Bol. Soc. Paran. Mat.
©SPM – E-ISSN-2175-1188
SPM: www.spm.uem.br/bspm

(3s.) **v. 2025 (43)** : 1–10.
ISSN-0037-8712
doi:10.5269/bspm.66527

Algebras for Monads in the Category of Subobjects

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ABSTRACT: For a given object Y in a category $\mathcal{C}$, we construct the category of T-Algebras (Eilenberg-Moore category) and Kleisli category corresponding to the monad defined on partial order category $Sub_{\mathcal{C}}[Y]$. We obtain sufficient condition for the right adjoint to be monadic for the string of adjunction $f(-) \dashv f^{-1} \dashv f^{\#}$. Finally, given any adjunction the sufficient condition for the comparison functor between the original category and the category of T-Algebras derived from monad to have a left adjoint is obtained.

Key Words: Adjunctions, natural transformation, monads, subobjects, eilenberg-moore category, kleisli Category.

Contents

1. Introduction and Preliminaries

It is very well known that corresponding to any map $f : Y \to Z$ between sets, there are induced maps $Imf : \wp(Y) \to \wp(Z)$ known as image map and $f^{-1} : \wp(Z) \to \wp(Y)$ called the inverse image map which are not only the set-theoretic maps but also there exist adjunction between these maps [2]. In [1], Arkhangel'skii used another induced map defined with the help of fibers namely, $f^{\#} : \wp(Y) \to \wp(Z)$, given by $f^{\#}(E) = \{k \in Z : f^{-1}(k) \subset E\}$ which gives rise to string of adjunction $f(-) \dashv f^{-1} \dashv f^{\#}$. In [2], for a given object Y in a category $\mathcal{C}$, the category of subobjects denoted by $Sub_{\mathcal{C}}[Y]$ was defined which again motivates to introduce induced maps namely $f(-)$, f^{-1} between the category of subobjects i.e. to find image and inverse image of a subobject corresponding to an arrow $f : Y \to Z$ in $\mathcal{C}$ such that there exist adjunction between them [3]. Motivated by this work in [4], the authors introduced another induced map $f^{\#} : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Z]$ and had given the complete interrelationship between the induced maps $f(-), f^{-1}$ and $f^{\#}$ and their interdependence using categorical concepts especially the concept of functor adjunctions. They had also given some interesting characterizations of the monicity and epicity of $f(-)$ map. Recently, in [5], Homayoun Nejah et. al. studied the category of partially ordered objects in a topos E. Further, a pair of adjoint functors give rise to a map of monads whose research was shaped by the applications to functional programming. This motivates us to study the concept of monad for the string of adjunction $f(-) \dashv f^{-1} \dashv f^{\#}$ and related categorical theoretic concepts.

In our present paper, for a given object Y in a category $\mathcal{C}$, we construct category of T-Algebras (Eilenberg-Moore category) and Kleisli category corresponding to the monads defined on partial order category $Sub_{\mathcal{C}}[Y]$ (Theorems 2.3, 2.7 and 2.9). Further, for the string of adjunction $f(-) \dashv f^{-1} \dashv f^{\#}$, we construct comparison functor between the partial order category of subobjects and the category of T-Algebras derived from monad (Theorem 2.4) and obtain the sufficient condition for the right adjoint to be monadic (Theorems 2.5 and 2.8). Finally, given any adjunction sufficient condition for the comparison functor between the original category and the category of T-Algebras derived from monad to have a left adjoint is obtained (Theorem 2.10).

Here we are assuming that the reader is familiar with basic concepts of Category Theory that can be found in [2,6,3]. Some concepts and results related with our paper are mentioned below. The readers are also advised to see our previous paper [4] for the better understanding of this paper.

2010 *Mathematics Subject Classification:* 18A20; 18A23; 18A40
Submitted January 02, 2023. Published July 12, 2025

Definition 1.1 *[2] "A monad on a category $\mathcal{C}$ consists of an endofunctor $T : \mathcal{C} \to \mathcal{C}$ and natural transformations $\eta : 1_{\mathcal{C}} \Rightarrow T$, $\mu : T^2 \Rightarrow T$ such that the following diagrams are commutative.*

$$\begin{array}{ccc} T^3 & \xrightarrow{T_\mu} & T^2 \\ {\scriptstyle \mu_T}\downarrow & & \downarrow{\scriptstyle \mu} \\ T^2 & \xrightarrow[\mu]{} & T \end{array}$$

i.e.

$$\mu \circ \mu_T = \mu \circ T_\mu$$

and

$$\begin{array}{ccccc} T & \xrightarrow{\eta_T} & T^2 & \xleftarrow{T_\eta} & T \\ & {\scriptstyle 1_T}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle 1_T} & \\ & & T & & \end{array}$$

i.e.

$$\mu \circ \eta_T = 1_T = \mu \circ T_\eta\text{"}$$

Theorem 1.1 *(Comparison Functor [2]) "Let $< F, G, \eta, \epsilon >: \mathcal{C} \to \mathcal{D}$ be an adjunction and (T, η, μ) be the corresponding monad and let $< F^T, G^T, \eta, \delta >: \mathcal{C} \to \mathcal{C}^T$ be the adjunction derived from monad, where $\mathcal{C}^T$ is the Eilenberg-Moore category. Then there exists a unique functor*

$$\phi : \mathcal{D} \to \mathcal{C}^T$$

with

$$G^T \circ \phi = G \text{ and}$$
$$\phi \circ F = F^T$$

known as comparison functor.

Note : If the above comparison functor is full, faithful and essentially surjective on objects i.e. $\mathcal{D}$ and $\mathcal{C}^T$ are equivalent categories then the functor G is known as monadic."

Lemma 1.1 *[4](Triangle identities for "$f(-) \dashv f^{-1} \dashv f^{\#}$"):*

"For a morphism $f : Y \to Z$ of $\mathcal{C}$, the following holds:

(i) $f^{-1} \circ f(-) \circ f^{-1} = f^{-1}$, $f(-) \circ f^{-1} \circ f(-) = f(-)$.

(ii) $f^{-1} \circ f^{\#} \circ f^{-1} = f^{-1}$, $f^{\#} \circ f^{-1} \circ f^{\#} = f^{\#}$."

Corollary 1.1 *[4] "Let $f(-) \dashv f^{-1}$ corresponding to an arrow $f : Y \to Z$ in a category $\mathcal{C}$. Then the following conditions are equivalent:*

(a) $f(-)$ is epi.

(b) $f(-)$ is left-adjoint left inverse of f^{-1} i.e. counit of adjunction is identity."

Corollary 1.2 *[4] "Let $f^{-1} \dashv f^{\#}$ corresponding to an arrow $f : Y \to Z$ in a category $\mathcal{C}$. Then the following conditions are equivalent:*

(a) $f(-)$ is monic or $f^{\#}$ is monic.

(b) f^{-1} is left-adjoint left inverse of $f^{\#}$ i.e. counit of adjunction is identity."

Lemma 1.2 *[4] "Let $\mathcal{C}$ and $\mathcal{D}$ be two partial order categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors such that $F \cong G$ i.e. F is iso to G then $F = G$."*

2. Algebras on a Monad

We begin by introducing the category of T-Algebras i.e. Eilenberg-Moore Category corresponding to a partial order category $Sub_{\mathcal{C}}[Y]$. Before this, for the sake of readability we first recall the brief introduction to category $Sub_{\mathcal{C}}[Y]$ and induced maps $f(-), f^{-1}$ and $f^{\#}$ for a given object Y and an arrow $f : Y \to Z$ in any category $\mathcal{C}$ from [4]. Here $Sub_{\mathcal{C}}[Y]$ is a partial order category having objects as the equivalence classes $[m]$ corresponding to a monic arrow $m : M \to Y$ of $\mathcal{C}$ with codomain Y with relation say $\leq$ (where $[m] \leq [n]$ if and only if there is an arrow $m \to n$ in the category of subobjects of Y). Further, if $\mathcal{C}$ is a category with pullbacks then for any arrow $f : Y \to Z$ in $\mathcal{C}$, there is a functor $f^{-1} : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]$, which is restriction of the pullback functor to $Sub_{\mathcal{C}}[Z]$ called the inverse image functor. Also $f(-) : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Z]$ called the image functor is left adjoint of f^{-1} and $f^{\#} : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Z]$ given by $f^{\#}[m] = max\{[n] \in Sub_{\mathcal{C}}[Z] : f^{-1}[n] \leq [m]\}$ is right adjoint of f^{-1} if the left and right adjoints of f^{-1} exist. Further, in [4] the authors introduced the concept of saturated objects and obtained its characterization using the string of adjunction $f(-) \dashv f^{-1} \dashv f^{\#}$ mentioned below. We utilize this concept in the construction of Eilenberg-Moore Category in this paper.

Definition 2.1 *[4] "Let $\mathcal{C}$ be a category with pullbacks and for any arrow $f : Y \to Z$ in $\mathcal{C}$, consider the inverse image functor $f^{-1} : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]$. Then an object $[m]$ of $Sub_{\mathcal{C}}[Y]$ is called saturated if f^{-1} is essentially surjective on that object i.e. there exists an object $[n]$ of $Sub_{\mathcal{C}}[Z]$ such that $f^{-1}[n] = [m]$."*

Theorem 2.1 *[4] "For an adjunction $f(-) \dashv f^{-1}$ and an object $[m]$ of $Sub_{\mathcal{C}}[Y]$, the following conditions are equivalent:*

(a) $[m]$ is saturated.

(b) $f^{-1} \circ f(-)[m] = [m]$."

Theorem 2.2 *[4] "For an adjunction $f^{-1} \dashv f^{\#}$ and an object $[m]$ of $Sub_{\mathcal{C}}[Y]$, the following conditions are equivalent:*

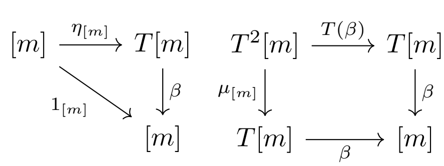
(a) $[m]$ is saturated.

(b) $f^{-1} \circ f^{\#}[m] = [m]$."

The following Theorem gives the construction of Eilenberg-Moore Category corresponding to monad on partial order category $Sub_{\mathcal{C}}[Y]$.

Theorem 2.3 *(Eilenberg-Moore category construction in $Sub_{\mathcal{C}}[Y]$)*
Let (T, η, μ) be the monad where $T : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]$ such that $T = f^{-1} \circ f(-)$ then there exists a category $\mathcal{D}$ and an adjunction $< F^T, U^T, \overline{\eta}, \delta >: Sub_{\mathcal{C}}[Y] \to \mathcal{D}$ such that it gives rise to the same monad.

Proof: For a given object Y in a category $\mathcal{C}$, let (T, η, μ) be the monad defined on the partial order category $Sub_{\mathcal{C}}[Y]$ where $T = f^{-1} \circ f(-)$. We define Eilenberg-Moore category for the monad $T : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]$ denoted by $Sub_{\mathcal{C}}[Y]^T$ as follows:
Let $([m], \beta)$ be an object of $Sub_{\mathcal{C}}[Y]^T$. Then there is an arrow $\beta : f^{-1} \circ f(-)[m] \to [m]$ and a natural component $\eta_{[m]} : [m] \to f^{-1} \circ f(-)[m]$, so $f^{-1} \circ f(-)[m] = [m]$, since $Sub_{\mathcal{C}}[Y]$ is a partial order category. Therefore, by Theorem 2.1, it follows that $[m]$ is saturated object of $Sub_{\mathcal{C}}[Y]$. Also



are commutative trivially.
Further, there is an arrow $k : ([m], \beta) \to ([m'], \gamma)$ in $Sub_{\mathcal{C}}[Y]^T$ if and only if there is an arrow $k : [m] \to [m']$ in $Sub_{\mathcal{C}}[Y]$. Hence $Sub_{\mathcal{C}}[Y]^T$ is the collection of all saturated objects of $Sub_{\mathcal{C}}[Y]$.

Next define $F^T : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]^T$ on objects by $F^T[m] = (T[m], \mu_{[m]})$ and on arrows by $F^T(k : [m] \to [m']) = T(k)$, since $f^{-1} \circ f(-)(T[m]) = f^{-1} \circ f(-) \circ f^{-1} \circ f(-)[m] = f^{-1} \circ f(-)[m] = T[m]$ by Lemma 1.1(i) and so $T[m]$ is a saturated object of $Sub_{\mathcal{C}}[Y]$. Therefore, F^T is well defined and define $U^T : Sub_{\mathcal{C}}[Y]^T \to Sub_{\mathcal{C}}[Y]$ on objects by $U^T([m], \beta) = [m]$ and on arrows by $U^T(k : ([m], \beta) \to ([m'], \gamma)) = k : [m] \to [m']$.

Therefore, $U^T \circ F^T[m] = U^T(T[m], \mu_{[m]}) = T[m]$ and $U^T \circ F^T(k) = U^T(T(k)) = T(k)$ and so $U^T \circ F^T = T$. Lastly, we have to prove that $F^T \dashv U^T$ and also need the counit and unit of adjunction.

$$\delta : F^T \circ U^T \Rightarrow 1_{Sub_{\mathcal{C}}[Y]^T}$$
$$\overline{\eta} : 1_{Sub_{\mathcal{C}}[Y]} \Rightarrow U^T \circ F^T.$$

As we have seen earlier that $U^T \circ F^T = T$. Therefore, we can take $\overline{\eta} = \eta$ as unit of adjunction. Further, for an object $([m], \beta)$ of $Sub_{\mathcal{C}}[Y]^T$, $F^T \circ U^T([m], \beta) = (T[m], \mu_{[m]})$ and $\beta : T[m] \to [m]$.
Therefore, define

$$\delta_{([m],\beta)} : (T[m], \mu_{[m]}) \to ([m], \beta)$$

by

$$\delta_{([m],\beta)} = \beta.$$

Finally, for proving the adjunction we prove the following bi-condition:

$$Hom_{Sub_{\mathcal{C}}[Y]^T}(F^T[m], ([m'], \beta)) \cong Hom_{Sub_{\mathcal{C}}[Y]}([m], U^T([m'], \beta))$$

Therefore, for any two objects $[m]$ and $([m'], \beta)$ of $Sub_{\mathcal{C}}[Y]$ and $Sub_{\mathcal{C}}[Y]^T$ respectively, we have:

$$\begin{aligned} Hom_{Sub_{\mathcal{C}}[Y]^T}(F^T[m], ([m'], \beta)) &= Hom_{Sub_{\mathcal{C}}[Y]^T}((T[m], \mu_{[m]}), ([m'], \beta)) \\ &\cong Hom_{Sub_{\mathcal{C}}[Y]}(T[m], [m']) \\ &= Hom_{Sub_{\mathcal{C}}[Y]}(T[m], T[m']) \\ &\cong Hom_{Sub_{\mathcal{C}}[Y]}([m], [m']) \\ &= Hom_{Sub_{\mathcal{C}}[Y]}([m], U^T([m'], \beta)) \end{aligned}$$

using the fact that $[m']$ is a saturated object of $Sub_{\mathcal{C}}[Y]$ and so $T[m'] = f^{-1} \circ f(-)[m'] = [m']$ and there is a natural transformation $\eta : 1_{Sub_{\mathcal{C}}[Y]} \Rightarrow T$.

Thus, we have $F^T \dashv U^T$ with $T = U^T \circ F^T$, unit of adjunction η and for an object $[m]$ of $Sub_{\mathcal{C}}[Y]$,

$$U^T(\delta_{F^T})_{[m]} = U^T(\delta_{F^T[m]}) = U^T(\delta_{(T[m], \mu_{[m]})}) = U^T(\mu_{[m]}) = \mu_{[m]}.$$

Hence $U^T(\delta_{F^T}) = \mu$ and so the adjunction $< F^T, U^T, \overline{\eta}, \delta >: Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]^T$ gives rise to the same monad (T, η, μ). □

For a given adjunction $f(-) \dashv f^{-1}$, the following Theorem gives the construction of comparison functor between the partial order category of subobjects and Eilenberg-Moore category constructed in Theorem 2.3.

Theorem 2.4 *Let $f(-) \dashv f^{-1}$ with η, ϵ be the unit and counit of adjunction respectively and (T, η, μ) be the corresponding monad. Consider the adjunction $< F^T, U^T, \eta, \delta >: Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]^T$ arises from monad then there exists a functor $\phi : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]^T$ with*

$$U^T \circ \phi = f^{-1} \text{ and } \phi \circ f(-) = F^T$$

Proof: Define $\phi : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]^T$ on objects by

$$\phi[n] = (f^{-1}[n], f^{-1}(\epsilon_{[n]}))$$

and on arrows by

$$\phi(k : [n] \to [n']) = f^{-1}(k) : (f^{-1}[n], f^{-1}(\epsilon_{[n]})) \to (f^{-1}[n'], f^{-1}(\epsilon_{[n']})),$$

since $f^{-1} \circ f(-) \circ f^{-1}[n] = f^{-1}[n]$ and so $f^{-1}[n]$ is saturated object of $Sub_{\mathcal{C}}[Y]$ and $\epsilon_{[n]} : f(-) \circ f^{-1}[n] \to [n]$ and so $f^{-1}(\epsilon_{[n]}) : f^{-1} \circ f(-) \circ f^{-1}[n] \to f^{-1}[n]$ and so $f^{-1}(\epsilon_{[n]}) : T(f^{-1}[n]) \to f^{-1}[n]$. Hence ϕ is well defined.
Further,

$$U^T \circ \phi[n] = U^T(f^{-1}[n], f^{-1}(\epsilon_{[n]})) = f^{-1}[n],$$
$$U^T \circ \phi(k) = U^T(f^{-1}(k)) = f^{-1}(k)$$

implies $U^T \circ \phi = f^{-1}$.
Also for an object $[m]$ in $Sub_{\mathcal{C}}[Y]$,

$$\phi \circ f(-)[m] = \phi(f(-)[m]) = (f^{-1} \circ f(-)[m], f^{-1}(\epsilon_{f(-)[m]})) = (T[m], \mu_{[m]}) = F^T[m]$$

and for an arrow $l : [m] \to [m']$ in $Sub_{\mathcal{C}}[Y]$,

$$\phi \circ f(-)(l : [m] \to [m']) = \phi(f(-)(l)) = f^{-1}(f(-)(l)) = f^{-1} \circ f(-)(l) = T(l) = F^T(l)$$

implies $\phi \circ f(-) = F^T$. □

As we have seen $Sub_{\mathcal{C}}[Z]$ and $Sub_{\mathcal{C}}[Y]^T$ are partial order categories, therefore, by Lemma 1.2, it follows that $Sub_{\mathcal{C}}[Z]$ and $Sub_{\mathcal{C}}[Y]^T$ are equivalent categories if and only if they are isomorphic. The following Theorem gives the necessary and sufficient condition for the above said categories to be isomorphic. Hence in particular, given an adjunction $f(-) \dashv f^{-1}$, it also gives an idea about the monadicity of right adjoint.

Theorem 2.5 *$Sub_{\mathcal{C}}[Z]$ and $Sub_{\mathcal{C}}[Y]^T$ are isomorphic categories if and only if $f(-)$ is epi.*

Proof: For given objects Y and Z in a category $\mathcal{C}$, the partial order categories of subobjects $Sub_{\mathcal{C}}[Z]$ and $Sub_{\mathcal{C}}[Y]^T$ will be isomorphic if and only if there exist functors $\psi : Sub_{\mathcal{C}}[Y]^T \to Sub_{\mathcal{C}}[Z]$ and $\phi : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]^T$ such that

$$\phi \circ \psi = 1 \text{ and } \psi \circ \phi = 1.$$

For this consider the comparison functor $\phi : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]^T$ and
define $\psi : Sub_{\mathcal{C}}[Y]^T \to Sub_{\mathcal{C}}[Z]$ on objects by

$$\psi([m], \beta) = f(-)[m]$$

and on arrows by

$$\psi(k : ([m], \beta) \to ([m'], \gamma)) = f(-)(k) : f(-)[m] \to f(-)[m'].$$

Next, we prove that $\phi \circ \psi = 1$ and $\psi \circ \phi = 1$ if and only if $f(-)$ is epi.
Therefore, for an object $([m], \beta)$ of $Sub_{\mathcal{C}}[Y]^T$, we have

$$\phi \circ \psi(([m], \beta)) = \phi(f(-)[m]) = (f^{-1} \circ f(-)[m], f^{-1}(\epsilon_{f(-)[m]})) = ([m], \beta),$$

since $[m]$ is a saturated object of $Sub_{\mathcal{C}}[Y]$ and so $\phi \circ \psi = 1$. Also for an object $[n]$ of $Sub_{\mathcal{C}}[Z]$,

$$\psi \circ \phi([n]) = \psi(f^{-1}[n], f^{-1}(\epsilon_{[n]})) = f(-) \circ f^{-1}[n].$$

Therefore, for any object $[n]$ of $Sub_{\mathcal{C}}[Z]$, $\psi \circ \phi([n]) = [n]$ if and only if $f(-) \circ f^{-1}[n] = [n]$ if and only if $f(-) \circ f^{-1} = 1_{Sub_{\mathcal{C}}[Z]}$ i.e. the counit of adjunction $f(-) \dashv f^{-1}$ is identity and so by Corollary 1.1, $\psi \circ \phi = 1$ if and only if $f(-)$ is epi. Hence $Sub_{\mathcal{C}}[Z]$ and $Sub_{\mathcal{C}}[Y]^T$ are isomorphic categories if and only if $f(-)$ is epi.

□

Remark 2.1 In particular, it follows that the functor $f^{-1} : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]$ is monadic if and only if $f(-)$ is epi.

Further, the concept of saturated objects motivates us also to introduce #-saturated objects and utilize it in the construction of Eilenberg-Moore Category for the monad (T, η, μ) where $T = f^{\#} \circ f^{-1}$.

Definition 2.2 *An object $[n]$ of $Sub_{\mathcal{C}}[Z]$ is called #-saturated if $f^{\#}$ is essentially surjective on that object i.e. there exists an object $[m]$ of $Sub_{\mathcal{C}}[Y]$ such that $f^{\#}[m] = [n]$.*

The following Theorem gives the characterization of #-saturated objects using unit of adjunction $f^{-1} \dashv f^{\#}$.

Theorem 2.6 *For an adjunction $f^{-1} \dashv f^{\#}$ and an object $[n]$ of $Sub_{\mathcal{C}}[Z]$, the following conditions are equivalent:*

(a) $[n]$ *is #-saturated.*

(b) $f^{\#} \circ f^{-1}[n] = [n]$.

Proof: (a)⇒(b) : Firstly, let $[n]$ be #-saturated object of $Sub_{\mathcal{C}}[Z]$ then there exists an object $[m]$ of $Sub_{\mathcal{C}}[Y]$ such that $f^{\#}[m] = [n]$. Therefore, for any two objects $[m]$ of $Sub_{\mathcal{C}}[Y]$ and $[n]$ of $Sub_{\mathcal{C}}[Z]$, the proof follows from the following UMP of adjunction :

$$\begin{array}{ccc} [n] & \xrightarrow{\;1_{[n]}\;} & \\ {\scriptstyle \alpha_{[n]}}\downarrow & \searrow & \\ f^{\#} \circ f^{-1}[n] & \longrightarrow & f^{\#}[m] \end{array}$$

where $\alpha_{[n]} : [n] \to f^{\#} \circ f^{-1}[n]$ is the natural component of unit of adjunction $f^{-1} \dashv f^{\#}$.

Conversely, let $f^{\#} \circ f^{-1}[n] = [n]$ then there is an object $[m] = f^{-1}[n]$ of $Sub_{\mathcal{C}}[Y]$ such that $f^{\#}[m] = [n]$. Therefore, $[n]$ is #-saturated and so (a) holds. □

The following Theorem gives the construction of Eilenberg-Moore Category for the monad (T, η, μ) where $T = f^{\#} \circ f^{-1}$.

Theorem 2.7 *Let (T, η, μ) be the monad where $T : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Z]$ such that $T = f^{\#} \circ f^{-1}$ then there exists a category $\mathcal{D}'$ and an adjunction $< F^T, U^T, \overline{\eta}, \delta >: Sub_{\mathcal{C}}[Z] \to \mathcal{D}'$ such that it gives rise to the same monad.*

Proof: Proof follows by taking objects of category $\mathcal{D}'$ as the collection of all #-saturated objects of $Sub_{\mathcal{C}}[Z]$ and is similar to that of Theorem 2.3 and hence omitted. □

For a given adjunction $f^{-1} \dashv f^{\#}$, the following theorem gives sufficient condition for the monadic behaviour of the right adjoint.

Theorem 2.8 *For the adjunction $f^{-1} \dashv f^{\#}$, $f^{\#} : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Z]$ is monadic if and only if $f^{\#}$ is monic.*

Proof: Proof is similar to that of Theorem 2.5 and hence omitted. □

Further, we construct Kleisli Category corresponding to any monad defined on the partial order category $Sub_{\mathcal{C}}[Y]$.

Theorem 2.9 *(Kleisli Category construction in $Sub_{\mathcal{C}}[Y]$)*
Let (T, η, μ) be any monad where $T : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]$. Then there exists a category $\mathcal{E}$ and an adjunction $< F_T, G_T, \overline{\eta}, \delta' >: Sub_{\mathcal{C}}[Y] \to \mathcal{E}$ such that it gives rise to the same monad.

Proof: For a given object Y in a category $\mathcal{C}$, let (T, η, μ) be the monad defined on the partial order category $Sub_{\mathcal{C}}[Y]$. Now, to define Kleisli Category for the monad $T : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]$ denoted by $Sub_{\mathcal{C}}[Y]_T$, define an object $[m]_T$ of $Sub_{\mathcal{C}}[Y]_T$ by $[m]_T = T[m]$ and an arrow $k^T : [m]_T \to [m']_T$ i.e. $k^T : T[m] \to T[m']$ by $k^T = \mu_{[m']} \circ T(k)$ corresponding to every object $[m]$, $[m']$ and arrow $k : [m] \to T[m']$ of $Sub_{\mathcal{C}}[Y]$.

Firstly, define the functor $F_T : Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]_T$ on objects by

$$F_T[m] = T[m] = [m]_T$$

and on arrows by

$$F_T(k : [m] \to [m']) = \mu_{[m']} \circ T(\eta_{[m']} \circ k) : T[m] \to T[m'],$$

since $k : [m] \to [m']$ implies $\eta_{[m']} \circ k : [m] \to T[m']$ and so $\mu_{[m']} \circ T(\eta_{[m']} \circ k) : T[m] \to T[m']$. Also by definition of monad $\mu_{[m']} \circ T(\eta_{[m']}) = 1_{T[m']}$ and so $\mu_{[m']} \circ T(\eta_{[m']} \circ k) = T(k)$. Therefore, $F_T(k) : [m]_T \to [m']_T$ is equivalent to $F_T(k) = T(k) : T[m] \to T[m']$. Hence $F_T = T$ on objects and arrows and so F_T is a functor.

Further, define $G_T : Sub_{\mathcal{C}}[Y]_T \to Sub_{\mathcal{C}}[Y]$ on objects by

$$G_T([m]_T) = G_T(T[m]) = T[m]$$

and on arrows by

$$G_T(g : [m]_T \to [m']_T) = G_T(g : T[m] \to T[m']) = g : T[m] \to T[m'].$$

Therefore, $G_T = 1_{Sub_{\mathcal{C}}[Y]_T}$ on objects and arrows and so G_T is a functor.

Now $G_T \circ F_T[m] = G_T(T[m]) = T[m]$ and $G_T \circ F_T(k) = G_T(T(k)) = T(k)$ and so $G_T \circ F_T = T$. Lastly, we have to prove that $F_T \dashv G_T$ and also need the counit and unit of adjunction,

$$\delta' : F_T \circ G_T \Rightarrow 1_{Sub_{\mathcal{C}}[Y]_T}$$
$$\overline{\eta} : 1_{Sub_{\mathcal{C}}[Y]} \Rightarrow G_T \circ F_T.$$

As we have seen earlier that $G_T \circ F_T = T$, therefore, we can take $\overline{\eta} = \eta$ as unit of adjunction. Further, for an object $[m]_T$ of $Sub_{\mathcal{C}}[Y]_T$, $F_T \circ G_T([m]_T) = F_T(T[m]) = T^2[m]$.
Therefore, define

$$\delta'_{[m]_T} : T^2[m] \to [m]_T \text{ i.e. } \delta'_{[m]_T} : T^2[m] \to T[m]$$

by

$$\delta'_{[m]_T} = \mu_{[m]}.$$

Finally, for proving adjunction we prove the following bi-condition:

$$Hom_{Sub_{\mathcal{C}}[Y]_T}(F_T[m], [m']_T) \cong Hom_{Sub_{\mathcal{C}}[Y]}([m], G_T([m']_T))$$

Therefore, for any two objects $[m]$ and $[m']_T$ of $Sub_{\mathcal{C}}[Y]$ and $Sub_{\mathcal{C}}[Y]_T$ respectively, define a mapping :

$$\phi : Hom_{Sub_{\mathcal{C}}[Y]_T}(F_T[m], [m']_T) \to Hom_{Sub_{\mathcal{C}}[Y]}([m], G_T([m']_T))$$

i.e.

$$\phi : Hom_{Sub_{\mathcal{C}}[Y]_T}([m]_T, [m']_T) \to Hom_{Sub_{\mathcal{C}}[Y]}([m], T[m'])$$

by $\phi(g) = g \circ \eta_{[m]}$ for any arrow $g : [m]_T \to [m']_T$, since $[m]_T = T[m]$. Also define a mapping

$$\psi : Hom_{Sub_{\mathcal{C}}[Y]}([m], T[m']) \to Hom_{Sub_{\mathcal{C}}[Y]_T}([m]_T, [m']_T)$$

by $\psi(k) = \mu_{[m']} \circ T(k)$ for any arrow $k : [m] \to T[m']$.
Now

$$\begin{aligned} \phi \circ \psi(k) &= \phi(\mu_{[m']} \circ T(k)) \\ &= \mu_{[m']} \circ T(k) \circ \eta_{[m]} \\ &= \mu_{[m']} \circ \eta_{T[m]} \circ k \\ &= 1_{T[m]} \circ k \\ &= k \end{aligned}$$

using the fact that $\eta : 1 \Rightarrow T$ and $\mu \circ \eta_T = 1$. Further,

$$\begin{aligned} \psi \circ \phi(g) &= \psi(g \circ \eta_{[m]}) \\ &= \mu_{[m']} \circ T(g \circ \eta_{[m]}) \\ &= \mu_{[m']} \circ T(g) \circ T(\eta_{[m]}) \\ &= \mu_{[m']} \circ \eta_{T[m']} \circ g \\ &= g. \end{aligned}$$

Therefore, we have $F_T \dashv G_T$ with $T = G_T \circ F_T$ and unit of adjunction η. Also for an object $[m]$ of $Sub_{\mathcal{C}}[Y]$,

$$G_T(\delta'_{F_T})_{[m]} = G_T(\delta'_{F_T[m]}) = G_T(\delta'_{[m]_T}) = G_T(\mu_{[m]}) = \mu_{[m]}.$$

Thus, $G_T(\delta'_{F_T}) = \mu$.
Hence the adjunction $< F_T, G_T, \overline{\eta}, \delta' >: Sub_{\mathcal{C}}[Y] \to Sub_{\mathcal{C}}[Y]_T$ gives rise to the same monad (T, η, μ). □

Finally, given an adjunction we obtain sufficient condition for the comparison functor between original category and category of T-Algebras derived from monad to have a left adjoint.

Theorem 2.10 *Let $< F, G, \eta, \epsilon >: \mathcal{C} \to \mathcal{D}$ be any adjunction and (T, η, μ) be the corresponding monad. Further consider $< F^T, G^T, \eta, \delta >: \mathcal{C} \to \mathcal{C}^T$ adjunction derived from the monad. If for every object (E, α) of $\mathcal{C}^T$, α and η_E are inverses of each other then the comparison functor has a left adjoint.*

Proof: Define the functor $\psi : \mathcal{C}^T \to \mathcal{D}$ by $\psi(E, \alpha) = F(E)$ for an object (E, α) and $\psi(h : (E, \alpha) \to (E', \alpha')) = F(h) : F(E) \to F(E')$ for an arrow $h : (E, \alpha) \to (E', \alpha')$ of $\mathcal{C}^T$. Then it can be easily checked that ψ is a functor.

Now we prove that $\psi \dashv \phi$ i.e. ψ is left adjoint to ϕ. For this firstly, we define the unit of adjunction $\nu : 1_{\mathcal{C}^T} \Rightarrow \phi \circ \psi$. Therefore, for an object (E, α) of $\mathcal{C}^T$

$$\phi \circ \psi(E, \alpha) = \phi(F(E)) = (GF(E), G(\epsilon_{F(E)})) = (T(E), \mu_E)$$

and so define $\nu_{(E,\alpha)} : (E, \alpha) \to \phi \circ \psi(E, \alpha)$ i.e. $\nu_{(E,\alpha)} : (E, \alpha) \to (T(E), \mu_E)$ by $\nu_{(E,\alpha)} = \eta_E$, since the following square is commutative:

$$\begin{array}{ccc} TE & \xrightarrow{T(\eta_E)} & T^2E \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\mu_E} \\ E & \xrightarrow[\eta_E]{} & TE \end{array}$$

using the fact that $\mu_E \circ T(\eta_E) = 1_{TE}$ and $\alpha : TE \to E$, $\eta_E : E \to TE$ are inverses of each other. Also $\eta : 1_{\mathcal{C}} \Rightarrow T$ implies that the following square is commutative:

$$\begin{array}{ccc} (E,\alpha) & \xrightarrow{\eta_E} & (TE,\mu_E) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \phi\circ\psi(h)=Th} \\ (K,\beta) & \xrightarrow[\eta_K]{} & (TK,\mu_K) \end{array}$$

and so $\nu : 1_{\mathcal{C}^T} \Rightarrow \phi \circ \psi$ is a natural transformation. Lastly, we need to prove the UMP of adjunction. Therefore, for any two objects (E,α) and D of $\mathcal{C}^T$ and $\mathcal{D}$ respectively and an arrow $f : (E,\alpha) \to \phi(D)$ i.e. $f : (E,\alpha) \to (GD, G(\epsilon_D))$ and so $f : E \to GD$, UMP of adjunction $F \dashv G$ implies that there exists a unique arrow $g : FE \to D$ and so $g : \psi(E,\alpha) \to D$ such that

$$\begin{array}{ccc} (E,\alpha) & & \\ {\scriptstyle \eta_{(E,\alpha)}}\downarrow & \searrow^{f} & \\ \phi\circ\psi(E,\alpha) & \xrightarrow[\phi(g)]{} & \phi(D) \end{array}$$

is commutative if and only if

$$\begin{array}{ccc} (E,\alpha) & & \\ {\scriptstyle \eta_E}\downarrow & \searrow^{f} & \\ (TE,\mu_E) & \xrightarrow[G(g)]{} & (G(D), G(\epsilon_D)) \end{array}$$

is commutative if and only if

$$\begin{array}{ccc} E & & \\ {\scriptstyle \eta_E}\downarrow & \searrow^{f} & \\ TE & \xrightarrow[G(g)]{} & G(D) \end{array}$$

is commutative and which is true by UMP of adjunction $F \dashv G$. Hence the UMP of adjunction $\psi \dashv \phi$ holds. □

Corollary 2.1 *For the adjunction $f(-) \dashv f^{-1}$, the comparison functor $\phi : Sub_{\mathcal{C}}[Z] \to Sub_{\mathcal{C}}[Y]^T$ has a left adjoint.*

Proof: Proof follows from Theorem 2.10 and from the fact that for an object $([m],\beta)$ of $Sub_{\mathcal{C}}[Y]^T$, $T[m] = [m]$ and so β and $\eta_{[m]}$ are inverses of each other. □

Conclusion

The most modern view of mathematics requires us to study the structures in mathematics in their most abstract form like functor adjunctions, monads and categorical operators etc. Monad (alias triple) is a fundamental tool in category theory. Every adjunction determines a monad. Conversely, every monad can be constructed from an adjunction in at least two different ways: the Kleisli and Eilenberg-Moore categories of a monad yield two adjunctions that induce the same monad. Within the category of all adjunctions inducing the same monad, the Kleisli category is an initial object whereas Eilenberg-Moore category is a terminal object. Monads, the Kleisli and Eilenberg-Moore categories have received an increasing attention in topology among many other fields including automata, fuzzy automata, programming language semantics, quantum computation and quantaloid enriched categories. Categorical closure operators in particular, provide a “unifying view of topological and discrete structure”. Convergence studied via either Kuratowski operator or Čech closure operator or through any other such operators is a specific example of the application of categorical closure operators. In fact, earlier work in the subject of Categorical Topology was primarily concerned with the categorical closure operators. Concepts like the extent of disconnectedness of a space can be viewed through "weakly hereditary closure operators". The next target is to study monadic topology and to extend convergence structures to a categorical setting using partially ordered monads on fuzzy sets. Also characterizations of various types of continuous, open and closed maps and strong and weak form of these mappings can be studied using the concept of adjunction between the induced functors $f(-)$, f^{-1} and $f^{\#}$.

Funding
No funding was received for conducting this study.

Compliance with ethical standards
Conflict of interest The author declares that they have no conflict of interest.

Authors' contributions
The author had the idea for the article, performed the literature search and drafted the work.

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