by buying and selling currencies in the foreign exchange market.

Suppose that we are given a mathematical market consisting of two investment possibilities:

A risk free security, (bond) where the price $S_0(t)$ evolves according to the ordinary differential equation:

$$\begin{cases} \mathrm{d}S_0(t) = \gamma_0(I_t)S_0(t)\,\mathrm{d}t, \; t \in [0,T], \\ S_0(0) > 0, \end{cases} \tag{4.3}$$

where I_t is a factor process with dynamics governed by a Brownian motion $B(\cdot)$, assumed to be non correlated with the Brownian motion $W(\cdot)$. We shall assume that the natural filtration generated by the observable factor process I_t is equal to the filtration $\mathcal{F}_t^B$ generated by $B(\cdot)$. Notice that the market is incomplete as the agent cannot trade in the factor process. The map $\gamma_0(\cdot) : [0,T] \to \mathbb{R}_+$ is a locally bounded continuous deterministic function.

A risky security (stock), where the price $S_1(t)$ at time t is given by

$$\begin{cases} \mathrm{d}S_1(t) = S_1(t)\left[(\varsigma(I_t) + \gamma_0(I_t))\,\mathrm{d}t + \sigma(I_t)\mathrm{d}W(t)\right] + \mathrm{d}\xi(t) + \sum_{j=1}^{n} \mathcal{L}^j(t), \\ S_1(0) > 0, \end{cases} \tag{4.4}$$

where $\mathcal{L}^j(t)$ is the power jump processes of $\Gamma(\cdot)$ given by (4.2).

Now, in order to ensure that $S_1(t) > 0$ for all $t \in [0,T]$, we assume the functions $\varsigma(\cdot) : [0,T] \to \mathbb{R}$, and $\sigma(\cdot) : [0,T] \to \mathbb{R}$ are bounded continuous deterministic maps such that

$$\varsigma(I_t),\ \sigma(I_t) \neq 0 \text{ and } \varsigma(I_t) - \gamma_0(I_t) > 0,\ \forall t \in [0,T].$$

Let $x(0) = x_0 > 0$ be an initial wealth process. By combining (4.3) and (4.4), we introduce the wealth dynamic

$$\begin{cases} \mathrm{d}x(t) = \gamma_0(I_t)(x(t) - \xi(t))\mathrm{d}t + u(t)\left[\varsigma(I_t)\mathrm{d}t + \sigma(I_t)\mathrm{d}W(t)\right] + \mathrm{d}\xi(t) + \sum_{j=1}^{n} \mathcal{L}^j(t), \\ x(0) = x_0. \end{cases} \tag{4.5}$$

where $\gamma_0(I_t)$: is the interest rate, $\varsigma(I_t)$: is the excess rate of return, and $\sigma(I_t)$: the volatility (or the dispersion) of the stock price with $\sigma(I_t) \geq \varepsilon$ for some $\varepsilon > 0$.are measurable bounded functions of I_t. The process $u = u(t)$ (the regular control process) represents the amount invested in the stock at time t, when the current wealth is $x(t)$ and based on the past partially observations $\mathcal{F}_t^B$ of the factor process, $\xi(t)$ is the intervention control.

The objective of the agent is to minimize over investment strategies a cost functional of the form:

$$J(u(\cdot), \xi(\cdot)) = \mathbb{E}^u\left[\frac{\delta}{2}\mathbb{V}_{ar}^u\left(x(T) - \xi(T) \mid B(T)\right) - \mathbb{E}^u(x(T) - \xi(T) \mid B(T))\right], \tag{4.6}$$

for some $\delta > 0$, with a dynamics for the wealth process $x(t)$ controlled by the amount $u(t)$.

If we denote $z(t) = x(t) - \xi(t) - \sum_{j=1}^{\eta} \mathcal{L}^j(t)$, then the dynamic (4.5) has the form:

$$\begin{cases} \mathrm{d}z(t) = \gamma_0(I_t)z(t)\mathrm{d}t + u(t)\left[\varsigma(I_t)\mathrm{d}t + \sigma(I_t)\mathrm{d}W(t)\right], \\ z(0) = x_0. \end{cases} \tag{4.7}$$

and the cost functional $J(u(\cdot), \xi(\cdot))$ has the form

$$J(u(\cdot), \xi(\cdot)) = \mathbb{E}^u\left[\frac{\delta}{2}\mathbb{V}_{ar}^u\left(z(T) \mid B(T)\right) - \mathbb{E}^u(z(T) \mid B(T))\right], \tag{4.8}$$

where $\mathbb{E}^u(z(t) \mid B(t))$ is the conditional expectation and $\mathbb{V}_{ar}^u(z(t) \mid B(t))$ is the conditional variance with respect to $\mathcal{P}^u$. We note that the law of total variance is given by

$$\mathbb{V}_{ar}^u(z(t)) = \mathbb{V}_{ar}^u(z(t) \mid B(t)) + \mathbb{V}_{ar}^u\left[\mathbb{E}^u(z(t) \mid B(t))\right].$$

By applying similar arguments developed in Pham [14], Li and Zhou [34] the optimal intervention control $u^*(t)$ of (4.7)-(4.8) is given in feedback form:

$$\begin{aligned} u^*(t) &= \frac{\varsigma(I_t)}{\sigma^2(I_t)}\left[\mathbb{E}^u(z^*(t) \mid B(t)) - z^*(t)\right] \\ &\quad + \frac{\varsigma(I_t)}{\sigma^2(I_t)c_t}\left[\frac{1}{2}b_t - a_t\mathbb{E}^u\left(z^*(t) \mid B(t)\right)\right], \end{aligned} \tag{4.9}$$

where $z(t)$ is given by Eq-(4.7), and $a_t, b_t\ c_t$ satisfy the linear BSDEs: $t\in[0,T]$

$$\begin{cases} \mathrm{d}a_t = \left[\frac{\varsigma^2(I_t)a_t^2}{\sigma^2(I_t)c_t} - 2\gamma_0(I_t)a_t\right]\mathrm{d}t + Z_t^a\mathrm{d}B(t),\ a_T = 0. \\ \mathrm{d}b_t = \left[\frac{\varsigma^2(I_t)a_t}{\sigma^2(I_t)c_t} - \gamma_0(I_t)\right]\mathrm{d}t + Z_t^b\mathrm{d}B(t),\ b_T = -1. \\ \mathrm{d}c_t = \left[\frac{\varsigma^2(I_t)}{\sigma^2(I_t)} - 2\gamma_0(I_t)\right]c_t\mathrm{d}t + Z_t^c\mathrm{d}B(t),\ c_T = \frac{\delta}{2}. \end{cases} \tag{4.10}$$

The explicit solutions of the above equations are given by

$$\begin{aligned} a_t &\equiv 0,\ \forall t\in[0,T], \\ b_t &= \mathbb{E}^u\left[-\exp\int_t^T \gamma_0(I_s)\mathrm{d}s \mid \mathcal{F}_t^B\right]. \\ c_t &= \mathbb{E}^u\left[\frac{\delta}{2}\exp\int_t^T (2\gamma_0(I_s) - \frac{\varsigma^2(I_s)}{\sigma^2(I_s)})\mathrm{d}s \mid \mathcal{F}_t^B\right]; \end{aligned} \tag{4.11}$$

Hence, substituting (4.11) into (4.9) yields

$$\begin{aligned} u^*(t) &= \frac{\varsigma(I_t)}{\sigma^2(I_t)}\left[x_0\exp\left(\int_0^t \gamma_0(I_\tau)\mathrm{d}\tau\right) - z^*(t)\right. \\ &\left. +\frac{1}{2}\int_0^t \frac{\varsigma^2(I_t)}{2\sigma^2(I_t)}\frac{|b_s|}{c_s}\exp\left(\int_0^t \gamma_0(I_\tau)\mathrm{d}\tau\right)\mathrm{d}s + \frac{|b_t|}{c_t}\right]. \end{aligned} \tag{4.12}$$

Finally, we deduce that the optimal control of the problem (4.5)-(4.6) is given in feedback form

$$\begin{aligned} u^*(t) &= \frac{\varsigma(I_t)}{\sigma^2(I_t)}\left[x_0\exp\left(\int_0^t \gamma_0(I_s)\mathrm{d}s\right) - x^*(t) + \xi(t) + \sum_{j=1}^n \mathcal{L}^j(t)\right. \\ &\left. +\frac{1}{2}\int_0^t \frac{\varsigma^2(I_t)}{2\sigma^2(I_t)}\frac{|b_s|}{c_s}\exp\left(\int_0^t \gamma_0(I_\tau)\mathrm{d}\tau\right)\mathrm{d}s + \frac{|b_t|}{c_t}\right]. \end{aligned} \tag{4.13}$$

Now, let $\xi^*(t)$ be $\mathcal{F}_t^Y-$adapted process satisfies Theorem 3.1, then for any $\xi(\cdot)\in\mathcal{U}_2^Y$ we get

$$\begin{aligned} &\mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\mathrm{d}\xi^*(t) \mid \mathcal{F}_t^Y\right] \\ \leq\ &\mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\mathrm{d}\xi(t) \mid \mathcal{F}_t^Y\right]. \end{aligned}$$

We define a subset $\mathcal{E}\subset\Omega\times[0,T]$ such that

$$\mathcal{E} = \{(t,w)\in[0,T]\times\Omega : M(t)+G(t)\Phi(t)>0\}, \tag{4.14}$$

and let $\xi(\cdot)\in\mathcal{U}_2^Y$ defined by

$$\mathrm{d}\xi(t) = \begin{cases} 0 : \text{if } (t,w)\in\mathcal{E}, \\ \mathrm{d}\xi^*(t) : \text{if } (t,w)\in\overline{\mathcal{E}}, \end{cases} \tag{4.15}$$

where $\overline{\mathcal{E}}$ is the complement of the set $\mathcal{E}$. We denote by $\boldsymbol{\chi}_{\mathcal{E}}$ the indicator function of $\mathcal{E}$. By a simple computations, we get

$$\begin{aligned}
0 &\leq \mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\mathrm{d}\left(\xi(t)-\xi^*(t)\right)\mid\mathcal{F}_t^Y\right] \\
&= \mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\mathcal{E}}(t,w)\mathrm{d}\left(-\xi^*\right)(t)\mid\mathcal{F}_t^Y\right] \\
&\quad +\mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\overline{\mathcal{E}}}(t,w)\mathrm{d}\left(\xi^*-\xi^*\right)(t)\mid\mathcal{F}_t^Y\right] \\
&= -\mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\mathcal{E}}(t,w)\mathrm{d}\xi^*(t)\mid\mathcal{F}_t^Y\right].
\end{aligned}$$

This implies that $\xi^*(\cdot)$ satisfies for any $t\in[0,T]$:

$$\mathbb{E}^u\left[\int_{[0,T]}(M(t)+G(t)\Phi(t))\boldsymbol{\chi}_{\mathcal{E}}(t,w)\mathrm{d}\xi^*(t)\right]=0.$$

From (4.14) and (4.15), we can easy shows that the optimal intervention control has the form:

$$\xi^*(t)=\xi(t)+\int_0^t\boldsymbol{\chi}_{\overline{\mathcal{E}}}(s,w)\mathrm{d}s,\;\; t\in[0,T]\,.$$

Finally, we give the explicit optimal portfolio section strategy for systems governed by Lévy measure associated with some Gamma process in feedback form by:

$$\begin{aligned}
u^*(t,x^*) &= \frac{\varsigma(I_t)}{\sigma^2(I_t)}\left[x_0\exp\left(\int_0^t\gamma_0(I_\tau)\mathrm{d}\tau\right)-x^*(t)+\xi(t)+\sum_{j=1}^{n}\mathcal{L}^j(t)\right. \\
&\quad\left.+\frac{1}{2}\int_0^t\frac{\varsigma^2(I_t)}{2\sigma^2(I_t)}\frac{|b_s|}{c_s}\exp\left(\int_0^t\gamma_0(I_\tau)\mathrm{d}\tau\right)\mathrm{d}s+\frac{|b_t|}{c_t}\right]. \\
\xi^*(t) &= \int_0^t\boldsymbol{\chi}_{\overline{\mathcal{E}}}(s,w)ds+\xi(t),\;\; t\in[0,T]\,. \\
\mathcal{L}^j(t) &= \frac{\sum_{0\leq s\leq t}\left(\Delta\Gamma(s)\right)^j-(j-1)!t}{t\Gamma_0(2j)},\;\; j:1,...,n.
\end{aligned}$$

5. Discussion and Conclusion

In this paper, a new set of general mean-field type necessary conditions for a class of optimal stochastic intervention control problem for partially observed random jumps on Wasserstein space of probability measures has been established. Girsanov's theorem and the L-derivatives with respect to probability law are applied to prove our main result. Conditional mean-variance portfolio selection problem with interventions is investigated. In order to assess the effectiveness of interventions, it is helpful to identify the motives of the government (or banks) activities in this area. Apparently, there are many problems left unsolved, and one possible problem is to obtain some optimality conditions for partial observed stochastic optimal intervention control for systems governed by general mean-field *backward stochastic differential equations with Lévy* process with moments of all orders with some applications to finance.

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Appendix

Proposition A1. Let $\mathcal{G}$ be the predictable $\sigma-$field on $\Omega \times [0,T]$, and g be a $\mathcal{G} \times \mathcal{B}(\Theta)-$measurable function such that $\mathbb{E}\int_0^T\int_\Theta |g(s,\theta)|^2 m(\mathrm{d}\theta)\mathrm{d}s < \infty$, then there exists a two positive constants $c_1(T,m(\Theta))$, and $C_2(T,m(\Theta))$ that depend only on T and $m(\Theta)$ such that

$$\begin{aligned} c_1(T,m(\Theta))\mathbb{E}\left[\int_0^T\int_\Theta |g(r,\theta)|^2 m(\mathrm{d}\theta)\mathrm{d}s\right] &\leq \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_\Theta g(s,\theta)\,\eta(\mathrm{d}\theta,\mathrm{d}s)\right|^2\right] \\ &\leq C_2(T,m(\Theta))\mathbb{E}\left[\int_0^T\int_\Theta |g(r,\theta)|^2 m(\mathrm{d}\theta)\mathrm{d}s\right]. \end{aligned}$$

Proof. See Bouchard and Elie [26, *Appendix*], Proposition 5.1, with $p = 2$).

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