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Neutrosophic Over Topological Spaces

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ABSTRACT: The concepts of Pythagorean and Fermatean sets are studied in a Neutrosophic environment and their topological structures are developed. The article also presents the idea of neutrosophic gradation of openness on neutrosophic subsets as well as the definition of Topological settings of Neutrosophic variants with examples.

Key Words: Pythagorean Neutrosophic Set (PNS); Fermatean Neutrosophic Set (FNS); Pythagorean Neutrosophic Topological Space (PNTS); Fermatean Neutrosophic Topological Space (FNTS); Neutrosophic gradation of openness ($N - go$);

Contents

1. Introduction

Zadeh proposed the concept of a Fuzzy Set(FS) [18] in 1965 and Chang proposed fuzzy topology [3] in 1968. Atanassov [2] proposed Intuitionistic Fuzzy Sets (IFSs) as a generalization of FSs. Coker proposed the concept of intuitionistic fuzzy topological spaces(IFTS) [4] in 1997 which opened an important area of research in IFSs. Coker has investigated its different properties and defined the concept of IFTS with referernce to Chang's concept.

Lowen developed the structure of fuzzy topological spaces [13]. He modified Chang's definition of fuzzy topology. He also introduced two functions, that will allow us to see the connection between fuzzy topological spaces and general topological spaces more clearly. After classifying FSs and topological spaces, I. Zahan and R. Nasrin [19] created relationships between the elements that compose them and also developed the relation between fuzzy sets and topological spaces.

An IFTS is defined by Tapas Kumar Mondal and S.K. Samanta [17] together with a knowledge of the intuitionistic grading of openness on nonempty fuzzy subsets. Lee. S. J. and Lee. E. P. (2000) [12] explored intuitionistic fuzzy topological spaces

Smarandache added indeterminacy to the notion of intuitionistic fuzzy and introduced the concept of Neutrosophic Set(NS) as a new tool to the world of mathematics in [15], [8].Also, Smarandache (2024) [9]expanded on revolutionary topologies and neutrosophic systems, offering key insights into handling uncertainty. Salama and Alblowi [14] explored the topological structure of the family of NSs and introduced the idea of Neutrosophic Topological Space (NTS) by using non-membership, indeterminacy, and membership functions all of which have a one-to-one correspondence between the members of a set and —]0, 1[+. As a more sophisticated tool than generic topological spaces, this new kind of topological space

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has been embraced. Scientists have been drawn to the concept of NT and a lot has been developed, still there are many subareas to be investigated.

Pythagorean Fuzzy sets(PFSs) and NSs are used in Pythagorean neutrosophic topology [11] to address vagueness and uncertainty in topological spaces. This approach offers a more thorough framework for defining and analyzing incomplete or uncertain information inside topological structures [1]. Fermatean neutrosophic topology's [10] main aim is to expand on classical topology by using the concepts of neutrosophic sets to examine spaces where unclear or indeterminacy is usual. It means developing new concepts, tools, and methods to handle unclear or incomplete data within the topological domain [16].

In many situations in our real world, the degree of membership is both more than and less than 1 [5]. A NS, which contains elements with a membership of more than 1 is called an Overset, introduced and studied by Smarandache. The set which contains elements whose membership concerning a set is less than 0, is known as Underset. A set that has elements with memberships above 1 and below 0 is referred to as being offset and underset, [6]- [7] that is, certain components have memberships that fall outside of the range [0, 1].

In this framework, the concepts of Pythagorean, and Fermatean Neutrosophic Sets have been studied. The ideas of PNTSs, and FNTSs are investigated and their basic properties are examined. The notion of a NTS and the concept of $(N - go)$ on neutrosophic subsets of a nonempty set are presented.

2. Preliminaries

Definition 2.1 *[15] Each element in a NS has degrees of truth, indeterminacy, and falsity associated with it. These degrees can be represented by one of three membership functions: the truth(T), indeterminacy(I), and falsity(F) membership functions. T, I, and F added up equals 3. These functions indicate, respectively, how much an element is part of the set, how ambiguous its membership is, or how much it does not belong to the set.*

Definition 2.2 *[8] If $G = \{\langle n, T_G(n), I_G(n), F_G(n)\rangle | n \in N\}$ and $H = \{\langle n, T_H(n), I_H(n), F_H(n)\rangle | n \in X\}$ are any two NSs of N then*

1. $(G) \subseteq (H)$ *iff* $T_G(n) \leq T_H(n); I_G(n) \leq I_H(n)$ *; and* $F_G(n) \geq F_H(n)$
2. $(G) = (B)$ *iff* $T_G(n) = T_H(n); I_G(n) = I_H(n)$ *; and* $F_G(n) = F_H(n) \forall n \in N$
3. $G = \{\langle n, F_G(n), 1 - I_G(n), T_G(n)\rangle | n \in N\}$
4. $G \cap H = \{\langle n, T_{G\cap H}(n), I_{G\cap H}(n), F_{G\cap H}(n)\rangle | n \in N\}$ *where,*
   - $T_{G\cap H}(n) = \min\{T_G(n), T_H(n)\}$
   - $I_{G\cap H}(n) = \min\{I_G(n), I_G(n)\}$
   - $F_{G\cap H}(n) = \max\{F_G(n), F_G(n)\}$
5. $G \cup H = \{\langle n, T_{G\cup H}(n), I_{G\cup H}(n), F_{G\cup H}(n)\rangle | n \in N\}$*where,*
   (a) $T_{G\cup H}(n) = \max\{T_G(n), T_H(n)\}$
   (b) $I_{G\cup H}(n) = \max\{I_G(n), I_H(n)\}$
   (c) $F_{G\cup H}(n) = \min\{F_G(n), F_H(n)\}$
6. *Let* $G = \{\langle n, T_G(n), I_G(n), F_G(n)\rangle | n \in N\}$ *Neutrosophic Set on* N*, then the complement of the set* G *(*$C(G)$ *for short), defined as*

$$C(G) = \{\langle n, F_G(n), 1 - I_G(n), T_G(n)\rangle | n \in N\}$$

Definition 2.3 *[7] Let E be a universe of discourse and the NS $N_1 \subset E$. Let $T_N(n), I_N(n), F_N(n)$ membership, indeterminate and non- membership function respectively, a element $n \in E$,*

$$A_1 \ : \ T_N(n), I_n(x), F_N(n) : E \to [0, \Omega]$$

where $0 < 1 < \Omega$ *and* Ω *is called over limit.*

A Single-Valued Neutrosophic Overset N_1 *is defined as:* $N_1 = \{(n, \langle T_N(n), I_N(n), F_N(n)\rangle), x \in E\}$ *such that there exists at least one element in* N_1 *that includes of at least one neutrosophic element* > 1 *and there are no elements with neutrosophic components that are* < 0.

3. Neutrosophic Set and its Variants

Pythagorean Neutrosophic Set

Definition 3.1 *Consider* P *be a not empty set. A PNS in* A_P *is defined*

$$A_P = \{\langle p : \ T_{A_P}(p), I_{A_P}(p), F_{A_P}(p)\rangle | p \in P\}$$

where T_{A_P} *is truth ,* I_{A_P} *is indeterminate,* F_{A_P} *is false degree membership. where*

$$T_{A_p}(p), I_{A_p}(p), F_{A_p}(p) | p \in P \to [0, 1],$$

$$0 \leq [T_{A_p}(p)]^2 + [I_{A_p}(p)]^2 + [F_{A_p}(p)]^2 \leq \sqrt{3}.$$

Definition 3.2 *Consider* P *be a non-empty set and the PNSs* A_p *and* B_p *be in the form*

$$A_p = \{\langle p, T_{A_p}(p), I_{A_p}(p), F_{A_p}(p)\rangle | p \in P\}$$

and $B_p = \{\langle p, T_{B_p}(p), I_{B_p}(p), F_{B_p}(p)\rangle | p \in P\}$

1. $(A_p) \subseteq (B_p)$ *iff* $T_{A_p}(p) \leq T_{B_p}(p); I_{A_p}(p) \geq I_{B_p}(p)$ *; and* $F_{A_p}(p) \geq F_{B_p}(p)$
2. $(A_p) = (B_p)$ *iff* $T_{A_p}(p) = T_{B_p}(p); I_{A_p}(p) = I_{B_p}(p)$ *; and* $F_{A_p}(p) = F_{B_p}(p)$ $\forall p \in P$
3. $A_p \cap B_p = \{\langle p, T_{A_p \cap B_p}(p), I_{A_p \cap B_p}(p), F_{A_p \cap B_p}(p)\rangle | p \in P\}$ *where,*
   (a) $T_{A_p \cap B_p}(p) = \min\{T_{A_p}(p), T_{B_p}(p)\}$
   (b) $I_{A_p \cap B_p}(p) = \max\{I_{A_p}(p), I_{B_p}(p)\}$
   (c) $F_{A_p \cap B_p}(p) = \max\{F_{A_p}(p), F_{B_p}(p)\}$
4. $A_p \cup B_p = \{\{p, T_{A_p \cup B_p}(p), I_{A_p \cup B_p}(p), F_{A_p \cup B_p}(p)\rangle | p \in P\}$ *where,*
   (a) $T_{A_p \cup B_p}(p) = \max\{T_{A_p}(p), T_{B_p}(p)\}$
   (b) $I_{A_p \cup B_p}(p) = \min\{I_{A_p}(p), I_{B_p}(p)\}$
   (c) $F_{A_p \cup B_p}(p) = \min\{F_{A_p}(p), F_{B_p}(p)\}$

Definition 3.3 *Let* $A_p = \{\langle p, T_{A_p}(p), I_{A_p}(p), F_{A_p}(p)\rangle | p \in P\}$ *PNS on* P*, then the complement of the set* A_P $(C(A_P))$*, defined as*

(C_1) $\quad C(A_p) = \{\langle p, F_{A_p}(p), 1 - I_{A_p}(p), T_{A_p}(p)\rangle | p \in P\}$
(C_2) $\quad C(A_p) = \{\langle p, F_{A_p}(p), (T_{A_p}(p) + F_{A_p}(p)) - I_{A_p}(p), T_{A_p}(p)\rangle | p \in P\}$

Definition 3.4 *The PNS* 0_{PN} *and* 1_{PN} *in* P *are outlined below:*
0_{PN} *it can be described as::*

$$(0_1) \quad 0_{PN} = \{\langle p, 0, 0, 1\rangle : p \in P\}$$

$$(0_2) \quad 0_{PN} = \{\langle p, 0, 0, 0\rangle : p \in P\}$$

1_{PN} *it can be described as:*

$$(1_1) \quad 1_{PN} = \{\langle p, 1, 0, 0\rangle \ : \ p \in P\}$$

$$(1_2) \quad 1_{PN} = \{\langle p, 0, 1, 0\rangle \ : \ p \in P\}$$

In this case, the integers 0 *and* 1 *belong to the constant mappings that translate each element of* P *to* 1 *and* 0.

Fermatean Neutrosophic set

Definition 3.5 *Consider a set F that is not empty. A Fermatean Neutrosophic set (FNS) in A_F is clarified $A_F = \{\langle f, (T_{A_F}(f), I_{A_F}(f), F_{A_F}(f))\rangle | f \in F\}$ where T_{A_F} is truth, I_{A_F} is indeterminate, F_{A_F} is false degree membership. Where, $T_{A_F}(f), I_{A_F}(f), F_{A_F}(f) | f \in F \to [0,1]$ and $0 \leq T^3_{A_F}(f) + I^3_{A_F}(f) + F^3_{A_F}(f) \leq \sqrt{3} \ \ \forall f \in F$.*

Definition 3.6 *Consider a set F is a non-empty set and the FNSs A_F and B_F is clarified $A_F = \{\langle f, T_{A_F}(f), I_{A_F}(f), F_{A_F}(f)\rangle | f \in F\}$ and $B_F = \{\langle f, T_{B_F}(f), I_{B_F}(f), F_{B_F}(f)\rangle | f \in F\}$*

1. $(A_F) \subseteq (B_F)$ *iff* $T_{A_F}(f) \leq T_{B_F}(f); I_{A_F}(f) \geq I_{B_F}(f)$; *and* $F_{A_F}(f) \geq F_{B_F}(f)$
2. $(A_F) = (B_F)$ *iff* $T_{A_F}(f) = T_{B_F}(f); I_{A_F}(f) = I_{B_F}(f)$; *and* $F_{A_F}(f) = F_{B_F}(f) \ \forall f \in F$
3. $A_F \cap B_F = \{\langle f, T_{A_F \cap B_F}(f), I_{A_F \cap B_F}(f), F_{A_F \cap B_F}(f)\rangle | f \in F\}$ *where,*
   (a) $T_{A_F \cap B_F}(f) = \min\{T_{A_F}(f), T_{B_F}(f)\}$
   (b) $I_{A_F \cap B_F}(f) = \max\{I_{A_F}(f), I_{B_F}(f)\}$
   (c) $F_{A_F \cap B_F}(f) = \max\{F_{A_F}(f), F_{B_F}(f)\}$
4. $A_F \cup B_F = \{\langle f, T_{A_F \cup B_F}(f), I_{A_F \cup B_F}(f), F_{A_F \cup B_F}(f)\rangle | f \in F\}$ *where,*
   (a) $T_{A_F \cup B_F}(f) = \max\{T_{A_F}(f), T_{B_F}(f)\}$
   (b) $I_{A_F \cup B_F}(f) = \min\{I_{A_F}(f), I_{B_F}(f)\}$
   (c) $F_{A_F \cup B_F}(f) = \min\{F_{A_F}(f), F_{B_F}(f)\}$

Definition 3.7 *Let $A_F = \{\langle f, T_{A_F}(f), I_{A_F}(f), F_{A_F}(f)\rangle | f \in F\}$ Fermatean Neutrosophic Set on X, then the complement of the set A_F ($C(A_F)$), defined as*
(C_1) $C(A_F) = \{\langle f, 1 - T_{A_F}(f), I_{A_F}(f), 1 - F_{A_F}(f)\rangle | f \in F\}$
(C_2) $C(A_F) = \{\langle f, F_{A_F}(f), I_{A_F}(f), T_{A_F}(f)\rangle | f \in F\}$
(C_3) $C(A_F) = \{\langle f, F_{A_F}(f), 1 - I_{A_F}(f), T_{A_F}(f)\rangle | f \in F\}$

Definition 3.8 *The FNS 0_{FN} and 1_{FN} in F are outlined below:*
0_{FN} may be defined as:

$$(0_1) \quad 0_{FN} = \{\langle f, 0, 0, 1\rangle : f \in F\}$$
$$(0_2) \quad 0_{FN} = \{\langle f, 0, 0, 0\rangle : f \in F\}$$

1_{FN} may be defined as:

$$(1_1) \quad 1_{FN} = \{\langle f, 1, 0, 0\rangle : f \in F\}$$
$$(1_2) \quad 1_{FN} = \{\langle f, 0, 1, 0\rangle : f \in F\}$$

In this case, the integers 0 and 1 belong to the constant mappings that translate each element of F to 1 and 0.

4. Neutrosophic gradation of openness: Neutrosophic over set

Definition 4.1 *Consider a Neutrosophic Over set (NOS) X is non-empty. A $(N - go)$ of neutrosophic over subsets of X, is a triplet $(\tau^T_{over}, \tau^I_{over}, \tau^F_{over})$ of functions $\tau^T_{over}, \tau^I_{over}, \tau^F_{over} : I^X \to I$ such that*

1. $\tau^T_{over}(\lambda) + \tau^l_{over}(\lambda) + \tau^F_{over}(\lambda) \leq 3, \forall \lambda \in I^X$
2. $\tau^T_{over}(0) = \tau^T_{over}(\Omega) = 1, \tau^I_{over}(0) = \tau^I_{over}(\Omega) = 1, \tau^F_{over}(0) = \tau^F_{over}(\Omega) = 0$
   $\tau^T_{over} \ (\lambda_1 \ \cap \ \lambda_2) \geq \tau^T_{over}(\lambda_1) \wedge \tau^T_{over}(\lambda_2)$,
   $\tau^I_{over}(\lambda_1 \ \cap \ \lambda_2) \geq \tau^I_{over}(\lambda_1) \wedge \tau^I_{over}(\lambda_2)$

3. $\tau^F_{over}\ (\lambda_1\ \cap\ \lambda_2) \leq \tau^F_{over}(\lambda_1) \vee \tau^F_{over}(\lambda_2), \lambda_i \in I^X, i = 1, 2$

4. $\tau^T_{Over}(\bigcup_{n\in\triangle} \lambda_i) \geq \bigwedge_{n\in\triangle} \tau^T_{over}(\lambda_i)$

   $\tau^I_{Over}(\bigcup_{n\in\triangle} \lambda_i) \geq \bigwedge_{n\in\triangle} \tau^I_{over}(\lambda_i)$,

   $\tau^F_{Over}(\bigcup_{n\in\triangle} \lambda_i) \leq \bigvee_{i\in\triangle}, \lambda_i \in I^X, i \in \Delta$.

Where $\tau^T_{Over}, \tau^I_{Over}$ and τ^F_{Over} are an independent component .The triplet $(\tau^T_{Over}, \tau^I_{Over}, \tau^F_{Over})$ is a $(N-go)$ on X. Then the collections $(X, \tau^T_{Over}, \tau^I_{Over}, \tau^F_{Over})$ is recognised as a NOTS.

Similarly can define the concepts of PNGO, and FNGO.

Example 4.1 *Let $X = \{a, b, c\}$ be a nonempty Neutrosophic Over set. A NGO of neutrosophic over subsets of X*

$$X_{N_{over}} = \left\{ \begin{array}{l} \langle x, (\frac{a}{1.3}, \frac{b}{0.4}, \frac{c}{0.6}), (\frac{a}{0.4}, \frac{b}{1.2}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.2})\rangle, \\ \langle y, (\frac{a}{0.7}, \frac{b}{0.3}, \frac{c}{0.8}), (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.8}, \frac{c}{1.1})\rangle : x, y \in [0, \Omega] \end{array} \right\}$$

$$\tau_{N_{over}} = \left\{ \begin{array}{l} \langle(\frac{a}{1.3}, \frac{b}{0.4}, \frac{c}{0.6}), (\frac{a}{0.4}, \frac{b}{1.2}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.5}, \frac{c}{0.2})\rangle \\ \langle(\frac{a}{0.7}, \frac{b}{0.3}, \frac{c}{0.8}), (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.8}, \frac{c}{1.1})\rangle \\ \langle(\frac{a}{0.7}, \frac{b}{0.3}, \frac{c}{0.6}), (\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.4}), (\frac{a}{0.5}, \frac{b}{0.8}, \frac{c}{0.2})\rangle \\ \langle(\frac{a}{1.3}, \frac{b}{0.4}, \frac{c}{0.8}), (\frac{a}{0.4}, \frac{b}{1.2}, \frac{c}{0.5}), (\frac{a}{0.6}, \frac{b}{0.8}, \frac{c}{1.1})\rangle : x, y \in [0, \Omega] \end{array} \right\}$$

Gradation of Openness

$$\tau^{T_1}_{N_{over}}\left(\frac{a}{0.7}, \frac{b}{0.6}, \frac{c}{1.1}\right) = 0.6$$

$$\tau^{I_1}_{N_{over}}\left(\frac{a}{1.3}, \frac{b}{0.7}, \frac{c}{0.2}\right) = 0.5$$

$$\tau^{F_1}_{N_{over}}\left(\frac{a}{0.1}, \frac{b}{1.2}, \frac{c}{0.4}\right) = 0.7$$

$$\tau^{T_2}_{N_{over}}\left(\frac{a}{0.7}, \frac{b}{0.3}, \frac{c}{0.6}\right) = 0.9$$

$$\tau^{I_2}_{N_{over}}\left(\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{0.4}\right) = 0.8$$

$$\tau^{F_2}_{N_{over}}\left(\frac{a}{0.5}, \frac{b}{0.6}, \frac{c}{1.1}\right) = 0.8$$

1. $\tau^{T_1}_{N_{over}}(\lambda) + \tau^{I_1}_{N_{over}}(\lambda) + \tau^{F_1}_{N_{over}}(\lambda) \leq 3, \forall \lambda \in I^X : X \to [0, \Omega]$

$$0.6 + 0.5 + 0.7 \leq 3$$

$$1.8 \leq 3$$

$$\tau^{T_2}_{N_{over}}(\lambda) + \tau^{I_2}_{N_{over}}(\lambda) + \tau^{F_2}_{N_{over}}(\lambda) \leq 3, \forall \lambda \in I^X : X \to [0, \Omega]$$

$$0.9 + 0.8 + 0.8 \leq 3$$

$$2.5 \leq 3$$

2. $\tau^T_{over}(0) = \tau^T_{over}(\Omega) = 1, \tau^I_{over}(0) = \tau^I_{over}(\Omega) = 1, \tau^F_{over}(0) = \tau^F_{over}(\Omega) = 0$

3. $\tau^T_{N_{over}}(\lambda_1\ \cap\ \lambda_2) \geq \tau^T_{N_{over}}(\lambda_1) \wedge \tau^T_{N_{over}}(\lambda_2), \lambda_i \in I^X, i = 1, 2$

$$\tau^T_{N_{over}}\left(\frac{a}{0.7}, \frac{b}{0.3}, \frac{c}{1.1}\right) \Rightarrow 0.8 \geq 0.6$$

$$\tau^I_{N_{over}}\ (\lambda_1\ \cap\ \lambda_2) \geq \tau^I_{N_{over}}(\lambda_1) \wedge \tau^I_{N_{over}}(\lambda_2), \lambda_i \in I^X, i = 1, 2$$

$$\tau^I_{N_{over}}\left(\frac{a}{0.3},\frac{b}{0.5},\frac{c}{0.4}\right) \Rightarrow 1 \geq 0.5$$

$$\tau^F_{N_{over}}(\lambda_1 \cap \lambda_2) \leq \tau^F_{N_{over}}(\lambda_1) \vee \tau^F_{N_{over}}(\lambda_2), \lambda_i \in I^X, i = 1,2$$

$$\tau^F_{N_{over}}\left(\frac{a}{0.5},\frac{b}{1.2},\frac{c}{0.4}\right) \Rightarrow 0.5 \leq 0.7$$

4.

$$\tau^T_{N_{over}}\left(\frac{a}{1.4},\frac{b}{0.6},\frac{c}{0.8}\right) \Rightarrow 0.8 \geq 0.6$$

$$\tau^I_{N_{Over}}\left(\bigcup_{i\in\triangle}\lambda_i\right) \geq \bigwedge_{i\in\triangle}\tau^I_{N_{Over}}(\lambda_i), \lambda_i \in I^X, i \in \triangle$$

$$\tau^I_{N_{over}}\left(\frac{a}{0.8},\frac{b}{1.1},\frac{c}{0.4}\right) \Rightarrow 0.7 \geq 0.5$$

$$\tau^F_{N_{Over}}\left(\bigcup_{i\in\triangle}\lambda_i\right) \leq \bigvee_{i\in\triangle}\tau^I_{N_{Over}}(\lambda_i), \lambda_i \in I^X, i \in \triangle.$$

$$\tau^F_{N_{over}}\left(\frac{a}{0.6},\frac{b}{0.7},\frac{c}{1.1}\right) \Rightarrow 0.9 \geq 0.7$$

Definition 4.2 *Consider Neutrosophic Over set X is nonempty and $F, F^* : I^X \to I$ be two mappings satisfying*

1. $F^T_{over}(\lambda) + F^I_{over}(\lambda) + F^F_{over}(\lambda) \leq 3, \forall\lambda \in I^X$
2. $F^T_{over}(0) = F^T_{over}(\Omega) = 1, F^I_{over}(0) = F^I_{over}(\Omega) = 1, F^F_{over}(0) = F^F_{over}(\Omega) = 0$
3. $F^T_{over}(\lambda_1 \cup \lambda_2) \geq F^T_{over}(\lambda_1) \wedge F^T_{over}(\lambda_2)$,
   $F^I_{over}(\lambda_1 \cup \lambda_2) \geq F^I_{over}(\lambda_1) \wedge F^I_{over}(\lambda_2)$
   $F^F_{over}(\lambda_1 \mathrm{U} \lambda_2) \leq F^F_{over}(\lambda_1) \vee F^F_{over}(\lambda_2),\ \lambda_i \in I^X,\ i = 1,2$
4. $F^T_{Over}\left(\bigcap_{i\in\triangle}\lambda_i\right) \geq \bigwedge_{i\in\triangle} F^T_{Over}(\lambda_i)$,

   $F^I_{Over}\left(\bigcap_{i\in\triangle}\lambda_i\right) \geq \bigwedge_{i\in\triangle} F^I_{Over}(\lambda_i)$

   $F^F_{Over}\left(\bigcap_{i\in\triangle}\lambda_i\right) \leq \bigvee_{i\in\triangle} F^F_{Over}(\lambda_i), \lambda_i \in I^X, i \in \Delta.$

Then the triplet $(\tau^T_{Over}, \tau^I_{Over}\tau^F_{Over})$ is a neutrosophic gradation of closedness on X (NGC on X) .
Similarly can define the concepts of PNGC, and FNGC.

Definition 4.3 *For two pairs of mappings (τ^T, τ^I, τ^F) and (F^T, F^I, F^F) from $I^X \to I$ define $\tau^T_{F^T}(\lambda) = F^T(\lambda^C), \tau^I_{F^I}(\lambda) = F^I(\lambda^C), \tau^F_{F^F}(\lambda) = F^F(\lambda^C)$ $F^T_{\tau^T}(\lambda) = \tau^T(\lambda^C), F^I_{\tau^I}(\lambda) = \tau^I(\lambda^C), F^F_{\tau^F}(\lambda) = \tau^F(\lambda^C)$*

Theorem 4.1 1. *(τ^T, τ^I, τ^F) is an NGO on X iff $(F^T_{\tau^T}{}_\tau F^l_{\tau^l}, F^F_{\tau^F})$ is an NGC on X,*

2. *(F^T, F^I, F^F) is an NGC on X iff $(\tau^T_{F}{}_\tau T^l_{F^l} T^F_{F^F})$ is an NGO on X,*
   $\tau^T_{F^T} = \tau^T, \tau^I_{F^I} = \tau^I, \tau^F_{F^F} = \tau^F$

3. $F^T_{\tau^T_{F^T}} = F^T, F^I_{\tau^I_{F^I}} = F^I, F^F_{\tau^F_{F^F}} = F^F$

Proof: $F^T_{\tau^T}(\mu) + F^I_{\tau^I}(\mu) + F^F_{\tau^F}(\mu) = \tau^T(\mu^C) + \tau^I(\mu^C) + \tau^F(\mu^C), \forall \mu \in I^X$

So

$$F^T_{\tau^T}(\mu) + F^I_{\tau^I}(\mu) + F^F_{\tau^F}(\mu) \leq 2 iff \tau^T(\mu^C) + \tau^I(\mu^C) + \tau^F(\mu^C) \leq 2, \forall \mu \in I^X \tag{4.1}$$

Clearly,

$F^T_{\tau^T}(0_X) = F^T_{\tau^T}(1_X) = 1 \iff \tau^T(0_X) = \tau^T(1_X) = 1$
$F^I_{\tau^I}(0_X) = F^I_{\tau^I}(1_X) = 1 \iff \tau^I(0_X) = \tau^I(1_X) = 1$
$F^F_{\tau^F}(0_X) = F^F_{\tau^F}(1_X) = 0 \iff \tau^F(0_X) = \tau^F(1_X) = 0$

$$F^T_{\tau^T}(\mu_1 \cup \mu_2) = \tau^T[(\mu_1 \cup \mu_2)^c] = \tau^T(\mu_1^c \cap \mu_2^c) \tag{4.2}$$

so

$F^T_{\tau^T}(\mu_1 \cup \mu_2) \geq F^T_{\tau^T}(\mu_1) \wedge F^T_{\tau^T}(\mu_2), \forall \mu_1, \mu_2 \in I^X$
$\iff \tau^T(\mu_1^C \cap \mu_2^C) \geq \tau^T(\mu_1^C) \wedge \tau^T(\mu_2^c), \forall \mu_1, \mu_2 \in I^X$

$$\iff \tau^T(\mu_1 \cap \mu_2) \geq \tau^T(\mu_1) \wedge \tau^T(\mu_2), \forall \mu_1, \mu_2 \in I^X (since (\mu^c)^c = \mu) \tag{4.3}$$

Similarly

$$F^I_{\tau^I}(\mu_1 \cup \mu_2) \geq F^I_{\tau^I}(\mu_1) \wedge F^I_{\tau^I}(\mu_2) \iff \tau^I(\mu_1 \cap \mu_2) \geq \tau^I(\mu_1) \wedge \tau^T(\mu_2), \forall \mu_1, \mu_2 \in I^X \tag{4.4}$$

$$F^F_{\tau^F}(\mu_1 \cap \mu_2) \geq F^F_{\tau^F}(\mu_1) \vee F^F_{\tau^F}(\mu_2) \iff \tau^F(\mu_1 \cap \mu_2) \leq \tau^F(\mu_1) \vee \tau^F(\mu_2), \forall \mu_1, \mu_2 \in I^X \tag{4.5}$$

$F^T_{\tau^T}\left(\bigcap_{i\in\Delta} \mu_i\right) = \tau^T\left[\left(\bigcap_{i\in\Delta} \mu_i\right)^c\right] = \tau^T\left(\bigcap_{i\in\Delta} \mu_i^c\right)$.

So

$F^{\tau_{T^T}}\left(\bigcap_{i\in\Delta} \mu_i\right) \geq \bigwedge_{i\in\Delta} F^I_{\tau_T}(\mu_i), \mu_i \in I^X, i \in \Delta$

$\iff \tau^T\left(\bigcup_{i\in\Delta} \mu_i^c\right) \geq \bigwedge_{i\in\Delta} \tau^T(\mu_i^c), \mu_i \in I^X, i \in \Delta$

$$\iff \tau^T\left(\bigcup_{i\in\Delta} \mu_i\right) \geq \tau^T(\mu_i), \mu_i \in I^X, i \in \Delta \tag{4.6}$$

Similarly

$$F^I_{\tau^I}\left(\bigcap_{i\in\triangle} \mu_i\right) \geq \bigwedge_{i\in\triangle} F^I_{\tau^I}(\mu_i) \iff \tau_I\left(\bigcup_{i\in\triangle} \mu_i\right) \geq \bigwedge_{i\in\triangle} \tau^I(\mu_i), \mu_i \in I^X, i \in \Delta \tag{4.7}$$

$$F^F_{\tau^F}\left(\bigcap_{i\in\triangle} \mu_i\right) \leq \bigvee_{i\in\triangle} F^F_{\tau^I}(\mu_i) \iff \tau^F\left(\bigcup_{i\in\triangle} \mu_i\right) \leq \bigvee_{i\in\triangle} \tau^F(\mu_i), \mu_i \in I^X, i \in \Delta \tag{4.8}$$

Hence by 4.1,4.2,4.3,4.4,4.5,4.6,4.7 and 4.8 (1) hold.
(2) The proof of (2) is similar to (1).
(3) The proof is straightforward.

Definition 4.4 *Let* $\{(\tau^T, \tau^I, \tau^F)\}_{i\in\Delta}$ *be a family of NGOs on* X*. Then their intersection is defined by*

$\bigcap_{i\in\Delta} (\tau^T, \tau^I, \tau^F) = \left(\bigwedge_{i\in\triangle} \tau_i^T, \bigwedge_{i\in\triangle} \tau_i^I, \bigvee_{i\in\triangle} \tau_i^T\right)$, *where*

$\left(\bigwedge_{i\in\triangle} \tau_i^T\right)(\mu) = \left(\tau_i^T(\mu)\right),$

$$\left(\bigwedge_{i\in\Delta}\tau_i^I\right)(\mu)=\bigwedge_{i\in\Delta}\left(\tau_i^I(\mu)\right),$$
$$\left(\bigvee_{i\in\triangle}\tau_i^F\right)(\mu)=\bigvee_{i\in\triangle}(\tau_i^F(\mu)),\ \ \mu\in I^X$$

Theorem 4.2 *An arbitrary intersection of NGOs is an NGO.*

Definition 4.5 *Let* $(\tau_1^T,\tau_1^I,\tau_1^F)$ *and* $(\tau_2^T,\tau_2^I,\tau_2^F)$ *be two NGOs on* X*. Then we define a relation '* $\leq$ *' by* $(\tau_1^T,\tau_1^I,\tau_1^F)\leq(\tau_2^T,\tau_2^l,\tau_2^F)\iff\tau_1^T\leq\tau_2^T,\tau_1^I\leq\tau_2^I,\tau_1^F\geq\tau_2^F$

5. Neutrosophic Topological Spaces and its Variants

Pythagorean Neutrosophic Topological Space

Definition 5.1 *Consider* P *is non-empty set and* $\tau\subset PNTS$*. Then* τ *is called a PNTS in the sense of chang if the next axioms are met:*

1. $0_{PN},1_{PN}\in\tau$ *for each* $T_{A_P},I_{A_P},F_{A_P}\in[0,1]$ *with* $0\leq T_{A_P}^2+I_{A_F}^2+F_{A_P}^2\leq\sqrt{2}$
2. *for any* $P_1,P_2\in\tau$*, implies* $P_1\cap P_2\in\tau$.
3. *for any* $\{P_n|n\in I\}$*, implies* $\bigcup_{n\in I}P_n\in\tau$.

Example 5.1 *Let* $P=\{a,b,c\}$

$$A=\left\langle p,\left(\frac{a}{1},\frac{b}{0.5},\frac{c}{0.3}\right),\left(\frac{a}{0.6},\frac{b}{1},\frac{c}{0.7}\right),\left(\frac{a}{0.2},\frac{b}{0.4},\frac{c}{0.8}\right)\right\rangle,$$

$$B=\left\langle p,\left(\frac{a}{0.9},\frac{b}{0.5},\frac{c}{0.5}\right),\left(\frac{a}{0.6},\frac{b}{0.8},\frac{c}{0.4}\right),\left(\frac{a}{0.3},\frac{b}{0.5},\frac{c}{1}\right)\right\rangle,$$

$$C=\left\langle p,\left(\frac{a}{1},\frac{b}{0.5},\frac{c}{0.5}\right),\left(\frac{a}{0.6},\frac{b}{1},\frac{c}{0.7}\right),\left(\frac{a}{0.2},\frac{b}{0.4},\frac{c}{0.8}\right)\right\rangle,$$

$$D=\left\langle p,\left(\frac{a}{0.9},\frac{b}{0.5},\frac{c}{0.3}\right),\left(\frac{a}{0.6},\frac{b}{0.8},\frac{c}{0.4}\right),\left(\frac{a}{0.3},\frac{b}{0.5},\frac{c}{1}\right)\right\rangle$$

$\tau=\{0_{PN},1_{PN},A,B,C,D\}$ *The collection* τ *is recognised as PNT on* P*, the pair* (P,τ) *a PNTS.*

Definition 5.2 *Consider* P *is non-empty set and* $\tau\subset PNTS$*. Then* τ *is called a PNTS in the sense of Lowen if the next axioms are met:*

1. $A_P\in\delta$ *for each* $T_{A_P},I_{A_P},F_{A_P}\in\delta\subseteq[0,1]$ *with* $0\leq T_{A_P}^2+I_{A_P}^2+F_{A_P}^2\leq\sqrt{2}$
2. *for any* $A_{P_1},A_{P_2}\in\tau$*, implies* $A_{P_1}\cap A_{P_2}\in\tau$.
3. *for any* $\{A_{P_n}|n\in I\}$*, implies* $\bigcup_{n\in I}A_{P_n}\in\tau$.

Then the pair (P,τ) *is called a PNTS.*

Example 5.2 *Let* $P=\{a,b,c\}$

$$A=\left\langle p,\left(\frac{a}{1},\frac{b}{0.5},\frac{c}{0.3}\right),\left(\frac{a}{0.6},\frac{b}{1},\frac{c}{0.7}\right),\left(\frac{a}{0.2},\frac{b}{0.4},\frac{c}{0.8}\right)\right\rangle,$$

$$B=\left\langle p,\left(\frac{a}{0.9},\frac{b}{0.5},\frac{c}{0.5}\right),\left(\frac{a}{0.6},\frac{b}{0.8},\frac{c}{0.4}\right),\left(\frac{a}{0.3},\frac{b}{0.5},\frac{c}{1}\right)\right\rangle,$$

$$C = \left\langle p, \left(\frac{a}{1}, \frac{b}{0.5}, \frac{c}{0.5}\right), \left(\frac{a}{0.6}, \frac{b}{1}, \frac{c}{0.7}\right), \left(\frac{a}{0.2}, \frac{b}{0.4}, \frac{c}{0.8}\right)\right\rangle,$$

$$D = \left\langle p, \left(\frac{a}{0.9}, \frac{b}{0.5}, \frac{c}{0.3}\right), \left(\frac{a}{0.6}, \frac{b}{0.8}, \frac{c}{0.4}\right), \left(\frac{a}{0.3}, \frac{b}{0.5}, \frac{c}{1}\right)\right\rangle$$

$\tau = \{A, B, C, D\}$

The collection τ is called a PNT on P, the pair (P, τ) a PNTS.

Definition 5.3 *The complement $C(A_P)$ of an PNS A_P in an PNTS (X, τ) is called a Pythagorean Neutrosophic closed set (PNCS) in X.*

Definition 5.4 *Let (P, τ) be a PNTS and $A_P = \{\langle P : T_{A_P}(p), I_{A_P}(p), F_{A_P}(p)\rangle | p \in P\}$ be an PNS in X. Then the Pythagorean Neutrosophic closure of A_P are defined by*

$Cl(A_P) = \cap\{\ K : K$ is a PNCS in P and $A_P \subseteq K\}$.

Definition 5.5 *Let (P, τ) be a PNTS and $A_P = \{\langle p : T_{A_P}(p), I_{A_P}(p), F_{A_P}(p)\rangle | p \in P\}$ be an PNS in P. Then the Pythagorean Neutrosophic Interior of A_P are defined by*

int $(A_P) = \cup\{\ G : G$ is a PNOS in P and $G \subseteq A_P\}$.

Fermatean Neutrosophic Topological Space

Definition 5.6 *Consider a non-empty set F and $\tau \subset FNTS$. Then τ is known as FNTS in the sense of chang if the next axioms are met:*

1. *$0_{FN}, 1_{FN} \in \tau$ for each $T_{A_F}, I_{A_F}, F_{A_F} \in [0, 1]$ with $0 \leq T^3_{A_F} + I^3_{A_F} + F^3_{A_F} \leq \sqrt{3}$*
2. *for any $F_1, F_2 \in \tau$, implies $F_1 \cap F_2 \in \tau$.*
3. *for any $\{F_n | n \in I\}$, implies $\bigcup_{n\in I} F_n \in \tau$.*

Example 5.3 *Let $F = \{a, b, c\}$*

$$A = \left\langle f, \left(\frac{a}{0.9}, \frac{b}{0.8}, \frac{c}{0.6}\right), \left(\frac{a}{0.9}, \frac{b}{1}, \frac{c}{0.9}\right), \left(\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.9}\right)\right\rangle,$$

$$B = \left\langle f, \left(\frac{a}{1}, \frac{b}{0.6}, \frac{c}{0.7}\right), \left(\frac{a}{0.9}, \frac{b}{0.9}, \frac{c}{0.8}\right), \left(\frac{a}{0.1}, \frac{b}{0.9}, \frac{c}{0.8}\right)\right\rangle,$$

$$C = \left\langle f, \left(\frac{a}{1.0}, \frac{b}{0.8}, \frac{c}{0.7}\right), \left(\frac{a}{0.9}, \frac{b}{1.0}, \frac{c}{0.9}\right), \left(\frac{a}{0.1}, \frac{b}{0.6}, \frac{c}{0.8}\right)\right\rangle,$$

$$D = \left\langle f, \left(\frac{a}{0.9}, \frac{b}{0.6}, \frac{c}{0.6}\right), \left(\frac{a}{0.9}, \frac{b}{0.9}, \frac{c}{0.8}\right), \left(\frac{a}{0.6}, \frac{b}{0.9}, \frac{c}{0.9}\right)\right\rangle$$

$\tau = \{0_{FN}, 1_{FN}, A, B, C, D\}$

The collection τ is called a FNT on F, the pair (F, τ) a FNTS.

Definition 5.7 *Consider a non-empty set F and $\tau \subset FNTS$. Then τ is known as FNTS in the sense of Lowen if it the next axioms are met:*

1. *$A_F \in \delta$ for each $T_{A_F}, I_{A_F}, F_{A_F} \in \delta \subseteq [0, 1]$ with $T^3_{A_F}(x) + I^3_{A_F}(x) + F^3_{A_F}(x) \leq \sqrt{3}$*
2. *for any $A_{F_1}, A_{F_2} \in \tau$, implies $A_{P_1} \cap A_{P_2} \in \tau$.*
3. *for any $\{A_{F_n} | n \in I\}$, implies $\bigcup_{n\in I} A_{F_n} \in \tau$.*

then the pair (X, τ) is called a FNTS.

Example 5.4 *Let $F = \{a, b, c\}$*

$$A = \left\langle f, \left(\frac{a}{0.9}, \frac{b}{0.8}, \frac{c}{0.6}\right), \left(\frac{a}{0.9}, \frac{b}{1}, \frac{c}{0.9}\right), \left(\frac{a}{0.6}, \frac{b}{0.6}, \frac{c}{0.9}\right)\right\rangle,$$

$$B = \left\langle f, \left(\frac{a}{1}, \frac{b}{0.6}, \frac{c}{0.7}\right), \left(\frac{a}{0.9}, \frac{b}{0.9}, \frac{c}{0.8}\right), \left(\frac{a}{0.1}, \frac{b}{0.9}, \frac{c}{0.8}\right)\right\rangle,$$

$$C = \left\langle f, \left(\frac{a}{1.0}, \frac{b}{0.8}, \frac{c}{0.7}\right), \left(\frac{a}{0.9}, \frac{b}{1.0}, \frac{c}{0.9}\right), \left(\frac{a}{0.1}, \frac{b}{0.6}, \frac{c}{0.8}\right)\right\rangle,$$

$$D = \left\langle f, \left(\frac{a}{0.9}, \frac{b}{0.6}, \frac{c}{0.6}\right), \left(\frac{a}{0.9}, \frac{b}{0.9}, \frac{c}{0.8}\right), \left(\frac{a}{0.6}, \frac{b}{0.9}, \frac{c}{0.9}\right)\right\rangle$$

$\tau = \{A, B, C, D\}$
The collection τ is called a FNT on F, the pair (F, τ) a FNTS.

Definition 5.8 *The complement $C(A_F)$ of a FNS A_F in an FNTS $(F,\ \tau)$ is known as Fermatean Neurosophic closed set(NSCS) in F*

Definition 5.9 *Consider (F, τ) be a FNTS and $A_F = \{\langle f : T_{A_F}(f), I_{A_F}(f), F_{A_F}(f)\rangle | f \in F\}$ be an FNS in F. Then the Fermatean Neurosophic closure of A_F are defined by*
$Cl(A_F) = \cap\{\ K : K$ *is a FNCS in* F *and* $A_F \subseteq K\}$.

Definition 5.10 *Let (F, τ) be a FNTS and*
$A_F = \{\langle f : T_{A_F}(f), I_{A_F}(f), F_{A_F}(f)\rangle | f \in F\}$ *be an FNS in F. Then the Fermatean Neurosophic Interior of A_F are defined by*
int $(A_F) = \cup\{\ G : G$ *is a FNOS in* F *and* $G \subseteq A_F\}$.

6. Conclusion

The topological settings of Pythagorean and Fermatean neutroposhic sets are examined and their topological structures are defined. Additionally, the topologies in the sense of Chang and Lowen are derived and suitable examples are provided. The idea of neutrosophic gradation of openness on neutrosophic subsets of a nonempty set as well as the idea of neutrosophic over topological space is discussed. Further the research can be extended with underset (degrees < 0), offset (some degrees > 1 and others < 0), refined neutrosophic topology, neutrosophic superhyper topology, hypersoft set, indetermsoft Set, Indetermhypersoft set, and TreeSoft Set".

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