Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–11.
ISSN-0037-8712
doi:10.5269/bspm.66510

Existence result of entropy solution for nonlinear elliptic problem without monotonicity condition in generalized sobolev spaces

Abdelkarim DERHAM, Badr EL HAJI* and Ouidad AZRAIBI

ABSTRACT: In the present paper we prove some existence results of entropy solution for nonlinear degenerate elliptic problems of the form $B(\upsilon) + H(x, \upsilon_n) = f$, in Musielak-Orlicz-Sobolev spaces, where $B(\upsilon) = -\text{div}\ (b(x, \upsilon, \nabla \upsilon))$ is a Leray-Lions, operator defined form the musielak-Orlicz-sobolev spaces $W_0^1 L_\Psi(\Omega)$ into its dual$f \in L^1(\Omega)$, and no monotonicity strict condition is assumed on the function $b(x, s, \xi)$, The tool we use to overcome this difficulty is to investigate some techniques introduced by Minty's lemma.

Key Words: Elliptic problem, Entropy solutions, Musielak-Orlicz-Sobolev spaces, Compact imbedding, Δ_2-condition..

Contents

1. Introduction

The purpose of this note, is to show an existence of entropy solutions for some nonlinear Dirichlet problem as

$$\begin{cases} B(\upsilon) + H(x, \upsilon_n) = f(x) & \text{in} \quad \Omega, \\ \upsilon = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where $B(\upsilon) = -\text{div}\ (b(x, \upsilon, \nabla \upsilon))$, Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$. $b(x, \upsilon, \nabla\upsilon) = (b_i(x, \upsilon, \nabla \upsilon))_{1\leq i \leq N}$, $b_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory functions such that for all ξ, ξ' in $\mathbb{R}^N$,

$$|b_i(x, s, \xi)| \leq |\phi_i(x)| + K_i \overline{P}^{-1}(\Psi(x, c_2|s|)) + K_i(\overline{\Psi}^{-1}\Psi(x, c_1 |\xi|)), \tag{1.2}$$

$$\Big(b(x, s, \xi) - b(x, s, \xi')\Big)(\xi - \xi') \geq 0, \tag{1.3}$$

$$b(x, s, \xi)\xi \geq \Psi(x, \lambda_1|\xi|), \tag{1.4}$$

with c_1, c_2, $\lambda_1, K_i > 0$. Ψ, P be two Musielak functions such that $P \prec\prec \Psi$. Furthermore $\overline{\Psi}$, $\overline{P}$ be the conjugate functions of Ψ and P respectively, and $\phi_i \in E_{\overline{\Psi}}(\Omega)$.

Furthermore, the nonlinear term $H(x, s)$ verify

* Corresponding author

Submitted December 30, 2022. Published April 20, 2025
2010 *Mathematics Subject Classification:* 35J25, 35J60, 46E30 .

$$H(x,s)s \geq 0, \tag{1.5}$$

$$\sup_{|s|\leq n} |H(x,s)| = h_n(x) \in L^1(\Omega), \tag{1.6}$$

$$f \in L^1(\Omega), \tag{1.7}$$

The Nonlinear elliptic problem as

$$\begin{cases} -\operatorname{div} b(x,\upsilon,\nabla\upsilon) = f - \operatorname{div} F & \text{in } \Omega \\ \upsilon = 0 & \text{on } \partial\Omega \end{cases}$$

with $f \in L^1(\Omega)$ and $F \in \left(L^{p'}(\Omega)\right)^N$, have been treated by Orsina et al. ([11]), by using the approach Minty's Lemma. Furthermore Badr EL HAJI et al. in their paper ([15]) and in weighted Orlicz-Sobolev spaces, they establish that the problem

$$\begin{cases} -\operatorname{div} b(x,\upsilon,\nabla\upsilon) = f(x) & \text{in} \quad \Omega, \\ \upsilon = 0 & \text{on} \quad \partial\Omega, \end{cases}$$

with $f \in L^1$. admits at least one solution by using the idea of Minty's lemma, on the others hand and by the similar proof Rhoudaf et al ([1]) have shown the entropy solutions for the problem looks like

$$\begin{cases} -\operatorname{div} b(x,\upsilon,\nabla\upsilon) = h & \text{in } \Omega \\ \upsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

with $h \in W^{-1,p'}(\Omega,\omega^*)$,

The mathematical researches dealing the existence of solutions to some problems parabolic and elliptic under a different assumptions is massive; we refer the reader to [3,4,5,6,7,13,14,15,16,17].

Our aim in this note is to solve the problem (1.1)(existence results) where the function $a_i(x,s,\xi)$ satisfy the large monotonicity condition and without adopting the almost everywhere convergence of the gradients, and in order to overcome this difficulties, we exploit the technique of Minty's lemma for proving the existence of an entropy solutions, However the approach that we used in the proof differs from that adopted by A. Benkirane et al used in [8]

The outline of this note is as follows. After giving the definition and some auxiliary results on anisotropic Sobolev space, and after recalling some essential assumptions which are necessary to have an existence solution, finally section 3 will be devoted to give our main results and their proofs.

2. Background

Here we give just some important definitions about Musielak-Orlicz spaces, for more details on this new space see ([22]).

2.1. Musielak-Orlicz-Sobolev spaces

Let Ω be an open subset of $\mathbb{R}^n$.

Definition 2.1 *A Musielak-Orlicz function Ψ is a real-valued function defined in $\Omega \times \mathbb{R}_+$ such that*
a) $\Psi(x,t)$ is an N-function i.e. convex, nondecreasing, continuous, $\Psi(x,0) = 0, \Psi(x,t) > 0$ for all $t > 0$ and

$$\lim_{t\to 0} \sup_{x\in\Omega} \frac{\Psi(x,t)}{t} = 0, \qquad \lim_{t\to\infty} \inf_{x\in\Omega} \frac{\Psi(x,t)}{t} = 0.$$

b) $\Psi(\cdot,t)$ is a Lebesgue measurable function.

For a Musielak-Orlicz function Ψ and a measurable function $\upsilon : \Omega \longrightarrow \mathbb{R}$, we put

$$\rho_{\Psi,\Omega}(\upsilon) = \int_{\Omega} \Psi(x, |\upsilon(x)|)dx.$$

The set $K_{\Psi}(\Omega) = \{\upsilon : \Omega \longrightarrow \mathbb{R} \text{ measurable } /\rho_{\Psi,\Omega}(\upsilon) < \infty\}$ is named the Musielak-Orlicz class. The Musielak-Orlicz space $L_{\Psi}(\Omega)$ is the vector space generated by $K_{\Psi}(\Omega)$, that is, $L_{\Psi}(\Omega)$ is the smallest linear space containing the set $K_{\Psi}(\Omega)$. That's to say

$$L_{\Psi}(\Omega) = \left\{\upsilon : \Omega \longrightarrow \mathbb{R} \text{ measurable } /\rho_{\Psi,\Omega}\left(\frac{\upsilon}{\lambda}\right) < \infty, \text{ for some } \lambda > 0\right\}.$$

For a Musielak-Orlicz function Ψ we put: $\Psi(x, s) = \sup_{t>0}\{st - \Psi(x, t)\}, \Psi$ is the conjugate Musielak-Orlicz function of Ψ in the sens of Young with respect to the variable s in the space $L_{\Psi}(\Omega)$

Let us define the two following equivalent norms as follows:

$$\|\upsilon\|_{\Psi,\Omega} = \inf\left\{\lambda > 0/\int_{\Omega} \Psi\left(x, \frac{|\upsilon(x)|}{\lambda}\right) dx \leq 1\right\}, (\textbf{ The Luxemburg norm })$$

$$\||\upsilon|\|_{\Psi,\Omega} = \sup_{\|v\|_{\Psi} \leq 1} \int_{\Omega} |\upsilon(x)v(x)|dx, \textbf{ So-called Orlicz norm}$$

where Ψ is the Musielak Orlicz function conjugate to Ψ.

We will need the space $E_{\Psi}(\Omega)$ given by

$$E_{\Psi}(\Omega) = \left\{w : \Omega \longrightarrow \mathbb{R} \text{ measurable } /\rho_{\Psi,\Omega}\left(\frac{w}{\lambda}\right) < \infty, \text{ for all } \lambda > 0\right\}.$$

A Musielak function Ψ is locally integrable on Ω if $\rho_{\Psi}(t\chi_D) < \infty$ for all $t > 0$ and all measurable $D \subset \Omega$ with meas $(D) < \infty$ Let Ψ a Musielak function locally integrable. Consequently $E_{\Psi}(\Omega)$ is separable ([22], Theorem 7.10) .

We say that sequence of functions $\upsilon_n \in L_{\Psi}(\Omega)$ is modular convergent to $\upsilon \in L_{\Psi}(\Omega)$ if there exists $\lambda > 0$ such that

$$\lim_{n\to\infty} \rho_{\Psi,\Omega}\left(\frac{\upsilon_n - \upsilon}{\lambda}\right) = 0.$$

2.2. Auxilary lemmas

Lemma 2.1 *[2]*
If

$$\bar{\Psi}(x, 1) \leq c_1 \quad a.e \text{ in } \Omega \text{ for some } c_1 > 0 \tag{2.1}$$

We said that Ψ verify the log-Hölder continuity where Ω be a bounded Lipschitz domain in $R^N (N \geq 2)$ Furthermore, $\mathfrak{D}(\Omega)$ is dense in $L_{\Psi}(\Omega)$ and in $W_0^1 L_{\Psi}(\Omega)$ for the modular convergence.

Lemma 2.2 *([23]) Let $F : \mathbb{R} \longrightarrow \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let Ψ be a Musielak-Orlicz function and let $\nu \in W_0^1 L_{\Psi}(\Omega)$. Then $F(\nu) \in W_0^1 L_{\Psi}(\Omega)$.*

Moreover, if the set D of discontinuity points of F' is finite, we have

$$\frac{\partial}{\partial x_i} F(\nu) = \begin{cases} F'(\nu)\frac{\partial \nu}{\partial x_i} & a.e \text{ in } \{x \in \Omega : \nu(x) \in D\} \\ 0 & a.e \text{ in } \{x \in \Omega : \nu(x) \notin D\} \end{cases}$$

Lemma 2.3 *[9] Assume that Ω verify the segment property and consider $\nu \in W_0^1 L_{\Psi}(\Omega)$. Consequently, there exists $(\nu_n) \subset \mathcal{D}(\Omega)$:*

$$\nu_n \to \nu \text{ in } W_0^1 L_{\Psi}(\Omega).$$

Moreover, if $\nu \in W_0^1 L_{\Psi}(\Omega) \cap L^{\infty}(\Omega)$ then $\|\nu_n\|_{\infty} \leq (N + 1)\|\nu\|_{\infty}$.

Lemma 2.4 *[21] Let* (h_n), $h \in L^1(\Omega)$ *satisfying*
i) $h_n \geq 0$ *a.e in* Ω,
ii) $h_n \longrightarrow h$ *a.e in* Ω,
iii) $\int_\Omega h_n(x)\,dx \longrightarrow \int_\Omega h(x)\,dx.$
Then $h_n \longrightarrow h$ *strongly in* $L^1(\Omega)$.

Lemma 2.5 *[10] Assume that* $h_n \in L_\Psi(\Omega)$ *converges in measure to* h *and if* h_n *bounded in* $L_\Psi(\Omega)$, *then* $h \in L_\Psi(\Omega)$ *and* $h_n \rightharpoonup h$ *for* $\sigma\left(\Pi L_\Psi, \Pi E_\psi\right)$.

Lemma 2.6 *[10] Suppose that* $\nu_n,\ \nu \in L_\Psi(\Omega)$. *If* $\nu_n \to \nu$ *for the modular convergence, then* $\nu_n \to \nu$ *for* $\sigma\left(L_\Psi(\Omega), L_\psi(\Omega)\right)$.

Lemma 2.7 *[18] If* $P \prec \Psi$ *and* $\nu_n \to \nu$ *for the modular convergence in* $L_\Psi(\Omega)$ *then* $\nu_n \to \nu \in P(\Omega)$.

Now let us consider the truncation function T_k defined as follows

$$k \geq 0, T_k(s) = \max(-k, \min(k, s)).$$

3. Main Existence Theorem

Let Y be a closed subspace of $W^1 L_\Psi(\Omega)$ for $\sigma(\prod L_\Psi, \prod E_{\overline{\Psi}})$ and let

$$Y_0 = Y \cap W^1 L_\Psi(\Omega),$$

such that Y is the closure of Y_0 for $\sigma(\prod L_\Psi, \prod E_{\overline{\Psi}})$. In the next, we consider the complementary system (Y, Y_0, Z, Z_0) generated by Y i.e. Y_0^* can be identified to Z and Z_0^* can be identified to Y by the means $\langle ., . \rangle$. Let the mapping T (associated to the operator A) defined from

$$D(T) = \{\upsilon \in Y, b_0(x, \upsilon, \nabla \upsilon) \in L_{\overline{\Psi}}(\Omega), b_i(x, \upsilon, \nabla \upsilon) \in L_{\overline{\Psi}}(\Omega)\}$$

into Z by the formula

$$b(\upsilon, v) = \int_\Omega b_0(x, \upsilon, \nabla \upsilon) v(x) dx + \sum_{1 \leq i \leq N} \int_\Omega b_i(x, \upsilon, \nabla \upsilon) \frac{\partial v(x)}{\partial x_i} dx \quad \forall v \in Y_0.$$

We consider the complementary system

$$(Y, Y_0, Z, Z_0) = (W_0^1 L_\Psi(\Omega), W_0^1 E_\Psi(\Omega), W^{-1} E_{\overline{\Psi}}(\Omega), W^{-1} L_{\overline{\Psi}}(\Omega)).$$

Let us give the notion of entropy solution as follows

Definition 3.1 *A function* υ *(measurable) is named an entropy solution of the problem* (1.1) *if* $T_k(\upsilon) \in W_0^1 L_\Psi(\Omega)$ *for every* $k > 0$ *and such that*

$$\int_\Omega b(x, \upsilon, \nabla \upsilon) \nabla T_k(\upsilon - \Theta) dx + \int_\Omega H(x, \upsilon_n) T_k(\upsilon - \Theta) dx \leq \int_\Omega f T_k(\upsilon - \Theta) dx$$

for every $\Theta \in W_0^1 E_\Psi(\Omega) \cap L^\infty(\Omega)$.

The existence results are stated as follows .

Theorem 3.1 *If the hypothesises (1.2)-(1.7) holds, the problem* (1.1) *admit at least an entropy solution in the sense of the definition 3.1 .*

3.1. Main Lemma

Lemma 3.1 *Suppose that υ be a mesurable function with $T_k(\upsilon) \in W_0^1 L_\Psi(\Omega)$ for every $k > 0$. Then*

$$\int_\Omega b(x, \upsilon_n, \nabla\Theta)\nabla T_k(\upsilon - \Theta)\ dx + \int_\Omega H(x, \upsilon_n)T_k(\upsilon - \Theta)dx \leq \int_\Omega f\ T_k(\upsilon - \Theta)dx, \tag{3.1}$$

is equivalent to

$$\int_\Omega b(x, \upsilon, \nabla\upsilon)\nabla T_k(\upsilon - \Theta)\ dx + \int_\Omega H(x, \upsilon_n)T_k(\upsilon - \Theta)dx = \int_\Omega f\ T_k(\upsilon - \Theta)dx, \tag{3.2}$$

for every $\Theta \in W_0^1 L_\Psi(\Omega)$, and for every $k > 0$.

Proof In fact (3.2) implies (3.1)is easily proved adding and subtracting

$$\int_\Omega b(x, \upsilon_n, \nabla\Theta)\nabla T_k(\upsilon - \Theta)\ dx,$$

and by thanking to assumption (1.3). Thus, it suffices to prove that (3.1) implies (3.2). Let $h, k > 0$, let us consider that $\lambda \in\ ]-1, 1[$ and $\vartheta \in W_0^1 L_\Psi(\Omega) \cap L^\infty(\Omega)$. By taking, $\Theta = T_h(\upsilon - \lambda T_k(\upsilon - \vartheta)) \in W_0^1 L_\Psi(\Omega) \cap L^\infty(\Omega)$ as test in (3.1), we may obtain:

$$I_{hk} \leq J_{hk}, \tag{3.3}$$

where

$$I_{hk} = \int_\Omega b(x, \upsilon_n, \nabla T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\nabla T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\ dx \\ + \int_\Omega H(x, \upsilon_n)T_k\left(\upsilon - T_h\left(\upsilon - \lambda T_k(\upsilon - \vartheta)\right)\right) dx = I'_{hk} + I''_{hkk}$$

and

$$J_{hk} = \int_\Omega f\ T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\ dx.$$

Set

$$A_{hk} = \{x \in \Omega, |\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),)| \leq k\},$$

and

$$B_{hk} = \{x \in \Omega, |\upsilon - \lambda T_k(\upsilon - \vartheta), | \leq h\}.$$

Then, we get

$$I'_{hk} = \int_{A_{kh}\cap B_{hk}} b(x, \upsilon_n, \nabla T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\nabla T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\ dx \\ + \int_{A_{kh}\cap B^C_{hk}} b(x, \upsilon_n, \nabla T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\nabla T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\ dx \\ + \int_{A^C_{kh}} b(x, \upsilon_n, \nabla T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\nabla T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\ dx.$$

Since $\nabla T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),)) \neq 0$ on A_{kh}, one has

$$\int_{A^C_{kh}} b(x, \upsilon_n, \nabla T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\nabla T_k(\upsilon - T_h(\upsilon - \lambda T_k(\upsilon - \vartheta),))\ dx = 0. \tag{3.4}$$

Furthermore, if $x \in B_{hk}^C$, we have $\nabla T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),)=0$ and by according to (1.4), we conclude that,

$$\int_{A_{kh}\cap B_{hk}^C} b(x,\upsilon_n,\nabla T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\ dx$$
$$=\int_{A_{kh}\cap B_{hk}^C} b(x,\upsilon_n,0)\nabla T_k(\upsilon-T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\ dx=0. \tag{3.5}$$

According to (3.4) and (3.5), one has

$$I'_{hk}=\int_{A_{kh}\cap B_{hk}} b(x,\upsilon_n,\nabla T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\ dx.$$

Letting $h\to+\infty, |\lambda|\leq 1$, we obtain

$$A_{kh}\to\{x,|\lambda||T_k(\upsilon-\vartheta)|\leq h\}=\Omega, \tag{3.6}$$

$$B_{hk}\to\Omega\quad \textit{which gives}\quad A_{kh}\cap B_{hk}\to\Omega. \tag{3.7}$$

By applying Lebesgue theorem, we deduce that

$$\lim_{h\to+\infty}\int_{A_{kh}\cap B_{hk}} b(x,\upsilon_n,\nabla T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\ dx$$
$$=\lambda\int_\Omega b(x,\upsilon_n,\nabla(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-\vartheta)\ dx. \tag{3.8}$$

Thus,

$$\lim_{h\to+\infty} I'_{hk}=\lambda\int_\Omega b(x,\upsilon_n,\nabla(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-\vartheta)\ dx.$$

Furthermore we can easily remark that

$$\lim_{h\to+\infty}\int_\Omega H(x,\upsilon_n)T_k\left(\upsilon-T_h\left(\upsilon-\lambda T_k(\upsilon-\psi)\right)\right)dx=\lambda\int_\Omega H(x,\upsilon_n)T_k[\upsilon-\vartheta]dx$$

consequently

$$\lim_{h\to+\infty} I_{hk}=\lambda\int_\Omega b(x,\upsilon_n,\nabla(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-\vartheta)\ dx+\lambda\int_\Omega H(x,\upsilon)T_k[\upsilon-\vartheta]dx. \tag{3.9}$$

On the other hand, we have,

$$J_{hk}=\int_\Omega f\ T_k(\upsilon-T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\ dx.$$

Then

$$\lim_{h\to+\infty}\int_\Omega f\ T_k(\upsilon-T_h(\upsilon-\lambda T_k(\upsilon-\vartheta),))\ dx=\lambda\int_\Omega fT_k(\upsilon-\vartheta)dx, \tag{3.10}$$

i.e.,

$$\lim_{h\to+\infty} J_{hk}=\lambda\int_\Omega fT_k(\upsilon-\vartheta)dx. \tag{3.11}$$

Tanking to (3.9), (3.11) and passing to the limit in (3.3), one has,

$$\lambda\left(\int_\Omega b(x,\upsilon_n,\nabla(\upsilon-\lambda T_k(\upsilon-\vartheta),))\nabla T_k(\upsilon-\vartheta)\ dx+\int_\Omega H(x,\upsilon_n)T_k[\upsilon-\vartheta]dx\right)\leq\lambda\left(\int_\Omega fT_k(\upsilon-\vartheta)dx\right)$$

for every $\vartheta \in W_0^1 L_\Psi(\Omega) \cap L^\infty(\Omega)$, and for every $k > 0$. Taking $\lambda > 0$ dividing by λ, and then letting $\lambda \longrightarrow 0$, we get

$$\int_\Omega b(x, \upsilon, \nabla\upsilon)\nabla T_k(\upsilon - \vartheta)\ dx + \int_\Omega H(x, \upsilon_n) T_k[\upsilon - \vartheta] dx \leq \int_\Omega f T_k(\upsilon - \vartheta) dx. \tag{3.12}$$

for $\lambda < 0$, dividing by λ, and then letting $\lambda \longrightarrow 0$, we have

$$\int_\Omega b(x, \upsilon, \nabla\upsilon)\nabla T_k(\upsilon - \vartheta)\ dx + \int_\Omega H(x, \upsilon_n) T_k[\upsilon - \vartheta] dx \geq \int_\Omega f T_k(\upsilon - \vartheta) dx. \tag{3.13}$$

According to (3.12) and (3.13), we deduce that :

$$\int_\Omega b(x, \upsilon, \nabla\upsilon)\nabla T_k(\upsilon - \vartheta)\ dx + \int_\Omega H(x, \upsilon_n) T_k[\upsilon - \vartheta] dx = \int_\Omega f T_k(\upsilon - \vartheta) dx. \tag{3.14}$$

This completes the proof of Lemma 3.1.

3.2. Proof of Theorem 3.1

3.2.1. Approximate problem and a priori estimate. For $n \in \mathbb{N}$, define $f_n := T_n(f)$. Let υ_n be solution in $W_0^1 L_\Psi(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}(b(x, \upsilon_n, \nabla\upsilon_n)) + H_n(x, \upsilon_n) = f_n & \text{in } \Omega \\ \upsilon_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.15}$$

where,

$$H_n(x, s) = \frac{H(x, s)}{1 + \frac{1}{n}|H(x, s)|},$$

which exists thanks to [20]. Choosing $T_k(\upsilon_n)$ as test function in (3.15), we have

$$\int_\Omega b(x, \upsilon_n, \nabla\upsilon_n)\nabla T_k(\upsilon_n)\ dx + \int_\Omega H_n(x, \upsilon_n) T_k(\upsilon_n)\, dx = \int_\Omega f_n T_k(\upsilon_n) dx,$$

Using $\nabla T_k(\upsilon_n) = \nabla \upsilon_{n\chi\{|\upsilon_n| \leq k\}}$ and in viw of (1.4), then since $H_n(x, \upsilon_n) T_k(\upsilon_n) \geq 0$, we obtain

$$\int_\Omega b(x, \upsilon_n, \nabla\upsilon_n)\nabla T_k(\upsilon_n)\ dx \geq \int_\Omega \Psi(x, \lambda_1 |\nabla T_k(\upsilon_n)|) dx,$$

then

$$\int_\Omega \Psi(x, \lambda_1 |\nabla T_k(\upsilon_n)|) dx \leq k \|f\|_{L^1(\Omega)}. \tag{3.16}$$

Then

$$\int_\Omega \Psi(x, \lambda_1 |\nabla T_k(\upsilon_n)|) dx \leq C_1 k, \tag{3.17}$$

where C_1 is a constant independently of n.

3.2.2. Locally convergence of υ_n in measure. Taking $\frac{1}{\lambda}|T_k(\upsilon_n)|$ in (3.15) and using (3.17), one has

$$\int_\Omega \Psi(x, \lambda_1 \frac{|\nabla T_k(\upsilon_n)|}{\lambda}) dx \leq \int_\Omega \Psi(x, \lambda_1 |\nabla T_k(\upsilon_n)|) dx \leq C_1 k. \tag{3.18}$$

Then we deduce by using (3.18), that

$$\begin{aligned} meas\{|\upsilon_n| > k\} &\leq \frac{1}{\inf\limits_k \Psi(x, \frac{k}{\lambda})} \int_{\{|\upsilon_n| > k\}} \Psi(x, \frac{|\upsilon_n(x)|}{\lambda}) dx \\ &\leq \frac{1}{\inf\limits_k \Psi(x, \frac{k}{\lambda})} \int_\Omega \Psi(x, \frac{1}{\lambda}|T_k(\upsilon_n)|) dx \\ &\leq \frac{C_1 k}{\inf\limits_k \Psi(x, \frac{k}{\lambda})} \quad \forall n, \quad \forall k \geq 0. \end{aligned} \tag{3.19}$$

For any $\beta > 0$, we have

$$meas\{|\upsilon_n - \upsilon_m| > \beta\} \leq meas\{|\upsilon_n| > k\} + meas\{|\upsilon_m| > k\} + meas\{|T_k(\upsilon_n) - T_k(\upsilon_m)| > \beta\},$$

and so that

$$meas\{|\upsilon_n - \upsilon_m| > \beta\} \leq \frac{2C_1 k}{\inf_k \Psi(x, \frac{k}{\lambda})} + meas\{|T_k(\upsilon_n) - T_k(\upsilon_m)| > \beta\}. \tag{3.20}$$

According to (3.17) and by using Poincaré inequality in generalized -Sobolev spaces, we may obtain that $(T_k(\upsilon_n))$ is bounded in $W_0^1 L_\Psi(\Omega)$, and then there exists $\omega_k \in W_0^1 L_\Psi(\Omega)$ such that $T_k(\upsilon_n) \rightharpoonup \omega_k$ weakly in $W_0^1 L_\Psi(\Omega)$ for $\sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}})$; strongly in $E_{\overline{\Psi}}(\Omega)$ and a.e. in Ω.

Thus, we can consider that $(T_k(\upsilon_n))_n$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then by (3.20) and in view of $\frac{2C_1 k}{\inf_k \Psi(x, \frac{k}{\lambda})} \to 0$ as $k \to +\infty$ there exists some $k = k(\varepsilon) > 0$ such that

$$meas\{|\upsilon_n - \upsilon_m| > \lambda\} < \varepsilon, \qquad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).$$

Consequently, υ_n is a Cauchy sequence in measure, thus, υ_n converges almost everywhere to some measurable function υ. Finally, there exist a subsequence of $\{\upsilon_n\}_n$, still indexed by n, and a function $\upsilon \in W_0^1 L_\Psi(\Omega)$ such that

$$\begin{cases} \upsilon_n u_n \rightharpoonup \upsilon & \text{weakly in } W_0^1 L_\Psi(\Omega) \text{ for } \sigma(\Pi L_\Psi, \Pi E_{\overline{\Psi}}) \\ \upsilon_n \longrightarrow \upsilon & \text{strongly in } E_\Psi(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{3.21}$$

3.2.3. Equi-integrability of nonlinearities. Here we try to prove that

$$H_n(x, \upsilon_n) \to H(x, \upsilon_n) \text{ strongly in } L^1(\Omega) \tag{3.22}$$

For this we choose $T_{l+1}(\upsilon_n) - T_l(\upsilon_n)$ as test in (3.15), we may get

$$\int_\Omega b(x, \upsilon_n, \nabla \upsilon_n) \nabla (T_{l+1}(\upsilon_n) - T_l(\upsilon_n)) dx + \int_\Omega H_n(x, \upsilon_n) (T_{l+1}(\upsilon_n) - T_l(\upsilon_n)) dx$$
$$= \int_\Omega f (T_{l+1}(\upsilon_n) - T_l(\upsilon_n)) dx$$

which implies that

$$\int_{\{l \leq |\upsilon_n| \leq l+1\}} b(x, \upsilon_n . \nabla \upsilon_n), \nabla \upsilon_n dx + \int_{\{|\upsilon_n| \geq l+1\}} |H_n(x, \upsilon_n)| dx$$
$$\leq c \int_{\{|\upsilon_n| \geq l\}} |f| dx.$$

Thus by (1.4), we have

$$\int_{\{|\upsilon_n| \geq l+1\}} |H_n(x, \upsilon_n)| dx \leq c \int_{\{|\upsilon_n| \geq l\}} |f_n| dx.$$

Let $\varepsilon > 0$, then there exist $l(\varepsilon) \geq 1$ such that

$$\int_{\{|\upsilon_n| > l(\varepsilon)\}} |H_n(x, \upsilon_n)| dx \leq \frac{\varepsilon}{2}, \tag{3.23}$$

For any measurable subset $E \subset \Omega$, we have

$$\int_E |H_n(x, \upsilon_n)| dx \leq \int_{E \cap \{|\upsilon_n| \leq l(\varepsilon)\}} |H_n(x, \upsilon_n)| dx + \int_{E \cap \{|\upsilon_n| > l(\varepsilon)\}} |H_n(x, \upsilon_n)| dx$$
$$\leq \int_E \left|h_{l(\varepsilon)}(x)\right| dx + \int_{E \cap \{|\upsilon_n| > l(\varepsilon)\}} |g_n(x, \upsilon_n)| dx.$$

According to (1.6) there exist $\eta(\varepsilon)>0$ such that

$$\int_E \left|h_{l(\varepsilon)}(x)\right| dx \leq \frac{\varepsilon}{2}, \tag{3.24}$$

for all E such that meas $(E)<\eta(\varepsilon)$

Finally, by combining (3.23) and (3.24) one easily has $\int_E |H_n(x,\upsilon_n)|\,dx \leq \varepsilon$, for all E such that meas $(E)<\eta(\varepsilon)$

3.2.4. An intermediate Inequality. In this step, we shall prove that for $\Theta \in W_0^1 L_\Psi(\Omega)\cap L^\infty(\Omega)$, we have

$$\int_\Omega b(x,\upsilon_n,\nabla\Theta)\nabla T_k(\upsilon_n-\Theta)\;dx+\int_\Omega H_n(x,\upsilon_n)\,T_k(\upsilon_n-\Theta)\,dx \leq \int_\Omega f_n T_k(\upsilon_n-\Theta)dx. \tag{3.25}$$

Now, it suffice to take $T_k(\upsilon_n-\Theta)$ as test in (3.15), with Θ in $W_0^1 L_\Psi(\Omega)\cap L^\infty(\Omega)$,in order to get

$$\int_\Omega b(x,\upsilon_n,\nabla\upsilon_n)\nabla T_k(\upsilon_n-\Theta)\;dx+\int_\Omega H_n(x,\upsilon_n)\,T_k(\upsilon_n-\Theta)\,dx = \int_\Omega f_n T_k(\upsilon_n-\Theta)dx. \tag{3.26}$$

Adding and substracting the term $\int_\Omega b(x,\upsilon_n,\nabla\Theta),\nabla T_k(\upsilon_n-\Theta)\;dx$ i.e.,

$$\int_\Omega b(x,\upsilon_n,\nabla\upsilon_n)\nabla T_k(\upsilon_n-\Theta)\;dx+\int_\Omega b(x,\upsilon_n,\nabla\Theta)\nabla T_k(\upsilon_n-\Theta)\;dx \tag{3.27}$$

$$-\int_\Omega b(x,\upsilon_n,\nabla\Theta)\nabla T_k(\upsilon_n-\Theta)\;dx+\int_\Omega H_n(x,\upsilon_n)\,T_k(\upsilon_n-\Theta)\,dx = \int_\Omega f_n T_k(\upsilon_n-\Theta)dx.$$

According to the hypothesis (1.3), we may obtain

$$\int_\Omega (b(x,\upsilon_n,\nabla\upsilon_n)-b(x,\upsilon_n,\nabla\Theta))\nabla T_k(\upsilon_n-\Theta)\;dx \geq 0. \tag{3.28}$$

According (3.26) and (3.28), we obtain (3.25).

3.2.5. Passing to the limit. Here we show that,

$$\int_\Omega b(x,\upsilon_n,\nabla\Theta)\nabla T_k(\upsilon-\Theta)\;dx+\int_\Omega H(x,\upsilon)\,T_k(\upsilon-\Theta)\,dx \leq \int_\Omega f T_k(\upsilon-\Theta)dx,\ \text{ for } \Theta\in W_0^1 L_\Psi(\Omega)\cap L^\infty(\Omega)$$

Firstly, we verify that

$$\int_\Omega b(x,\upsilon_n,\nabla\Theta)\nabla T_k(\upsilon_n-\Theta)\;dx \to \int_\Omega b(x,\upsilon_n,\nabla\Theta)\nabla T_k(\upsilon-\Theta)\;dx \text{ as } n\to+\infty.$$

Since $T_{\mathbb{E}}(\upsilon_n)\rightharpoonup T_{\mathbb{E}}(\upsilon)$ weakly in $W_0^1 L_\Psi(\Omega)$, with $\mathbb{E}=k+\|\Theta\|_\infty$, then

$$T_k(\upsilon_n-\Theta)\rightharpoonup T_k(\upsilon-\Theta) \text{ in } W_0^1 L_\Psi(\Omega), \tag{3.29}$$

this entails that
$$\frac{\partial T_k}{\partial x_i}(\upsilon_n-\Theta)\rightharpoonup \frac{\partial T_k}{\partial x_i}(\upsilon-\Theta) \text{ weakly in } L_\Psi(\Omega) \qquad \forall i=1,..,N. \tag{3.30}$$

Show that
$$b(x,T_{\mathbb{E}}(\upsilon_n),\nabla\Theta)\to b(x,T_{\mathbb{E}}(\upsilon),\nabla\Theta) \text{ strongly in } (L_{\overline{\Psi}}(\Omega))^N.$$

According to hypothesis (1.2), we may get

$$|b_i(x,T_{\mathbb{E}}(\upsilon_n),\nabla\Theta)| \leq |\Theta_i(x)|+K_i\overline{P}^{-1}(\Psi(x,c_2|T_{\mathbb{E}}(\upsilon_n)|))+K_i\overline{\Psi}^{-1}\Psi(x,c_1|\nabla\Theta|),$$

with β and μ are positive constants. Since $T_{\mathbb{E}}(\upsilon_n) \rightharpoonup T_{\mathbb{E}}(\upsilon)$ weakly in $W_0^1 L_\Psi(\Omega)$ and $W_0^1 L_\Psi(\Omega) \hookrightarrow\hookrightarrow L_{\overline{\Psi}}(\Omega)$, then $T_{\mathbb{E}}(\upsilon_n) \rightharpoonup T_{\mathbb{E}}(\upsilon)$ strongly in $L_\Psi(\Omega)$ and a.e. in Ω, hence

$$|b(x, T_{\mathbb{E}}(\upsilon_n), \nabla\Theta)| \to |b(x, T_{\mathbb{E}}(\upsilon), \nabla\Theta)| \text{ a.e. in } \Omega.$$

and

$$|\Theta_i(x)| + K_i \overline{P}^{-1}(\Psi(x, c_2|T_{\mathbb{E}}(\upsilon_n)|)) + K_i \overline{\Psi}^{-1}\Psi(x, c_1 |\nabla\Theta|) \to$$

$$|\Theta_i(x)| + K_i \overline{P}^{-1}(\Psi(x, c_2|T_{\mathbb{E}}(\upsilon)|)) + K_i \overline{\Psi}^{-1}\Psi(x, c_1 |\nabla\Theta|),$$

a.e. in Ω. then, Vitali's theorem implies that

$$b(x, T_{\mathbb{E}}(\upsilon_n), \nabla\Theta) \to b(x, T_{\mathbb{E}}(\upsilon), \nabla\Theta) \text{ strongly in } (L_{\overline{\Psi}}(\Omega))^N, \text{ as } n \to \infty. \quad (3.31)$$

According (3.30) and (3.31), we obtain

$$\int_\Omega b(x, \upsilon_n, \nabla\Theta)\nabla T_k(\upsilon_n - \Theta)\, dx \to \int_\Omega b(x, \upsilon_n, \nabla\Theta)\nabla T_k(\upsilon - \Theta)\, dx \text{ as } n \to +\infty. \quad (3.32)$$

Furthermore, we establish that

$$\int_\Omega f_n T_k(\upsilon_n - \Theta)dx \to \int_\Omega f T_k(\upsilon - \Theta)dx. \quad (3.33)$$

We have $f_n T_k(\upsilon_n - \Theta) \to f T_k(\upsilon - \Theta)$ a.e. in Ω and $|f T_k(\upsilon_n - \Theta)| \leq k|f|$,

By applying Vitali's theorem, we may obtain (3.33).

Similarly in view of (3.22) we may prove that

$$\int_\Omega H_n(x, \upsilon_n)\, T_k(\upsilon_n - \Theta)\, dx \to \int_\Omega H(x, \upsilon_n) T_k(\upsilon - \Theta) dx \text{ as } n \to \infty. \quad (3.34)$$

According to (3.32), (3.33) and (3.34) we may pass to the limit in (3.25), so that $\forall\Theta \in W_0^1 L_\Psi(\Omega) \cap L^\infty(\Omega)$, we deduce

$$\int_\Omega b(x, \upsilon_n, \nabla\Theta)\nabla T_k(\upsilon - \Theta)\, dx + \int_\Omega H(x, \upsilon_n) T_k(\upsilon - \Theta) dx \leq \int_\Omega f T_k(\upsilon - \Theta) dx.$$

According to the Lemma 3.1, we may deduce that υ is an entropy solution of the problem (1.1). This achieve the proof of theorem 3.1.

Acknowledgements The authors would like to thank the reviewers for their helpful and constructive comments that greatly contributed to improving this version of the paper. They would also like to thank the Editors for their generous comments.

References

1. Y. Akdim, ,E. Azroul, M.Rhoudaf; *Existence of T-solution for degenerated problem via Minty's Lemma.* Acta Math. sinica (English Ser.) $24, 431 - 438(2008)$.
2. M. Ait Khellou, A. Benkirane,S.M. Douiri, *Some properties of Musielak spaces with only the log-Hölder continuity condition and application* Annals of Functional Analysis,Tusi Mathematical Research Group (TMRG) 2020.DOI: 10.1007/s43034-020-00069-7.
3. S. Ait Dada Alla, O. Azraibi, and B. El Haji, On some nonlinear elliptic equations with measurable boundary conditions in anisotropic weighted Sobolev spaces. Gulf Journal of Mathematics, 17(1), 130-141. https://doi.org/10.56947/gjom.v17i1.1712
4. O. Azraibi, D. Abdelkarim and B. El Haji, ; On some anisotropic elliptic problem with measure data. Int. J. Nonlinear Anal. Appl. 16 (2025) 5, 1-11 ISSN: 2008-6822 (electronic) http://dx.doi.org/10.22075/ijnaa.2024.33483.4992.
5. O. Azraibi, S. Ait Dada Alla and B. El Haji, On some nonlinear elliptic probles with boundary conditions in generalized sobolev spaces, J. Nonlinear Funct. Anal. 2024 (2024) 19 https://doi.org/10.23952/jnfa.2024.19

6. O. Azraibi, B.EL haji and M. Mekkour, Nonlinear parabolic problem with lower order terms in Musielak-Sobolev spaces without sign condition and with Measure data, Palestine Journal of Mathematics,Vol. 11(3)(2022) , 474-503.

7. O. Azraibi, B.EL haji and M. Mekkour, *On some Nonlinear elliptic problems with large monotonicity in Musielak-Orlicz-Sobolev spaces*, , Journal of Mathematical Physics, Analysis, Geometry 2022, Vol. 18, No. 3,1-18.

8. A. Benkirane, A. Elmahi; *Almost everywhere convergence of the gradient of solutions to elliptic equations in Orlicz spaces*, Nonlinear Anal. T.M.A. 28(11)(1997), 1769-1784.

9. A. Benkirane and M. Sidi El Vally, *Some approximation properties in Musielak-Orlicz- Sobolev spaces,*: Thai.J. Math. 10, (2012) 371-381.

10. A. Benkirane, and M. Sidi El Vally, *Variational inequalities in Musielak-Orlicz-Sobolev spaces,* Bull. Belg. Math. Soc. Simon Stevin 21, (2014) 787-811.

11. L. Boccardo, L.Orsina, *Existence results for Dirichlet problems in L1 via Minty's lemma. Appl. Anal.* $76(3-4), 309-317(2000)$.

12.

13. B. El Haji and M. El Moumni and A. Talha, *Entropy solutions for nonlinear parabolic equations in Musielak Orlicz spaces without* $Delta_2$*-conditions*, Gulf Journal of Mathematics Vol 9, Issue 1 (2020) 1-26.

14. B. El Haji and M. El Moumni; Entropy solutions of nonlinear elliptic equations with L^1-data and without strict monotonocity conditions in weighted Orlicz-Sobolev spaces, Journal of Nonlinear Functional Analysis, Vol. 2021 (2021), Article ID 8, pp. 1-17

15. B. El Haji, M. El Moumni, K. Kouhaila; *On a nonlinear elliptic problems having large monotonocity with* L^1-data *in weighted Orlicz-Sobolev spaces* ,Moroccan J. of Pure and Appl. Anal. (MJPAA) Volume 5(1), 2019, Pages 104-116 DOI 10.2478/mjpaa-2019-0008

16. B. El Haji and M. El Moumni and K. Kouhaila; Existence of entropy solutions for nonlinear elliptic problem having large monotonicity in weighted Orlicz-Sobolev spaces , LE MATEMATICHE Vol. LXXVI (2021) - Issue I, pp. 37-61, https://doi.org/10.4418/2021.76.1.3.

17. N. El Amarti, B. El Haji and M. El Moumni. Entropy solutions for unilateral parabolic problems with L^1-data in Musielak-Orlicz-Sobolev spaces Palestine Journal of Mathematics, Vol. 11(1)(2022) , 504-523.

18. R. Elarabi, M. Rhoudaf, H. Sabiki, *Entropy solution for a nonlinear elliptic problem with lower order term in Musielak-Orlicz spaces.* Ric. Mat. (2017). https://doi.org/10.1007/s11587-017-0334-z

19. J.-P. Gossez, *Some approximation properties in Orlicz-Sobolev spaces,* Studia Math. 74 (1982), no. 1, 17-24.

20. J.-P. Gossez and V. Mustonen; *Variational inequalities in Orlicz-Sobolev spaces,* Nonlinear Anal. 11 (1987), no. 3, 379-392.

21. E.Hewitt, K.Stromberg, *Real and Abstract Analysis.* Springer, Berlin (1965)

22. J. Musielak, *Modular spaces and Orlicz spaces,* Lecture Notes in Math. 1034 (1983).

23. A. Porretta, *Existence results for strongly nonlinear parabolic equations via strong conver- gence of truncations,* Annali di matematica pura ed applicata. (IV), Vol. CLXXVII (1999) 143-172.

Abdelkarim DERHAM,

Laboratory LaR2A, Department of Mathematics,

Faculty of Sciences Tetouan, Abdelmalek Essaadi University, BP 2121, Tetouan,

Morocco.

E-mail address: `abdederham@gmail.com`

and

Badr EL HAJI,

Laboratory LaR2A, Department of Mathematics,

Faculty of Sciences Tetouan, Abdelmalek Essaadi University, BP 2121, Tetouan,

Morocco.

E-mail address: `badr.elhaji@gmail.com`

and

Ouidad AZRAIBI,

Laboratory LaR2A, Department of Mathematics,

Faculty of Sciences Tetouan, Abdelmalek Essaadi University, BP 2121, Tetouan,

Morocco.

E-mail address: `azraibi.ouidad@gmail.com`