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The notes on the special tube surfaces generated by normal curves in E_2^4

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ABSTRACT: In this work, the special tube surfaces generated by normal curves with Frenet frame in E_2^4 are examined and some certain results of describing the surface characterizations on the surfaces are presented in detail. Moreover, using the Gaussian curvatures and mean curvatures of tube surfaces with normal curve generated by Frenet frame in E_2^4, the conditions being Weingarten surface and HK-quadric surface, minimal surface, flat, ... etc are examined in detail.

Key Words: Semi-Euclidean Space E_2^4, tube surfaces, weingarten surface, normal curve.

Contents

1. Introduction

Today, many studies have been done that deal with canal surfaces from algebraic and geometric aspects. The canal surfaces are parameterized with the help of the Frenet frame of the central curve of the spheres that form it. The canal surface is defined as an envelope of a variable radius moving sphere. The studies on the canal surface show that if the orbit of the centers of these spheres has a helix and radius function constant, the resulting surface is the helical canal surface, if the orbit of the centers of the spheres is straight line, the resulting surface becomes a rotating surface. If the radius function of the moving sphere forming the canal surface is constant, the canal surface is called the tube surface. Also, a surface is called a Weingarten if there is a smooth relation $U(k_1, k_2) = 0$ between two principle curvatures k_1 and k_2. If K and H denote the Gauss curvature and the mean curvatures, respectively, then $U(k_1, k_2) = 0$ implies a relation as $\Phi(K, H) = 0$. The existence of a non-trivial functional relation $\Phi(K, H) = 0$ on a surface, which is parameterized by a patch $x(w, v)$, is equivalent to the following Jacobian determinant,

$$\frac{\partial(K, H)}{\partial(w, v)} = 0. \tag{1.1}$$

Furthermore, if the equations $U = a_1k_1 + a_2k_2 - a_3$ or $\Phi = a_1H + a_2K - a_3$ hold, the surfaces are called linear Weingarten surfaces, where $a_i, i \in \mathbb{R}$ with $a_1^2 + a_2^2 \neq 0$. Also, if a surface satisfies the following equation

$$a_1H^2 + 2a_2HK + a_3K^2 = \text{constant}, a_1 \neq 0, \tag{1.2}$$

then the surface is said to be a $HK-$quadric surface, [12].

It is well known that a curve is a normal curve in E^3 if for all s the orthogonal complement of $T(s)$ contains a fixed point, in [14], the author gave some characterizations of space-like normal curves with space-like, time-like or null principal normal in the Minkowski 3-space E_1^3. The curves for which the

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position vector always lie in their normal plane, are for simplicity called normal curves. By definition for a normal curve, the position vector x satisfies

$$x(s) = \lambda N(s) + \mu B(s), s \in I \subset \mathbb{R},$$

for some differentiable functions λ, μ, [14]. In [5], the author defined characterizations of semi-real quaternionic Bertrand curves in the four dimensional space E_2^4 and he studied the Serret-Frenet formulae of the curve in E_2^4 and then applying quaternionic Bertrand curves. In [6], authors obtained explicit parameter equations of spacelike rectifying curves and normal curve whose projection onto spacelike, timelike and lightlike plane of Minkowski 3-space. In [7], they gave the necessary and sufficient conditions for non-null curves with non-null normal in 4-dimensional Semi-Euclidean space with index 2 to be osculating curves. In [10,11], the authors studied spacelike curves with osculating curves in null cone space. In papers [1,2,3,4], some mathematical characterizations of surfaces in Galilean space were studied by the authors.

2. Preliminaries

Let E_2^4 denote the $4-$dimensional pseudo-Euclidean space with signature $(2,4)$, that is, the real vector space $\mathbb{R}^4$ endowed with the metric $\langle , \rangle_{E_2^4}$ which is defined by

$$\langle , \rangle_{E_2^4} = -\sum_{i=1}^{2} (dx_i)^2 + \sum_{i=3}^{4} (dx_i)^2 , \tag{2.1}$$

or

$$\langle , \rangle_{E_2^4} = -dx_1^2 - dx_2^2 + dx_3^2 + dx_4^2,$$

where (x_1, x_2, x_3, x_4) is a standard rectangular coordinate system in E_2^4.

Recall that an arbitrary vector $v \in E_2^4 \backslash \{0\}$ can have one of three characters: it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null if $g(v,v) = 0$ and $v \neq 0$.

The norm of a vector v is given by $\| v \| = \sqrt{g(v,v)}$ and two vectors v and w are said to be orthogonal if $g(v,w) = 0$. An arbitrary curve $x(s)$ in E_2^4 can locally be space-like, time-like or null.

A space-like or time-like curve $x(s)$ has unit speed, if $g(x', x') = \pm 1$.

Let $(x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4)$ be any three vectors in E_2^4. The pseudo Euclidean cross product is given as

$$x \wedge y \wedge z = \begin{pmatrix} -i_1 & -i_2 & i_3 & i_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}, \tag{2.2}$$

where $i_1 = (1,0,0,0)\,, i_2 = (0,1,0,0)\,, i_3 = (0,0,1,0)\,, i_4 = (0,0,0,1)$, [13].

The pseudo-Riemannian sphere $S_2^3(m,r)$ centered at $m \in E_2^4$, with radius $r > 0$ of E_2^4 is defined by

$$S_2^3(m,r) = \left\{ x \in E_2^4 : \langle x - m, x - m \rangle = r^2 \right\}.$$

The pseudo-hyperbolic space $H_1^3(m,r)$ centered at $m \in E_2^4$, with radius $r > 0$ of E_2^4 is defined by

$$H_1^3(m,r) = \left\{ x \in E_2^4 : \langle x - m, x - m \rangle = -r^2 \right\}.$$

The pseudo-Riemannian sphere $S_2^3(m,r)$ is diffeomorfic to $\mathbb{R}^2 \times S$ and the pseudo-hyperbolic space $H_1^3(m,r)$ is diffeomorfic to $S^1 \times \mathbb{R}^2$. The hyperbolic space $H^3(m,r)$ is given by

$$H^3(m,r) = \left\{ x \in E_2^4 : \langle x - m, x - m \rangle = -r^2, x_1 > 0 \right\}.$$

Let $\Psi : M \to E_2^4$ be an isometric immersion of an oriented pseudo-Riemannian submanifold M into E_2^4. Henceforth, a submanifold in E_2^4 always means pseudo-Riemannian. Let $\bar{\nabla}$ be the Levi-Civita connection of E_2^4 and ∇ be the induced connection on M. Also, for any vector fields X, Y tangent to M, we get the Gaussian formula

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.3}$$

where h is the second fundamental form which is symmetric in X and Y. For a unit normal vector field ξ, the Weingarten formula is defined by

$$\bar{\nabla}_X \xi = -A_\xi X + D_\xi X, \tag{2.4}$$

where A_ξ is the Weingarten map or the shape operator with respect to ξ. The Weingarten map A_ξ is a self-adjoint endomorphism of TM which cannot be diagonalized generally. It is known that, h and A_ξ are related by

$$\langle h(X, Y), \xi \rangle = \langle A_\xi X, Y \rangle . \tag{2.5}$$

The covariant derivative $\tilde{\nabla} h$ of the second fundamental form h is given by

$$\tilde{\nabla}_X h\left(Y, Z\right) = \nabla_X^{\perp} h\left(Y, Z\right) - h\left(\nabla_X Y, Z\right) - h\left(Y, \nabla_X Z\right), \tag{2.6}$$

where $\nabla^{\perp}$ indicates the linear connection induced on the normal bundle $T^{\perp}M$. Also, Codazzi equation is given by

$$\tilde{\nabla}_X h\left(Y, Z\right) = \tilde{\nabla}_Y h\left(X, Z\right). \tag{2.7}$$

Let $e_1, e_2, ..., e_m$ be a local orthonormal frame field in E_s^m such that $e_1, e_2, ..., e_n$ are tangent to M^n and $\{e_{n+1}, ..., e_m\}$ are normal to M^n. Let $w_1, w_2, ..., w_m$ be the co-frame of $e_1, e_2, ..., e_m$. We'll make use of the following convention on the ranges of indices $1 \leq i, j, ... \leq n, n+1 \leq s, t, ... \leq 4, 1 \leq A, B, ... \leq 4$. Also, $w_A\left(e_B\right) = \delta_{AB}$ and the pseudo-Riemannian metric on E_s^m is given by

$$ds^2 = \sum_i^n \varepsilon_A w_A^2; \varepsilon_A = \langle e_A, e_A \rangle = \pm 1. \tag{2.8}$$

Let w_A be the dual 1-form of e_A defined by $w_A X = \langle e_A, X \rangle$. Also, the connection forms w_{AB} are defined by

$$de_A = \sum \varepsilon_B w_{AB} e_B; w_{AB} + w_{BA} = 0. \tag{2.9}$$

After, the structure equations of E_2^4 are written as follows

$$dw_A = \sum_B \varepsilon_B w_{AB} \wedge w_B; dw_A = \sum_C \varepsilon_C w_{AC} \wedge w_{CB}. \tag{2.10}$$

The canonical forms $\{w_A\}$ and the connection forms $\{w_{AB}\}$ restricted to M^n are also indicated by the same symbols. Also, we get

$$w_s = 0, s = n+1, ..., 4$$

and since w_s are zero forms on M^n, there are symmetric tensor h_{ij}^s by Cartan's lemma such

$$w_{is} = \sum_j \varepsilon_j h_{ij}^s w_j; h_{ij}^s = h_{ji}^s. \tag{2.11}$$

The mean curvature vector H of M^n in E_s^m is given by

$$H = \frac{1}{2} \sum_{s=n+1}^{m} \sum_{i=1}^{n} \varepsilon_j \varepsilon_s h_{ij}^s e_s. \tag{2.12}$$

Also, the covariant differentiation of e_i is given by

$$de_i = \sum_A \varepsilon_A w_{iA} e_A \text{ or } \bar{\nabla}_{e_i} e_j = \sum_B \varepsilon_B w_{jB}\left(e_i\right) e_B.$$

Let denote by E, F, G the coefficients of the first fundamental form of M^n. If $\Psi(u,v)$ is a smooth function, the second differential parameter of Laplacian of a function $\Psi(u,v)$ with respect to the first fundamental form of M^n is the operator Δ which is defined by

$$\Delta\Psi = -\frac{1}{\sqrt{|EF-G^2|}}\left[\left(\frac{G\Psi_u - F\Psi_v}{\sqrt{|EF-G^2|}}\right)_u - \left(\frac{F\Psi_u - E\Psi_v}{\sqrt{|EF-G^2|}}\right)_v\right]$$

[13].

Let $\{T, N, B_1, B_2\}$ be the non-null moving Frenet frame along a unit speed non-null curve β in E_2^4, consisting of the tangent, principal normal, first binormal and second binormal vector field, respectively. If β is a non-null curve with non-null vector fields, then $\{T, N, B_1, B_2\}$ is a orthonormal frame and the Frenet equations are given as

$$\begin{aligned} T' &= k_1 N \\ N' &= -\epsilon_0\epsilon_1 k_1 T + k_2 B_1 \\ B_1' &= -\epsilon_1\epsilon_2 k_2 N + k_3 B_2 \\ B_2' &= -\epsilon_2\epsilon_3 k_3 B_1, \end{aligned} \tag{2.13}$$

where the following conditions are satisfied

$$\begin{aligned} g(T,N) &= g(T,B_1) = g(T,B_2) = g(N,B_2) = g(N,B_1) = g(B_1,B_2) = 0 \\ g(T,T) &= \epsilon_0, g(N,N) = \epsilon_1, g(B_1,B_1) = \epsilon_{2,} g(B_2,B_2) = \epsilon_3 \\ \epsilon_i &\in \{-1,1\}, i \in I = \{0,1,2,3\}, \end{aligned}$$

[8].

Let β be a non-null curve in E_2^4. We define that β is the normal curve in E_2^4, if its position vector with respect to some chosen origin always lies in the orthogonal complement $T^\perp$. The orthogonal complement $T^\perp$ is non-degenerate hyperplanes of E_2^4, spanned by $\{N, B_1, B_2\}$. By definition, for a normal curve in E_2^4, the position vector β satisfies

$$\beta(s) = \mu(s)N(s) + \gamma(s)B_1(s) + \theta(s)B_2(s) \text{ or } g(\beta, T) = 0, \tag{2.14}$$

for $\mu, \gamma, \theta \in C^\infty$ of $s \in I \subset \mathbb{R}$.

Proposition 2.1 *Laplacian function of the differentiable function given by $g : M \subset \mathbb{R}^2 \to \mathbb{R}$ is defined as*

$$\Delta g = \left(\frac{d^2 g}{du^2}\right) + \left(\frac{d^2 g}{dv^2}\right), (u,v) \in M. \tag{2.15}$$

If $\Delta g = 0$, the function g is harmonic in M, [9].

3. Tube Surfaces in E_2^4

In this section, we will obtain some characterizations of the tube surface generated by the orthonormal frame for the center normal curve of $\beta(s)$ in E_2^4. First of all, we will obtain the tube surface and thus the tubular surface formed by the center normal curve $\beta(s)$ obtained using the Serret-Frenet frame in E_2^4, then it will be flat, minimal surface, (K,H) -Weingarten surface and KH-quadric surface on these surfaces we will examine the conditions.

A canal surface is expressed as the envelope of a setting out sphere with exchanging radius, which is described by the orbit $\beta(w(s))$ (spine curve) with its center and a radius function ρ in addition to its parametrized through Frenet frame of the spine curve $\beta(w(s))$. If the radius function ρ is a constant, then the canal surface is called as a tube or tubular surface. Let one denotes by ρ the vector connecting the point from the arc-length parametrized curve $\beta(w(s))$ with the point from the surface and since ρ lies in the Euclidean normal plane of the curve $\beta(w(s))$, the points at a distance A_1 to a point of $\beta(w(s))$

form an Euclidean circle in G_3, clearly, one can have the position vector R of a point on the surface in the following from

$$R = \beta\,(w(s)) + \rho \Rightarrow \rho = A_1(\cos v_1 \vec{n} + \sin v_1 \vec{b}), \tag{3.1}$$

where v_1 is the Euclidean angle between the isotropic vectors; $\vec{n}$ and $\vec{\rho}$ lie in the Euclidean normal plane of the curve $\beta\,(w(s))$.

Theorem 3.1 *Let β be a unit speed normal curve for each $s \in I \subset \mathbb{R}$ and for $k_1, k_2, k_3 \neq 0$ in E_2^4. Then, the principal normal, the first binormal and the second binormal components of the position vector of the curve are given respectively by*

$$g(\beta, N) = \frac{-1}{\epsilon_0\epsilon_1 k_1}; g(\beta, B_1) = \frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2}; g(\beta, B_2) = -\int \frac{k_1' k_3}{\epsilon_0\epsilon_2 k_1^2 k_2} ds.$$

In this case, the vector equation is given as

$$\beta(s) = \left(\frac{-1}{\epsilon_0\epsilon_1 k_1}\right) N(s) + \left(\frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2}\right) B_1(s) - \left(\int \frac{k_1' k_3}{\epsilon_0\epsilon_2 k_1^2 k_2} ds\right) B_2(s). \tag{3.2}$$

Proof: Let β be a unit speed non-null normal curve in E_2^4, with non zero k_1, k_2, k_3. From definition, for the position vector of the curve β using the Frenet equations (2.13) and the equation (2.14), one gets

$$T = (-\epsilon_0\epsilon_1 k_1 \mu)T + \left(\begin{array}{c} \mu' \\ -\gamma\epsilon_1\epsilon_2 k_2 \end{array}\right) N + \left(\begin{array}{c} k_2\mu + \gamma' \\ -\epsilon_3\epsilon_2 k_3 \theta \end{array}\right) B_1 + \left(\begin{array}{c} k_3\gamma \\ +\theta' \end{array}\right) B_2. \tag{3.3}$$

Using equation (3.3), one can write

$$\begin{aligned} -\epsilon_0\epsilon_1 k_1 \mu &= 1 \\ (\mu' - \gamma\epsilon_1\epsilon_2 k_2)\,\epsilon_1 &= 0 \\ (k_2\mu + \gamma' - \epsilon_3\epsilon_2 k_3\theta)\,\epsilon_2 &= 0 \\ (k_3\gamma + \theta')\epsilon_3 &= 0. \end{aligned} \tag{3.4}$$

Therefore, from the first equation, one gets

$$\mu = \frac{-1}{\epsilon_0\epsilon_1 k_1}. \tag{3.5}$$

If this last equation is used in equation $(\mu' - \gamma\epsilon_1\epsilon_2 k_2)\,\epsilon_1 = 0$, one gets

$$\gamma = \frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2}. \tag{3.6}$$

Finally, by considering the value of γ in the equation $(k_3\gamma + \theta')\epsilon_3 = 0$, one has

$$\theta = \frac{-1}{\epsilon_0\epsilon_2} \int \frac{k_1' k_3}{k_1^2 k_2} ds. \tag{3.7}$$

Finally, using (2.14) and (3.5), (3.6), (3.7) we easily obtain (3.2). □

3.1. Representation of the tube surfaces generated by normal curve in E_2^4

Let us assume that Ω is an envelope of parameterizing the spheres that define canal surfaces. In this case, Let β be a curve formed by the centers of the Galilean spheres with $k_1, k_2, k_3, \neq 0$ and for $\Sigma^1, \Sigma^2, \Sigma^3, \Sigma^4 \in C^\infty$, and let the radius of the canal surface be the function d. Hence, using the Frenet-Serret frame $\{T, N, B_1, B_2\}$, we can write

$$\Omega(s,\xi) - \beta(s) = \Sigma^1(s,\xi)\overrightarrow{T} + \Sigma^2(s,\xi)\overrightarrow{N} + \Sigma^3(s,\xi)\overrightarrow{B_1} + \Sigma^4(s,\xi)\overrightarrow{B_2}. \tag{3.8}$$

Furthermore, for $x_i \in C^\infty, i \in \{1,2,3,4\}$ and for the vector $x(s)$ in E_2^4, we can write

$$\langle x(s), x(s)\rangle_{E_2^4} = -x_1^2(s) - x_2^2(s) + x_3^2(s) + x_4^2(s). \tag{3.9}$$

Thus, if the vector $\Omega(s,\xi) - \beta(s) = \left(\Sigma^1(s,\xi), \Sigma^2(s,\xi), \Sigma^3(s,\xi), \Sigma^4(s,\xi)\right)$ given in (3.8) is taken into account in (3.9), we write the following equation

$$\langle x(s), x(s)\rangle_{E_2^4} = -x_1^2(s) - x_2^2(s) + x_3^2(s) + x_4^2(s) = d^2(s). \tag{3.10}$$

Furthermore, from (3.10), we can say that the surface $\Omega(s,\xi)$ is on the sphere with an radius $d(s)$ centered at $\beta(s)$. Hence, mathematical equations between the vector $\Omega(s,\xi) - \beta(s)$ being normal to the canal surface and the vectors Ω_s, Ω_ξ which are tangent to the sphere on the surface, are given by

$$\langle \Omega(s,\xi) - \beta(s), \Omega_s\rangle = 0; \langle \Omega(s,\xi) - \beta(s), \Omega_\xi\rangle = 0. \tag{3.11}$$

In this case, using the metric given in (3.8) and (3.10), we write

$$-\left(\Sigma^1\right)^2 - \left(\Sigma^2\right)^2 + \left(\Sigma^3\right)^2 + \left(\Sigma^4\right)^2 = d^2,$$

and derivativing with respect to the parameter s in this last expression, we get

$$-\Sigma_s^1\Sigma^1 - \Sigma_s^2\Sigma^2 + \Sigma_s^3\Sigma^3 + \Sigma_s^4\Sigma^4 = dd_s. \tag{3.12}$$

Hence, differentiating with respect to ξ parameter and using Serret-Frenet frame in (3.8), one gets

$$\Omega_\xi = \Sigma_\xi^1(s,\xi)\overrightarrow{T} + \Sigma_\xi^2(s,\xi)\overrightarrow{N} + \Sigma_\xi^3(s,\xi)\overrightarrow{B_1} + \Sigma_\xi^4(s,\xi)\overrightarrow{B_2}. \tag{3.13}$$

Also, If the expression (3.8) and the last equation are used in the equation $\langle \Omega(s,\xi) - \beta(s), \Omega_\xi\rangle_{E_2^4} = 0$, one has

$$-\Sigma^1\Sigma_\xi^1 - \Sigma^2\Sigma_\xi^2 + \Sigma^3\Sigma_\xi^3 + \Sigma^4\Sigma_\xi^4 = 0. \tag{3.14}$$

Also, let's try to express the canal surface given in E_2^4 in a different way by finding the values of Σ^2, Σ^3 with the help of equations (2.13) and (3.11). Thus, we can write

$$\begin{aligned}
\Omega(s,\xi) - \beta(s) &= \Sigma^1(s,\xi)\overrightarrow{T} + \Sigma^2(s,\xi)\overrightarrow{N} + \Sigma^3(s,\xi)\overrightarrow{B_1} + \Sigma^4(s,\xi)\overrightarrow{B_2} \\
\Omega_s(s,\xi) - \beta'(s) &= \Sigma_s^1\overrightarrow{T} + \Sigma^1\overrightarrow{T'} + \Sigma_s^2\overrightarrow{N} + \Sigma^2\overrightarrow{N'} + \Sigma_s^3\overrightarrow{B_1} + \Sigma^3\overrightarrow{B_1'} + \Sigma_s^4\overrightarrow{B_2} + \Sigma^4\overrightarrow{B_2'} \\
&= \left(\Sigma_s^1 + 1\right)\overrightarrow{T} + \Sigma^1 k_1\overrightarrow{N} + \Sigma_s^2\overrightarrow{N} + \Sigma^2\left(-\epsilon_0\epsilon_1 k_1\overrightarrow{T} + k_2\overrightarrow{B_1}\right) \\
&\quad + \Sigma_s^3\overrightarrow{B_1} + \Sigma^3\left(-\epsilon_1\epsilon_2 k_2\overrightarrow{N} + k_3\overrightarrow{B_2}\right) + \Sigma_s^4\overrightarrow{B_2} + \Sigma^4\left(-\epsilon_2\epsilon_3 k_3\overrightarrow{B_1}\right)
\end{aligned}$$

and hence, we get

$$\begin{aligned}
\Omega_s(s,\xi) = &\left(\Sigma_s^1 + 1 - \Sigma^2\epsilon_0\epsilon_1 k_1\right)\overrightarrow{T} + \left(\Sigma^1 k_1 + \Sigma_s^2 - \Sigma^3\epsilon_1\epsilon_2 k_2\right)\overrightarrow{N} \\
&+ \left(\Sigma_s^3 + \Sigma^2 k_2 - \Sigma^4\epsilon_2\epsilon_3 k_3\right)\overrightarrow{B_1} + \left(\Sigma^3 k_3 + \Sigma_s^4\right)\overrightarrow{B_2}.
\end{aligned} \tag{3.15}$$

Therefore, using following equation

$$\Rightarrow \langle \Omega(s,\xi) - \alpha(s), \Omega_s \rangle = 0,$$

we can obtain

$$\begin{aligned} 0 &= -\Sigma^1 \left(\Sigma^1_s + 1 - \Sigma^2 \epsilon_0 \epsilon_1 k_1\right) - \Sigma^2 \left(\Sigma^1 k_1 + \Sigma^2_s - \Sigma^3 \epsilon_1 \epsilon_2 k_2\right) \\ &+ \Sigma^3 \left(\Sigma^3_s + \Sigma^2 k_2 - \Sigma^4 \epsilon_2 \epsilon_3 k_3\right) + \Sigma^4 \left(\Sigma^3 k_3 + \Sigma^4_s\right) \end{aligned} \tag{3.16}$$

and considering this algebraic equation, we can write

$$\begin{aligned} \Sigma^1_s + 1 - \Sigma^2 \epsilon_0 \epsilon_1 k_1 &= 0 \\ \Sigma^1 k_1 + \Sigma^2_s - \Sigma^3 \epsilon_1 \epsilon_2 k_2 &= 0 \\ \Sigma^3_s + \Sigma^2 k_2 - \Sigma^4 \epsilon_2 \epsilon_3 k_3 &= 0 \\ \Sigma^3 k_3 + \Sigma^4_s &= 0. \end{aligned} \tag{3.17}$$

Also, from (3.16), we may write

$$\begin{aligned} 0 \;=\; & -\Sigma^1\Sigma^1_s - \Sigma^1 + \Sigma^1\Sigma^2\epsilon_0\epsilon_1 k_1 - \Sigma^2\Sigma^1 k_1 - \Sigma^2\Sigma^2_s + \Sigma^2\Sigma^3\epsilon_1\epsilon_2 k_2 \\ & -\Sigma^3\Sigma^3_s - \Sigma^3\Sigma^2 k_2 - \Sigma^3\Sigma^4\epsilon_2\epsilon_3 k_3 + \Sigma^4\Sigma^3 k_3 + \Sigma^4\Sigma^4_s \end{aligned}$$

$$\begin{aligned} 0 \;=\; & -\Sigma^1\Sigma^1_s - \Sigma^2\Sigma^2_s + \Sigma^3\Sigma^3_s + \Sigma^4\Sigma^4_s - \Sigma^1 + \Sigma^1\Sigma^2\epsilon_1\epsilon_0 k_1 - \Sigma^2\Sigma^1 k_1 + \Sigma^2\Sigma^3\epsilon_2\epsilon_1 k_2 \\ & -\Sigma^3\Sigma^2 k_2 - \Sigma^3\Sigma^4\epsilon_3\epsilon_2 k_3 + \Sigma^4\Sigma^3 k_3. \end{aligned}$$

Thus, with the help of the equation (3.12), ϵ_0, ϵ_1 values are considered to be time-like and ϵ_2, ϵ_3 values are considered to be space-like, we can write

$$0 = -\Sigma^1 - 2\Sigma^2\Sigma^3 k_2 - \Sigma^1\Sigma^1_s - \Sigma^2\Sigma^2_s + \Sigma^3\Sigma^3_s + \Sigma^4\Sigma^4_s \tag{3.18}$$

$$\Rightarrow -\Sigma^1 - 2\Sigma^2\Sigma^3 k_2 + dd_s = 0 \Rightarrow \Sigma^1 = -2\Sigma^2\Sigma^3 k_2 + dd_s.$$

Therefore, the following equation can be written

$$\begin{aligned} -\left(\Sigma^1\right)^2 - \left(\Sigma^2\right)^2 + \left(\Sigma^3\right)^2 + \left(\Sigma^4\right)^2 &= d^2 \\ \left(-2\Sigma^2\Sigma^3 k_2 + dd_s\right)^2 - \left(\Sigma^2\right)^2 + \left(\Sigma^3\right)^2 + \left(\Sigma^4\right)^2 &= d^2 \\ \left(\Sigma^3\right)^2 + \left(\Sigma^4\right)^2 - \left(\Sigma^2\right)^2 &= d^2 - \left(-2\Sigma^2\Sigma^3 k_2 + dd_s\right)^2 \\ \left(d\cos\theta\cosh\xi\right)^2 + \left(d\sin\theta\cosh\xi\right)^2 - \left(d\sinh\xi\right)^2 &= d^2 - \left(-2\Sigma^2\Sigma^3 k_2 + dd_s\right)^2 \\ \left(d\cosh\xi\right)^2 - \left(d\sinh\xi\right)^2 &= d^2 - \left(-2\Sigma^2\Sigma^3 k_2 + dd_s\right)^2 \end{aligned}$$

Considering the equation $-2\Sigma^2\Sigma^3 k_2 + dd_s = 0$ in the last equation, one writes

$$d = \sqrt{c + 4\int \Sigma^2\Sigma^3 k_2 ds}; c \in IR \tag{3.19}$$

Hence,

$$\Sigma^2 = d\sinh\xi; \Sigma^3 = d\cosh\xi\cos\theta; \Sigma^4 = d\cosh\xi\sin\theta; \Sigma^1 = -2\Sigma^2\Sigma^3 k_2 + dd_s;$$

$$d = \sqrt{c + 4\int \Sigma^2\Sigma^3 k_2 ds}; c \in \mathbb{R} k_2 d\cos\theta\sinh 2\xi = d_s; d = \text{sabit ise } k_2 = 0 \tag{3.20}$$

previous values are written in the equation (3.8), and the canal surface Ω can be written as

$$\Omega(s,\xi)=\beta(s)+\left(-2\Sigma^{2}\Sigma^{3}k_{2}+dd_{s}\right)\overrightarrow{T}+d\begin{pmatrix}\sinh\xi\overrightarrow{N}\\+\cosh\xi\cos\theta\overrightarrow{B_1}\\+\cosh\xi\sin\theta\overrightarrow{B_2}\end{pmatrix}. \tag{3.21}$$

In the last equation, the surface obtained in condition to $d=\sqrt{c+4\int\Sigma^{2}\Sigma^{3}k_{2}ds}; c\in\mathbb{R}$ is as follows

$$\Omega(s,\xi)=\beta(s)+d\left(s\right)\left(\sinh\xi\overrightarrow{N}+\cosh\xi\cos\theta\overrightarrow{B_1}+\cosh\xi\sin\theta\overrightarrow{B_2}\right). \tag{3.22}$$

Hence, the following theorem is given.

Theorem 3.2 *Let the center curve of the tube surface in E_2^4 be a unit speed curve $\beta:I\to E_2^4$ with the curvatures k_1,k_2 and $k_3\neq 0$. Then, the tube surface in E_2^4 is parametrized as follows*

$$\Omega(s,\xi)=\beta(s)+\left(-2\Sigma^{2}\Sigma^{3}k_{2}+dd_{s}\right)\overrightarrow{T}+\begin{pmatrix}\sinh\xi\overrightarrow{N}+\cosh\xi\cos\theta\overrightarrow{B_1}\\+\cosh\xi\sin\theta\overrightarrow{B_2}\end{pmatrix}$$

or

$$\begin{aligned}\Omega(s,\xi)&=\beta(s)+\left(-\cos\theta\sinh 2\xi k_{2}+dd_{s}\right)\overrightarrow{T}\\&\quad+d\left(\sinh\xi\overrightarrow{N}+\cosh\xi\cos\theta\overrightarrow{B_1}+\cosh\xi\sin\theta\overrightarrow{B_2}\right)\end{aligned}$$

and

$$\Omega(s,\xi)=\beta(s)+d\left(\sinh\xi\overrightarrow{N}+\cosh\xi\cos\theta\overrightarrow{B_1}+\cosh\xi\sin\theta\overrightarrow{B_2}\right); k_{2}=0$$

where $d=e^{-\cos\theta\sinh 2\xi\int k_{2}ds+c}; c\in\mathbb{R}$, θ =constant and where T,N and B_1,B_2 denote the tangent, principal normal and the first binormal, the second binormal of the curve β.

Theorem 3.3 *Let β be a unit speed non-null normal curve and the center curve of the canal surface in E_2^4. Then, the following expressions are provided for the canal surface defined in E_2^4.*

1) The canal surface Ω generated by the normal curve β is given by

$$\Omega(s,\xi)=\begin{pmatrix}\frac{-1}{\epsilon_0\epsilon_1k_1}\\+d\sinh\xi\end{pmatrix}\overrightarrow{N}+\begin{pmatrix}\frac{k_1'}{\epsilon_0\epsilon_2k_1^2k_2}\\+d\cosh\xi\cos\theta\end{pmatrix}\overrightarrow{B_1}+\begin{pmatrix}-\int\frac{k_1'k_3}{\epsilon_0\epsilon_2k_1^2k_2}ds\\+d\cosh\xi\sin\theta\end{pmatrix}\overrightarrow{B_2}.$$

2) The mean curvature vector and the mean curvature of tube surface Ω are given as

$$H=\frac{1}{2}\left\{\begin{array}{c}k_3\left(-d\sinh\xi+L_1''\right)\\+\epsilon_2\epsilon_1k_2\left(-d\cosh\xi\sin\theta+L_3''\right)\end{array}\right\}\left(0,k_3,0,-\epsilon_2\epsilon_1k_2\right);$$

$$H=\frac{1}{2}\left\{k_3\begin{pmatrix}-d\sinh\xi\\+\frac{1}{\epsilon_0\epsilon_1}\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right)\end{pmatrix}+\epsilon_2\epsilon_1k_2\begin{pmatrix}-d\cosh\xi\sin\theta\\+\frac{1}{\epsilon_0\epsilon_2}\frac{d}{ds}\left(\frac{k_1'k_3}{k_1^2k_2}\right)\end{pmatrix}\right\}.$$

3) The Gaussian curvature of the surface Ω is given as

$$K=-d\left(\frac{k_3}{\epsilon_0\epsilon_1}\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right)+\frac{k_2\epsilon_1}{\epsilon_0}\frac{d}{ds}\left(\frac{k_1'k_3}{k_1^2k_2}\right)\right)\begin{pmatrix}k_3\sinh\xi\\+\epsilon_2\epsilon_1k_2\cosh\xi\sin\theta\end{pmatrix},$$

where for $L_1=\frac{-1}{\epsilon_0\epsilon_1k_1}, L_2=\frac{k_1'}{\epsilon_0\epsilon_2k_1^2k_2}, L_3=-\int\frac{k_1'k_3}{\epsilon_0\epsilon_2k_1^2k_2}ds$, the following equations are satisfied

$$\xi=\operatorname{arccot}h\left\{-\frac{\epsilon_1}{\epsilon_2}\left(\frac{k_1''k_1k_2-2k_1'^2k_2-k_1'k_1k_2'}{k_1k_2^2k_1'}\cos\theta+\frac{k_3}{k_2}\sin\theta\right)\right\}$$

and

$$\begin{aligned} k_1 &= \frac{2}{\epsilon_0\epsilon_1 d \sinh\xi}; \int k_2 ds = \frac{-c_1}{\cos\theta \sinh 2\xi}; \\ 0 &= k_1' - c_2\epsilon_0\epsilon_2 k_1^2 k_2 e^{\int\left(\epsilon_2\epsilon_1 k_2 \frac{\tanh\xi}{\cos\theta} + k_3\tan\theta\right)ds} \\ k_3^2 - k_2^2 &= 1; \frac{k_3}{k_2} = -\epsilon_2\epsilon_1 \tanh\xi \sin\theta \ . \end{aligned}$$

4) The canal surface Ω *is minimal* $\Leftrightarrow$ *the following equation is satisfied*

$$d = \frac{k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3''}{k_3 \sinh\xi + \epsilon_2\epsilon_1 k_2 \cosh\xi \sin\theta}.$$

Proof: Using the normal curve β given by

$$\beta(s) = \left(\frac{-1}{\epsilon_0\epsilon_1 k_1}\right) N(s) + \frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2} B_1(s) - \left(\int \frac{k_1' k_3}{\epsilon_0\epsilon_2 k_1^2 k_2} ds\right) B_2(s),$$

and hence, the tube surface can be written as

$$\begin{aligned} \Omega(s,\xi) = &\left(\frac{-1}{\epsilon_0\epsilon_1 k_1} + d(s)\sinh\xi\right)\overrightarrow{N} + \left(\frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2} + d(s)\cosh\xi\cos\theta\right)\overrightarrow{B_1} \\ &+ \left(-\int \frac{k_1' k_3}{\epsilon_0\epsilon_2 k_1^2 k_2} ds + d(s)\cosh\xi\sin\theta\right)\overrightarrow{B_2} \end{aligned} \tag{3.23}$$

In addition, for ease of operation, the surface given in (3.23) can be written as

$$\begin{aligned} \Omega(s,\xi) = &\left(L_1 + d(s)\sinh\xi\right)\overrightarrow{N} + \left(L_2 + d(s)\cosh\xi\cos\theta\right)\overrightarrow{B_1} \\ &+ \left(L_3 + d(s)\cosh\xi\sin\theta\right)\overrightarrow{B_2}, \end{aligned} \tag{3.24}$$

so that where $L_1 = \frac{-1}{\epsilon_0\epsilon_1 k_1}, L_2 = \frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2}, L_3 = -\int \frac{k_1' k_3}{\epsilon_0\epsilon_2 k_1^2 k_2} ds$. In this case, taking the derivative for $\Omega(s,\xi)$ according to the s and ξ parameters, respectively, we get

$$\Omega_s = \left(0, L_1' + d_s\sinh\xi, L_2' + d_s\cosh\xi\cos\theta, L_3' + d_s\cosh\xi\sin\theta\right); \tag{3.25a}$$

$$\Omega_\xi = \left(0, d\cosh\xi, d\cos\theta\sinh\xi, d\sinh\xi\sin\theta\right). \tag{3.25b}$$

Let's find the components of the first fundamental form for the surface using (3.25b). Hence, we obtain

$$\begin{aligned} E &= \langle\Omega_s,\Omega_s\rangle = -\left(L_1' + d_s\sinh\xi\right)^2 + \left(L_2' + d_s\cosh\xi\cos\theta\right)^2 \\ &\quad + \left(L_3' + d_s\cosh\xi\sin\theta\right)^2 \\ F &= \langle\Omega_\xi,\Omega_s\rangle = -d\left(L_1' + d_s\sinh\xi\right)d\cosh\xi \\ &\quad + d\left(L_2' + d_s\cosh\xi\cos\theta\right)d\sinh\xi\cos\theta \\ &\quad + d\left(L_3' + d_s\cosh\xi\sin\theta\right)d\sinh\xi\sin\theta \\ G &= \langle\Omega_\xi,\Omega_\xi\rangle = -d^2 \end{aligned}$$

and if $F = 0$, the following equations can be written

$$-L_1' + L_2'\tanh\xi\cos\theta + L_3'\tanh\xi\sin\theta = d_s\begin{pmatrix} \sinh\xi - \cosh\xi\cos^2\theta\tanh\xi \\ -\cosh\xi\sin^2\theta\tanh\xi \end{pmatrix} \tag{3.26}$$

$$L_1' = L_2'\tanh\xi\cos\theta + L_3'\tanh\xi\sin\theta. \tag{3.27}$$

So that L_1, L_2, L_3 values are written instead of (3.27), we get

$$\frac{-1}{\epsilon_1}\frac{d\left(\frac{1}{k_1}\right)}{ds} = \frac{1}{\epsilon_2}\frac{d\left(\frac{k_1'}{k_1^2 k_2}\right)}{ds}\tanh\xi\cos\theta + \frac{1}{\epsilon_1}\frac{d\left(-\int\frac{k_1' k_3}{k_1^2 k_2}ds\right)}{ds}\tanh\xi\sin\theta. \tag{3.28}$$

Thus, the equation (3.29) can be written equivalent to the last equation

$$\xi = \operatorname{arccot} h\left\{-\frac{\epsilon_1}{\epsilon_2}\left(\frac{k_1''k_1k_2 - 2{k_1'}^2k_2 - k_1'k_1k_2'}{k_1k_2^2k_1'}\cos\theta + \frac{k_3}{k_2}\sin\theta\right)\right\}. \tag{3.29}$$

Also, the components of the normal curve are as follows

$$\mu = L_1 = \frac{-1}{\epsilon_0\epsilon_1 k_1}, \gamma = L_2 = \frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2}, \theta = L_3 = -\int\frac{k_1'k_3}{\epsilon_0\epsilon_2 k_1^2 k_2}ds \tag{3.30}$$

and these equations can be written as follows

$$\begin{aligned} \mu' &= L_1' = \epsilon_2\epsilon_1 k_2 L_2 \\ \theta' &= L_3' = -k_3 L_2 \\ \gamma' &= L_2' = -k_2 L_1 + \epsilon_3\epsilon_2 k_3 L_3. \end{aligned} \tag{3.31}$$

First of all, for ease of operation, the following equation can be written by assuming that $E = \langle\Omega_s, \Omega_s\rangle = 1$

$$-L_1'^2 + L_2'^2 + L_3'^2 + d_s^2 + 2d_s\left(\cosh\xi\left(L_2'\cos\theta + L_3'\sin\theta\right) - \sinh\xi L_1'\right) + d_s^2 = 1. \tag{3.32}$$

Considering that $L_1' = L_2'\tanh\xi\cos\theta + L_3'\tanh\xi\sin\theta$, and therefore $\coth\xi L_1' = L_2'\cos\theta + L_3'\sin\theta$, we have

$$d_s^2 + \frac{2L_1'}{\sinh\xi}d_s - L_1' + L_2' + L_3' - 1 = 0. \tag{3.33}$$

Therefore, corresponding to the value d_s given in this last equation, for the second order differential equation, we find $\Delta = \left(\frac{2L_1'}{\sinh\xi}\right)^2 - 4\left(-L_1' + L_2' + L_3' - 1\right)$ and we get

$$d_s = \frac{1}{\sinh\xi}\left(-L_1' \pm \sqrt{\left(L_1'\right)^2 - \sinh^2\xi\left(-L_1' + L_2' + L_3' - 1\right)}\right).$$

Here, assuming that $-L_1' + L_2' + L_3' = 1$, we can obtain

$$d_s\left(d_s + \frac{2L_1'}{\sinh\xi}\right) = 0 \Rightarrow d_s = 0 \text{ or } d_s = \frac{-2}{\sinh\xi}L_1', \tag{3.34}$$

and hence, we can write

$$\begin{aligned} d = \text{constant or } d &= \frac{-2}{\sinh\xi}L_1 = \frac{2}{\epsilon_0\epsilon_1\sinh\xi}\frac{1}{k_1} \\ k_1 &= \frac{2}{\epsilon_0\epsilon_1 d\sinh\xi}. \end{aligned} \tag{3.35}$$

After that, we will perform surface characterizations according to the $d =$ constant state. Also, since $d = e^{-\cos\theta\sinh 2\xi\int k_2 ds + c}$ the following statement is obtained

$$-\cos\theta\sinh 2\xi\int k_2 ds = \frac{\ln d}{e^c} = c_1 \Rightarrow \int k_2 ds = \frac{-c_1}{\cos\theta\sinh 2\xi}; c_1 \in \mathbb{R}_0. \tag{3.36}$$

Using the equations obtained above, let's find the vectors e_1, e_2 tangent to the surface Ω in E_2^4. So that, using $d =$constant and the following equations

$$\Omega_s = (0, L_1', L_2', L_3')$$

$$\Omega_\xi = (0, d\cosh\xi, d\sinh\xi\cos\theta, d\sinh\xi\sin\theta),$$

we can write the equations in (3.37)

$$e_1 = (0, L_1', L_2', L_3') = (0, \epsilon_2\epsilon_1 k_2 L_2, k_2 L_1 + \epsilon_2\epsilon_3 k_3 L_3, -k_3 L_2)$$

$$e_1 = A(0, \epsilon_2\epsilon_1 k_2 L_2, L_2', -k_3 L_2); \langle e_1, e_1\rangle = 1 \tag{3.37}$$

$$e_2 = \frac{1}{d}(0, \cosh\xi, \sinh\xi\cos\theta, \sinh\xi\sin\theta); \langle e_2, e_2\rangle = -1$$

$$A = \frac{1}{(k_3^2 - k_2^2) L_2 + L_2'^2}.$$

Considering being orthogonal here, we obtain

$$\langle e_1, e_2\rangle = \frac{A}{d}(-\epsilon_2\epsilon_1 k_2 L_2\cosh\xi + L_2'\sinh\xi\cos\theta - k_3 L_2\sinh\xi\sin\theta) = 0$$

$$\frac{L_2'}{L_2} = \left(\epsilon_2\epsilon_1 k_2\frac{\coth\xi}{\cos\theta} + k_3\tan\theta\right). \tag{3.38}$$

Here, by making the necessary calculations, we get

$$L_2 = c_2 e^{\int\left(\epsilon_2\epsilon_1 k_2\frac{\coth\xi}{\cos\theta} + k_3\tan\theta\right)ds} \tag{3.39}$$

and

$$k_1' - c_2\epsilon_0\epsilon_2 k_1^2 k_2 e^{\int\left(\epsilon_2\epsilon_1 k_2\frac{\tanh\xi}{\cos\theta} + k_3\tan\theta\right)ds} = 0. \tag{3.40}$$

Secondly, e_3 and e_4 vectors can be selected to be normal to the surface. Primarily, with the terms $k_3^2 - k_2^2 = 1$ and $\frac{k_3}{k_2} = -\epsilon_2\epsilon_1\tanh\xi\sin\theta$, such that $\langle e_3, e_3\rangle = -1$ and $\langle e_1, e_3\rangle = 0$, $\langle e_2, e_3\rangle = 0$

$$e_3 = (0, k_3, 0, -\epsilon_2\epsilon_1 k_2) \tag{3.41}$$

normal vector can be selected. Finally, a fourth vector perpendicular to these last three vectors is as

$$\begin{aligned} e_4 &= e_1\wedge e_2\wedge e_3 = \begin{pmatrix} -e_1 & -e_2 & e_3 & e_4 \\ 0 & A\epsilon_2\epsilon_1 k_2 L_2 & AL_2' & -Ak_3 L_2 \\ 0 & \frac{\cosh\xi}{d} & \frac{\sinh\xi\cos\theta}{d} & \frac{\sinh\xi\sin\theta}{d} \\ 0 & k_3 & 0 & -\epsilon_2\epsilon_1 k_2 \end{pmatrix} \\ &= \left(\begin{array}{c} \frac{A}{d}\left(\begin{array}{c} L_2\sinh\xi\cos\theta\left(k_3^2 - k_2^2\right) \\ +L_2'\left(k_3\sinh\xi\sin\theta + \epsilon_2\epsilon_1 k_2\cosh\xi\right) \end{array}\right), \\ 0,0,0 \end{array}\right), \end{aligned}$$

where $\frac{A}{d}(L_2\sinh\xi\cos\theta\left(k_3^2 - k_2^2\right) + L_2'(k_3\sinh\xi\sin\theta + \epsilon_2\epsilon_1 k_2\cosh\xi) = B$. In this case, so that $\langle e_4, e_4\rangle = 1$. let's take the vector

$$e_4 = \frac{1}{B}(B, 0, 0, 0). \tag{3.42}$$

In this case, we can talk about the frame $\{e_1, e_2, e_3, e_4\}$ which is given and by the equations

$$e_1 = \frac{1}{(k_3^2 - k_2^2) L_2 + L_2'^2}(0, \epsilon_2\epsilon_1 k_2 L_2, L_2', -k_3 L_2)$$

$$e_2 = \frac{1}{d}(0, \cosh\xi, \sinh\xi\cos\theta, \sinh\xi\sin\theta) \tag{3.43}$$

$$e_3 = (0, k_3, 0, -\epsilon_2\epsilon_1 k_2); e_4 = \frac{1}{B}(B, 0, 0, 0) \tag{3.44}$$

$$\langle e_1, e_1\rangle = 1; \langle e_2, e_2\rangle = -1; \langle e_3, e_3\rangle = -1; \langle e_4, e_4\rangle = 1 \tag{3.45}$$

and moves on the surface. Let's find the curvatures of the surface using this frame. Primarily, we can write following equations

$$\Omega_s = (0, L_1', L_2', L_3')\,; \Omega_{ss} = (0, L_1'', L_2'', L_3'')\,; \Omega_{s\xi} = 0 \tag{3.46}$$

$$\Omega_\xi = \begin{pmatrix} 0, \\ d\cosh\xi, \\ d\sinh\xi\cos\theta, \\ d\sinh\xi\sin\theta \end{pmatrix}; \Omega_{\xi\xi} = \begin{pmatrix} 0, \\ d\sinh\xi, \\ d\cosh\xi\cos\theta, \\ d\cosh\xi\sin\theta \end{pmatrix}$$

Thus, using the equations (2.10) and (3.43), (3.44), let's find the components of w_{AB} connection forms and h the second fundamental form. So, $\{e_3, e_4\}$ are normal to the surface, we get

$$c_{11}^1 = h_{11}^3 = c_{11}^3 = \langle \Omega_{ss}, e_3 \rangle = -k_3 L_1'' - \epsilon_2\epsilon_1 k_2 L_3'' \tag{3.47}$$

$$c_{12}^1 = h_{12}^3 = c_{12}^3 = \langle \Omega_{s\xi}, e_3 \rangle = 0 \tag{3.48}$$

$$c_{22}^1 = h_{22}^3 = c_{22}^3 = \langle \Omega_{\xi\xi}, e_3 \rangle = -k_3 d\sinh\xi - \epsilon_2\epsilon_1 k_2 d\cosh\xi\sin\theta \tag{3.49}$$

$$c_{11}^2 = h_{11}^4 = c_{11}^4 = \langle \Omega_{ss}, e_4 \rangle = 0 \tag{3.50a}$$

$$c_{12}^2 = h_{12}^4 = c_{12}^4 = \langle \Omega_{s\xi}, e_4 \rangle = 0 \tag{3.50b}$$

$$c_{22}^2 = h_{22}^4 = c_{22}^4 = \langle \Omega_{\xi\xi}, e_3 \rangle = 0. \tag{3.50c}$$

Furthermore, using the expressions $w_A(X) = \langle e_A, X \rangle$, $de_A = \sum e_B w_{AB} e_B$, $\langle de_A, e_B \rangle = \sum e_B w_{AB}$ and $w_{is} = \sum\limits_j \varepsilon_j h_{ij}^s w_j$

$$w_{12} = \varepsilon_1 h_{11}^2 w_1 + \varepsilon_2 h_{12}^2 w_2 = 0; w_{14} = \varepsilon_1 h_{11}^4 w_1 + \varepsilon_2 h_{12}^4 w_2 = 0 \tag{3.51a}$$

$$w_{13} = \varepsilon_1 h_{11}^3 w_1 + \varepsilon_2 h_{12}^3 w_2 = (-k_3 L_1'' - \epsilon_2\epsilon_1 k_2 L_3'')\, w_1 \tag{3.51b}$$

$$w_{23} = \varepsilon_1 h_{21}^3 w_1 + \varepsilon_2 h_{22}^3 w_2 = (-k_3 d\sinh\xi - \epsilon_2\epsilon_1 k_2 d\cosh\xi\sin\theta)\, w_2 \tag{3.51c}$$

$$w_{24} = \varepsilon_1 h_{21}^4 w_1 + \varepsilon_2 h_{22}^4 w_2 = 0; w_{34} = \varepsilon_1 h_{31}^4 w_1 + \varepsilon_2 h_{32}^4 w_2 = 0 \tag{3.51d}$$

connection forms are obtained. Hence, the mean curvature vector H of the surface is given by

$$H = \frac{1}{2}\sum_{s=n+1}^{m}\sum_{i=1}^{n} \varepsilon_j \varepsilon_s h_{ij}^s e_s = \frac{1}{2}\left\{ \begin{array}{c} \varepsilon_1\varepsilon_3 h_{11}^3 e_3 + \varepsilon_2\varepsilon_3 h_{22}^3 e_3 \\ +\varepsilon_1\varepsilon_4 h_{11}^4 e_4 + \varepsilon_2\varepsilon_4 h_{22}^4 e_4 \end{array} \right\} \tag{3.52}$$

$$H = \frac{1}{2}\left\{ -\varepsilon_1\varepsilon_3 \begin{pmatrix} k_3 L_1'' \\ +\epsilon_2\epsilon_1 k_2 L_3'' \end{pmatrix} - \varepsilon_2\varepsilon_3 d \begin{pmatrix} k_3\sinh\xi \\ +\epsilon_2\epsilon_1 k_2\cosh\xi\sin\theta \end{pmatrix} \right\} e_3 \tag{3.53}$$

and considering the following equations

$$\varepsilon_{1,4} = \langle e_1, e_1 \rangle = 1; \varepsilon_{2,3} = \langle e_2, e_2 \rangle = -1, \tag{3.54}$$

we get

$$H = \frac{1}{2}\left\{ \begin{array}{c} (k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3'') - k_3 d\sinh\xi \\ -\epsilon_2\epsilon_1 k_2 d\cosh\xi\sin\theta \end{array} \right\} e_3 = \varpi . e_3. \tag{3.55}$$

So that, considering the vector $e_3 = (0, k_3, 0, -\epsilon_2\epsilon_1 k_2)$, the mean curvature vector H can be written as

$$H = \frac{1}{2}\left\{ \begin{array}{c} k_3(-d\sinh\xi + L_1'') \\ +\epsilon_2\epsilon_1 k_2(-d\cosh\xi\sin\theta + L_3'') \end{array} \right\} (0, k_3, 0, -\epsilon_2\epsilon_1 k_2) \tag{3.56}$$

$$H = (0, \varpi k_3, 0, -\varpi\epsilon_2\epsilon_1 k_2)\,. \tag{3.57}$$

Hence, the mean curvature of the surface is given by

$$H = \frac{1}{2}\{k_3(-d\sinh\xi + L_1'') + \epsilon_2\epsilon_1 k_2(-d\cosh\xi\sin\theta + L_3'')\} \tag{3.58}$$

or for the equations $L_1 = \frac{-1}{\epsilon_0\epsilon_1 k_1}, \gamma = L_2 = \frac{k_1'}{\epsilon_0\epsilon_2 k_1^2 k_2}, \theta = L_3 = -\int \frac{k_1' k_3}{\epsilon_0\epsilon_2 k_1^2 k_2} ds$, we can write

$$H = \frac{1}{2}\left\{ k_3 \begin{pmatrix} -d\sinh\xi \\ +\frac{1}{\epsilon_0\epsilon_1}\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right) \end{pmatrix} + \epsilon_2\epsilon_1 k_2 \begin{pmatrix} -d\cosh\xi\sin\theta \\ +\frac{1}{\epsilon_0\epsilon_2}\frac{d}{ds}\left(\frac{k_1' k_3}{k_1^2 k_2}\right) \end{pmatrix} \right\}. \tag{3.59}$$

Finally, the Gaussian curvature K of the surface can be written as

$$K = \sum_{s=3}^{4} \varepsilon_s \left(h_{11}^s h_{22}^s - h_{12}^s h_{21}^s\right) = \varepsilon_3 \begin{pmatrix} h_{11}^3 h_{22}^3 \\ -h_{12}^3 h_{21}^3 \end{pmatrix} + \varepsilon_4 \begin{pmatrix} h_{11}^4 h_{22}^4 \\ -h_{12}^4 h_{21}^4 \end{pmatrix}$$

$$K = -d\left(k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3''\right)\left(k_3\sinh\xi + \epsilon_2\epsilon_1 k_2\cosh\xi\sin\theta\right) \tag{3.60}$$

or

$$K = -d \begin{pmatrix} \frac{k_3}{\epsilon_0\epsilon_1}\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right) \\ +\frac{k_2\epsilon_1}{\epsilon_0}\frac{d}{ds}\left(\frac{k_1' k_3}{k_1^2 k_2}\right) \end{pmatrix} \begin{pmatrix} k_3\sinh\xi \\ +\epsilon_2\epsilon_1 k_2\cosh\xi\sin\theta \end{pmatrix}. \tag{3.61}$$

Also, the requirement and sufficient condition to be minimal of the surface Ω is that the mean curvature H is zero. Therefore, the following equality is satisfied

$$H = \frac{1}{2}\left\{\left(k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3''\right) - k_3 d\sinh\xi - \epsilon_2\epsilon_1 k_2 d\cosh\xi\sin\theta\right\} = 0 \tag{3.62}$$

$$d = \frac{k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3''}{k_3\sinh\xi + \epsilon_2\epsilon_1 k_2\cosh\xi\sin\theta}. \tag{3.63}$$

□

Theorem 3.4 *If the surface Ω given E_2^4 is minimal $\Longleftrightarrow$ for $i = 2,3,4$, coordinate functions Ω^i are harmonic and the following equations are satisfied*

$$\begin{aligned} k_1 k_1'' - 2k_1'^2 &= -k_1^3\epsilon_0\epsilon_1 d\sinh\xi; \\ \frac{d^2}{ds^2}\left(\frac{k_1'}{k_1^2 k_2}\right) &= -\epsilon_0\epsilon_2 d\cosh\xi\cos\theta; \\ \frac{d^2}{ds^2}\left(\int \frac{k_3 k_1'}{k_1^2 k_2} ds\right) &= -\epsilon_0\epsilon_1 d\cosh\xi\sin\theta. \end{aligned}$$

Proof: Let us assume that the surface Ω is a minimal surface. In this case, for the mean curvature H, we write $H = 0$. So that, the equation (3.57) is satisfied. Furthermore, the requirement to suggest that Ω^i, $i = 2,3,4$ coordinate functions be harmonic requires $\Delta\Omega^i = 0$. So,

$$\Omega(s,\xi) = (0, L_1 + d\sinh\xi, L_2 + d\cosh\xi\cos\theta, L_3 + d\cosh\xi\sin\theta)$$

$$\Omega(s,\xi) = (0, \Omega^2, \Omega^3, \Omega^4), \tag{3.64}$$

using previous equation for the surface, we can write

$$\Delta\Omega_s^2 = \frac{\partial^2\Omega^2}{\partial s^2} = L_1'', \Delta\Omega_\xi^2 = \frac{\partial^2\Omega^2}{\partial\xi^2} = d\sinh\xi \tag{3.65a}$$

$$\Delta\Omega^2 = L_1'' + d\sinh\xi = 0 \tag{3.65b}$$

$$\Delta\Omega_s^3 = \frac{\partial^2\Omega^3}{\partial s^2} = L_2'', \Delta\Omega_\xi^3 = \frac{\partial^2\Omega^3}{\partial\xi^2} = d\cosh\xi\cos\theta \tag{3.66a}$$

$$\Delta\Omega^3 = L_2'' + d\cosh\xi\cos\theta = 0 \tag{3.66b}$$

$$\Delta\Omega_s^4 = \frac{\partial^2\Omega^4}{\partial s^2} = L_3'', \Delta\Omega_\xi^4 = \frac{\partial^2\Omega^4}{\partial \xi^2} = d\cosh\xi\sin\theta \tag{3.67a}$$

$$\Delta\Omega^4 = L_3'' + d\cosh\xi\sin\theta = 0 \tag{3.67b}$$

Also, from (3.65b), (3.66b) and (3.67b), we get

$$L_1'' = -d\sinh\xi; L_2'' = -d\cosh\xi\cos\theta; L_3'' = -d\cosh\xi\sin\theta \tag{3.68}$$

and respectively, from the equation $L_1'' = -d\sinh\xi$, we get

$$\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right) = -\epsilon_0\epsilon_1 d\sinh\xi \Rightarrow k_1k_1'' - 2k_1'^2 = -k_1^3\epsilon_0\epsilon_1 d\sinh\xi. \tag{3.69}$$

Also, from the equation $L_2'' = -d\cosh\xi\cos\theta$, we write

$$\frac{d^2}{ds^2}\left(\frac{k_1'}{k_1^2k_2}\right) = -\epsilon_0\epsilon_2 d\cosh\xi\cos\theta. \tag{3.70}$$

Finally, from the equation $L_3'' = -d\cosh\xi\sin\theta$, we get

$$\frac{d^2}{ds^2}\left(\int\frac{k_3k_1'}{k_1^2k_2}ds\right) = -\epsilon_0\epsilon_1 d\cosh\xi\sin\theta. \tag{3.71}$$

□

Theorem 3.5 *Necessary and sufficient conditions for the surface Ω given in E_2^4 to be a flat surface, the following equations are satisfied*

$$k_3\frac{d^2\left(\frac{1}{k_1}\right)}{ds^2} - \epsilon_2\epsilon_1 k_2\frac{d\left(\frac{k_3k_1'}{k_1^2k_2}\right)}{ds} = 0$$

$$\xi = \operatorname{arccot} h\left(\frac{-k_3}{k_2\epsilon_2\epsilon_1\sin\theta}\right).$$

Proof: Let's find the curvatures K and H for the surface Ω given in E_2^4. For ease of operation, let's first consider the following equations

$$f(s) = k_3L_1'' + \epsilon_2\epsilon_1k_2L_3''; p(s,\xi) = k_3\sinh\xi + \epsilon_2\epsilon_1k_2\cosh\xi\sin\theta. \tag{3.72}$$

Hence, the curvatures K and H, respectively, are written as

$$K = \varepsilon_3 d\left(k_3L_1'' + \epsilon_2\epsilon_1k_2L_3''\right)\left(k_3\sinh\xi + \epsilon_2\epsilon_1k_2\cosh\xi\sin\theta\right) = \varepsilon_3 df(s)p(s,\xi) \tag{3.73}$$

and

$$H = \frac{1}{2}\left\{k_3\left(L_1'' - d\sinh\xi\right) + \epsilon_2\epsilon_1k_2(L_3'' - d\cosh\xi\sin\theta)\right\} = \frac{1}{2}\left\{f(s) - dp(s,\xi)\right\}. \tag{3.74}$$

Then, for the curvature K of the surface Ω using the following equation

$$K = -df(s)p(s,\xi), \tag{3.76}$$

$$K = -df(s)p(s,\xi) = 0 \Rightarrow f(s) \neq 0, p(s,\xi) = 0$$

the previous equation is obtained, and from the last expression, it can be written

$$k_3\sinh\xi + \epsilon_2\epsilon_1k_2\cosh\xi\sin\theta = 0 \Rightarrow \xi = \operatorname{arccot} h\left(\frac{-k_3}{k_2\epsilon_2\epsilon_1\sin\theta}\right). \tag{3.77}$$

Also, if the following equation is satisfied,

$$K = -df(s)p(s,\xi) = 0 \Rightarrow f(s) = 0, p(s,\xi) \neq 0, \tag{3.78}$$

one gets

$$k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3'' = 0 \Rightarrow k_3 \frac{d^2}{ds^2}\left(\frac{1}{k_1}\right) = \epsilon_2\epsilon_1 k_2 \frac{d^2}{ds^2}\left(\int \frac{k_3 k_1'}{k_1^2 k_2} ds\right). \tag{3.79}$$

□

Theorem 3.6 *Necessary and sufficient conditions to be the Weingarten surface of the surface Ω in E_2^4, the following statements are satisfied*

1) If $f(s)$ =constant, the surface Ω is the (H,K)-Weingarten surface corresponds to the following equation

$$\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right) + \epsilon_2\epsilon_1 k_2 \frac{d^2}{ds^2}\left(\int \frac{k_3 k_1'}{k_1^2 k_2} ds\right) = constant.$$

2) If $p'(s,\xi) = 0$ or $p(s,\xi) = h(\xi)$, corresponds to the equation $d = -\frac{f(s)}{p(s,\xi)} = -\frac{f(s)}{h(\xi)}$, the surface Ω is the (H,K)-Weingarten surface.

3) If $p_\xi(s,\xi) = 0$, the surface Ω is the $\Phi(H,K)$-Weingarten surface.

Proof: We know that if $\Phi(H,K) = 0$, the surface is called the Weingarten surface. In that case,

$$f(s) = k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3''; \; p(s,\xi) = k_3 \sinh\xi + \epsilon_2\epsilon_1 k_2 \cosh\xi \sin\theta$$

corresponding to the previous values, using the equations of the Gaussian curvature K and mean curvature H as follows, and if partial derivatives are taken according to s and ξ parameters, respectively, one gets

$$K_s = -d\left(f'(s)p(s,\xi) + f(s)p'(s,\xi)\right), K_\xi = -df(s)p_\xi(s,\xi) \tag{3.80a}$$

$$H_s = \frac{1}{2}\left\{-f'(s) + dp'(s,\xi)\right\}, H_\xi = \frac{1}{2}dp_\xi(s,\xi). \tag{3.80b}$$

Hence, from definition, with the help of equations in (3.80), we get

$$\Phi(H,K) = \frac{\partial(H,K)}{\partial(s,\xi)} = K_\xi H_s - K_s H_\xi$$

$$\Phi(H,K) = -\frac{d^2}{2}\begin{pmatrix} f'(s)p(s,\xi) \\ +f(s)p'(s,\xi) \end{pmatrix} p_\xi(s,\xi) + \frac{d}{2}f(s)p_\xi(s,\xi)\left\{\begin{matrix} f'(s) \\ -dp'(s,\xi) \end{matrix}\right\}. \tag{3.81}$$

Here, we can say that

1) If $f(s)$ =constant, $f(s) = k_3 L_1'' + \epsilon_2\epsilon_1 k_2 L_3''$ =constant and in response to the following equation

$$\frac{d^2}{ds^2}\left(\frac{1}{k_1}\right) + \epsilon_2\epsilon_1 k_2 \frac{d^2}{ds^2}\left(\int \frac{k_3 k_1'}{k_1^2 k_2} ds\right) = \text{constant}, \tag{3.82}$$

$\Phi(K,H) = 0$ is obtained. Hence, the surface Ω is (H,K)-Weingarten surface.

2) If $p'(s,\xi) = 0$ or $p(s,\xi) = h(\xi)$, in response to the equation $d = -\frac{f(s)}{p(s,\xi)} = -\frac{f(s)}{h(\xi)}$, we get

$$\Phi(H,K) = \frac{d^2}{2}f'(s)p(s,\xi)p_\xi(s,\xi) + \frac{d}{2}f(s)f'(s)p_\xi(s,\xi) = 0$$

and hence, the surface Ω is (H,K)-Weingarten surface.

3) If $p_\xi(s,\xi) = 0$, $\Phi(K,H) = 0$ and hence the surface Ω is (H,K)-Weingarten surface. □

Theorem 3.7 *If the surface Ω in E_2^4 is a linear Weingarten surface, the following expressions are satisfied.*

1) If $f(s) = A =$constant, $a_3 = -A^2 a_1$.
2) If $p'(s,\xi) = 0$ that is if $p(s,\xi) = h(\xi)$, $a_2^2 + 4a_1a_3 = 0$.
3) If $p_\xi(s,\xi) = 0$ that is if $p(s,\xi) = t(s)$, $a_2^2 + 4a_1a_3 = 0$.

Proof: Let the surface Ω given in E_2^4 be the Weingarten surface. Then, from the definition of the Weingarten surface and using the equations (3.73) and (3.75), we get

$$a_1\left(-df(s)p(s,\xi)\right) + a_2\frac{1}{2}\left\{f(s) - dp(s,\xi)\right\} = a_3. \tag{3.83}$$

So that,
1) If $f(s) = A =$ constant, we can write

$$p(s,\xi)\left(-a_1dA - \frac{d}{2}a_2\right) + \left(\frac{a_2A}{2} - a_3\right) = 0. \tag{3.84}$$

Also, using the linear independence of vectors, we write

$$p(s,\xi) \neq 0 \ ve \ a_1dA + \frac{d}{2}a_2 = 0 \Rightarrow \frac{a_1}{a_2} = \frac{-1}{2A} \tag{3.85}$$

$$\frac{a_2A}{2} - a_3 = 0 \Rightarrow \frac{a_3}{a_2} = \frac{A}{2}. \tag{3.86}$$

Finally, we get

$$\frac{a_1}{a_2} = \frac{-1}{2A} \ and \ \frac{a_3}{a_2} = \frac{A}{2} \Rightarrow a_3 = -A^2a_1. \tag{3.87}$$

2) If $p'(s,\xi) = 0$, that is if $p(s,\xi) = h(\xi)$, the equation $d = -\frac{f(s)}{p(s,\xi)} = -\frac{f(s)}{h(\xi)}$ is written. So that we can write

$$a_1\left(-df(s)h(\xi)\right) + a_2\frac{1}{2}\left\{f(s) - dh(\xi)\right\} = a_3$$

$$f(s)\left(-a_1dh(\xi) + a_2\frac{1}{2}\right) + \left(-\frac{a_2}{2}dh(\xi) - a_3\right) = 0 \tag{3.88}$$

and using the linear independent of vectors, we get

$$f(s) \neq 0 \quad ve \ -a_1dh(\xi) + a_2\frac{1}{2} = 0 \Rightarrow \frac{2a_1}{a_2} = \frac{1}{dh(\xi)} \tag{3.89}$$

$$-\frac{a_2}{2}dh(\xi) - a_3 = 0 \Rightarrow \frac{-a_2}{2a_3} = \frac{1}{dh(\xi)}. \tag{3.90}$$

Hence, from the previous equations, we get

$$\frac{2a_1}{a_2} = \frac{-a_2}{2a_3} \Rightarrow a_2^2 + 4a_1a_3 = 0 \text{ or } a_3 = -d^2h^2(\xi)a_1. \tag{3.91}$$

3) If $p_\xi(s,\xi) = 0$ that is if $p(s,\xi) = t(s)$, we can write

$$a_1\left(-df(s)t(s)\right) + a_2\frac{1}{2}\left\{f(s) - dt(s)\right\} = a_3$$

$$f(s)\left(-da_1t(s) + \frac{a_2}{2}\right) + \left(-dt(s)a_2\frac{1}{2} - a_3\right) = 0 \tag{3.92}$$

and using the linear independent of the vectors, we get

$$f(s) \neq 0 \quad and \ -da_1t(s) + \frac{a_2}{2} = 0 \Rightarrow \frac{a_1}{a_2} = \frac{1}{2dt(s)} \tag{3.93}$$

$$-dt(s)a_2\frac{1}{2}-a_3=0\Rightarrow\frac{a_2}{a_3}=\frac{-2}{dt(s)}. \tag{3.94}$$

Therefore, from the last equations, we have

$$\frac{2a_1}{a_2}=\frac{-a_2}{2a_3}\Rightarrow a_2^2+4a_1a_3=0 \ or \ a_3=-d^2t^2(s)a_1. \tag{3.95}$$

□

Theorem 3.8 *If the surface Ω in E_2^4 is a linear Weingarten surface, the following statements are satisfied*
1) If $f(s)=A$=constant, the following equation is satisfied

$$p(s,\xi)=\frac{A\left(a_1+2a_2A\right)}{d\left(a_1+4a_2A+4a_3A^2\right)}.$$

2) If $p'(s,\xi)=0($ or $p(s,\xi)=h(\xi))$ or $p_\xi(s,\xi)=0($ or $p(s,\xi)=t(s))$, k_2,k_3 =constant and for the requirement to be HK-quadric surface

$$\frac{k_3}{k_2}=-\epsilon_2\epsilon_1\tanh\xi\sin\theta$$

the previous equality is satisfied.

Proof: Let us assume that the tubular surface Ω formed by a normal curve is HK-quadric. In this case, taking the necessary differential calculations from the definition of HK-quadric surface, we get

$$a_1HH_\xi+a_2(H_\xi K+HK_\xi)+a_3KK_\xi=0. \tag{3.96}$$

Then, from the equations K and H, we can write following equations

$$K_s=-d\left(f'(s)p(s,\xi)+f(s)p'(s,\xi)\right),K_\xi=-df(s)p_\xi(s,\xi) \tag{3.97}$$

$$H_s=\frac{1}{2}\left\{f'(s)-dp'(s,\xi)\right\},H_\xi=\frac{-1}{2}dp_\xi(s,\xi). \tag{3.98}$$

Let's examine some assumptions using the previous equations.
1) If $f(s)=A$ =constant, we get

$$K_s=-dAp'(s,\xi),K_\xi=-dAp_\xi(s,\xi);H_s=\frac{-d}{2}p'(s,\xi),H_\xi=\frac{-d}{2}p_\xi(s,\xi). \tag{3.99}$$

So, if this last equation is replaced in equation (3.96), we get

$$0=\left\{-a_1\frac{dA}{4}-a_2\frac{dA^2}{2}+\left(\begin{array}{c}\frac{a_1d^2}{4}+\frac{1}{2}a_2Ad^2\\+a_2\frac{d^2A}{2}+a_3d^2A^2\end{array}\right)p(s,\xi)\right\}p_\xi(s,\xi)$$

$$0=\left\{-a_1\frac{dA}{4}-a_2\frac{dA^2}{2}+d^2\left(\frac{a_1}{4}+a_2A+a_3A^2\right)p(s,\xi)\right\}p_\xi(s,\xi). \tag{3.100}$$

In this case, for the equation

$$p_\xi(s,\xi)\neq 0,a_2^2+4a_1a_3=0, \tag{3.101}$$

we obtain

$$0=-a_1\frac{dA}{4}-a_2\frac{dA^2}{2}+d^2\left(\frac{a_1}{4}+a_2A+a_3A^2\right)g(s,\xi)$$

and hence the surface given with the condition of providing the following equation

$$p(s,\xi)=\frac{A\left(a_1+2a_2A\right)}{d\left(a_1+4a_2A+4a_3A^2\right)} \tag{3.102}$$

is HK-quadric.

2) If $p'(s,\xi)=0$ or $p(s,\xi)=h(\xi)$, from the equations K and H, we get

$$K_\xi = -df(s)p_\xi(s,\xi), H_\xi = \frac{-d}{2}p_\xi(s,\xi).$$

and by substituting in the following equation

$$a_1HH_\xi + a_2(H_\xi K + HK_\xi) + a_3KK_\xi = 0,$$

we get

$$\frac{-a_1}{4}d\left\{f(s)-dh(\xi)\right\}p_\xi(s,\xi) + a_2\begin{pmatrix} \frac{d^2}{2}p_\xi(s,\xi)f(s)h(\xi) \\ -\frac{d}{2}\left\{f(s)-dh(\xi)\right\}f(s)p_\xi(s,\xi)\end{pmatrix}$$
$$+a_3d^2f^2(s)h(\xi)p_\xi(s,\xi) = 0 \quad (3.103)$$
$$p_\xi(s,\xi)\left(\frac{-a_1}{4}\left\{f(s)-dh(\xi)\right\} + a_2\left\{\begin{array}{c}\frac{d^2}{2}f(s)h(\xi) \\ -\frac{d}{2}f(s)\left(f(s)-dh(\xi)\right) \\ +a_3d^2f^2(s)h(\xi)\end{array}\right\}\right) = 0$$
$$\left(f(s)\left(-a_1+4da_2h(\xi)\right)+da_1h(\xi)+f^2(s)\left(-2a_2+4a_3dh(\xi)\right)\right) = 0.$$

So that, if necessary arrangements are made in the last equality, we obtain

$$a_1\left(-f(s)+dh(\xi)\right) + a_2\left(4dh(\xi)f(s)-2f^2(s)\right) + a_3\left(4dh(\xi)f^2(s)\right) = 0. \quad (3.104)$$

Thus, from the algebraic equations given in (3.104), we get

$$\begin{array}{rcl} -f(s)+dh(\xi) &=& 0 \\ 4dh(\xi)f(s)-2f^2(s) &=& 0 \\ d.h(\xi)f^2(s) &=& 0. \end{array}$$

Also, from the last equality system, we write $p(s,\xi)=h(\xi)=0$ and with the help of the equation $p(s,\xi)=k_3\sinh\xi+\epsilon_2\epsilon_1k_2\cosh\xi\sin\theta$, the following equation is satisfied

$$\frac{k_3}{k_2} = -\epsilon_2\epsilon_1\coth\xi\sin\theta.$$

Also, if $p_\xi(s,\xi)=0$ or $p(s,\xi)=t(s)$, from the equations (3.73) and (3.74), we write

$$\begin{array}{rcl} K_s &=& -d\left(f'(s)t(s)+f(s)t'(s)\right), K_\xi = 0 \\ H_s &=& \frac{1}{2}\left(f'(s)-dt'(s)\right), H_\xi = 0. \end{array}$$

Finally, if the last equations are written in the equation (3.105)

$$a_1HH_s + a_2(H_sK + HK_s) + a_3KK_s = 0, \quad (3.105)$$

we obtain

$$\begin{array}{rcl} 0 &=& a_1\left\{\frac{1}{4}\left(f(s)-dt(s)\right)f'(s) - \frac{d}{4}\left(f(s)-dt(s)\right)t'(s)\right\} \\ && +a_2\begin{pmatrix} -\frac{d}{2}f(s)f'(s)t(s)+\frac{d^2}{2}f(s)t(s)t'(s) \\ -\frac{d}{2}f(s)f'(s)t(s)-\frac{d}{2}f(s)^2t'(s)+\frac{d^2}{2}f'(s)t^2(s) \\ +\frac{d^2}{2}t(s)f(s)t'(s)\end{pmatrix} \\ && +a_3d^2\left(f(s)t^2(s)f'(s)+f^2(s)t(s)t'(s)\right). \end{array}$$

Here, for $p'(s,\xi) = t'(s) \neq 0$,

$$0 = a_1 \left\{ -\frac{1}{4} \left(f(s) - dt(s) \right) \right\} + a_2 \left(df(s)t(s) - \frac{1}{2} f^2(s) \right) + a_3 d \left(f^2(s) t(s) \right) \tag{3.106}$$

the previous equation can be written. Hence, since $a_1, a_2, a_3 \neq 0$, the following equations can be written

$$\begin{aligned} f(s) - dt(s) &= 0 \\ f(s) \left(dt(s) - \frac{1}{2} f(s) \right) &= 0 \\ d \left(f^2(s) t(s) \right) &= 0. \end{aligned}$$

Therefore, $t(s) = 0$ is obtained and if the surface given with the help of the equation $p(s,\xi) = k_3 \sinh \xi + \epsilon_2 \epsilon_1 k_2 \cosh \xi \sin \theta$ is $HK-$quadric, the following equation is satisfied

$$\frac{k_3}{k_2} = -\epsilon_2 \epsilon_1 \coth \xi \sin \theta.$$

□

4. Conclusion

In this study, the special tube surfaces generated by normal curves with frenet frame in Pseudo-Euclidean space E_2^4 are examined and some certain results of describing the surface characterizations on the surfaces are presented in detail. As a first instance, it is explored that the conditions of tube or canal surface, in which the surfaces that can be chosen to be tube surface generated by normal curve. Moreover, using the Gaussian curvatures and mean curvatures of tube surfaces with normal curve generated frenet frame in E_2^4, Weingarten surface and HK-quadric surface, minimal surface, flat, ... etc. conditions are examined.

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